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Context-sensitive Innermost Reachability is Decidable for Linear Right-shallow Term Rewriting Systems

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The reachability problem for given an initial term, a goal term, and a term rewriting system (TRS) is to decide whether the initial one is reachable to the goal one by the TRS or not. A term is shallow if each variable in the term occurs at depth 0 or 1. Innermost reduction is a strategy that rewrites innermost redexes, and context-sensitive reduction is a strategy in which rewritable positions are indicated by specifying arguments of function symbols. In this paper, we show that the reachability problem under context-sensitive innermost reduction is decidable for linear right-shallow TRSs. Our approach is based on the tree automata technique that is commonly used for analysis of reachability and its related properties. We show a procedure to construct tree automata accepting the sets of terms reachable from a given term by context-sensitive innermost reduction of a given linear right-shallow TRS.

1. Introduction

The reachability problem for given two terms s, t, and a term rewriting system (TRS) R is to decide whether s is reachable to t by R or not. Decision procedures of the problem are applicable to security protocol verification and solving other problems of TRSs. Since it is known that this problem is undecidable for general TRSs, efforts have been made to find subclasses of TRSs in which the reachability is decidable. Reachability properties for several subclasses of TRSs have been proved to be decidable[3),4),6),11)–16)] as shown in **Fig. 1**.

Innermost reduction, a strategy that rewrites innermost redexes, is known as call-by-value computation widely used in most programming languages. Context-sensitive reduction[9)] is a strategy in which rewritable positions are indicated by specifying arguments of function symbols, and is used in evaluating `if` ··· `then` ··· `else` ··· or `case` structures. Therefore, the languages that adopt call-by-value computation and `if` ··· `then` ··· `else` ··· structure (e.g., C language) have computation models defined by context-sensitive innermost reduction. For innermost reduction and context-sensitive reduction, some decidable classes of reachability are known[5),8)], but it is not known for context-sensitive innermost reduction.

The reachability relation by context-sensitive innermost reduction is not equal to the intersection of reachability relations for innermost reductions and context-sensitive reduction, as shown in the following Example 1. Thus the decidability of reachability by context-sensitive innermost reductions cannot be obtained by the direct combination of the two existing results.

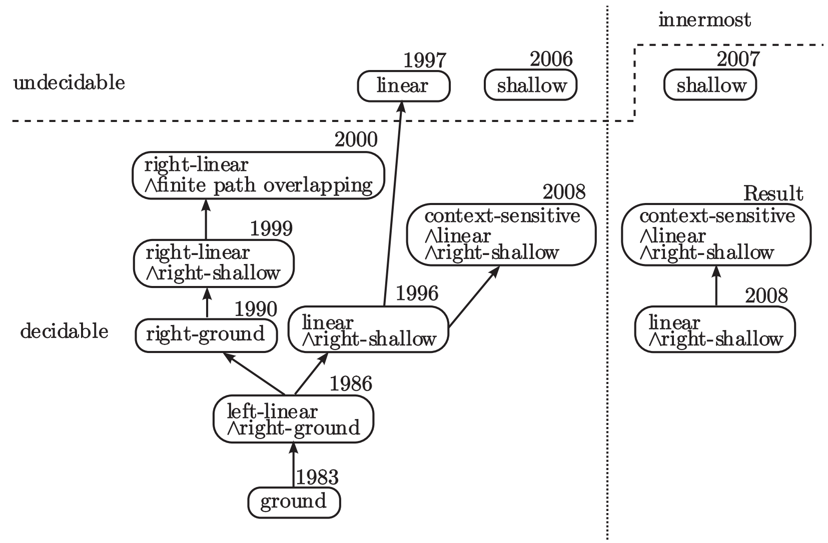

Fig. 1 Major subclasses of TRS in which the reachability is decidable or undecidable.

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Example 1 Consider the following context-sensitive TRS:

$$\left\{\begin{array}{c} \mathtt{if}(\mathtt{true}, x, y) \to x \\ \mathtt{if}(\mathtt{false}, x, y) \to y \\ a \to b \\ f(x) \to g(x) \end{array}\right\} \quad \begin{array}{c} \mu(\mathtt{if}) = \mu(g) = \{1\} \\ \mu(f) = \emptyset \end{array}$$

where reductions below f are inhibited. Then $g(a)$ is reachable from $\mathtt{if}(\mathtt{true}, f(a), a)$ by context-sensitive innermost reduction, but not reachable by innermost reduction. □

This paper shows a procedure to construct a tree automaton (TA) accepting the set of terms reachable from a given term if a given TRS is linear and right-shallow, which means that the reachability and the joinability are both decidable. The procedure basically consists of both constructions[8] for innermost reduction and context-sensitive reduction. However, the natural merger of the two procedures is not enough as shown later (Example 10).

2. Preliminary

We use usual notations of term rewriting system[1] and tree automata[2]. Let F be a set of function symbols with fixed arity and X be an enumerable set of variables. The arity of function symbol f is denoted by $\mathrm{ar}(f)$. Function symbols with $\mathrm{ar}(f) = 0$ are *constants*. The set of *terms*, defined in the usual way, is denoted by $\mathcal{T}(F, X)$. A term is *linear* if no variable occurs more than once in the term. The set of variables occurring in t is denoted by $\mathrm{Var}(t)$. A term t is *ground* if $\mathrm{Var}(t) = \emptyset$. The set of ground terms is denoted by $\mathcal{T}(F)$.

A *position* in a term t is defined, as usual, as a sequence of positive integers, and the set of all positions in a term t is denoted by $\mathrm{Pos}(t)$, where the empty sequence ε is used to denote root position. The *depth* of a position p is defined as $|p|$. The *height* $|t|$ of a term t is defined as $max(\{|p| \mid p \in \mathrm{Pos}(t)\})$. A term t is *shallow* if depths of variable occurrences in t are all 0 or 1. The *subterm* of t at position p is denoted by $t|_p$, and $t[t']_p$ represents the term obtained from t by replacing the subterm $t|_p$ by t'. If a term s is a subterm of t and $s \neq t$, s is a *proper* subterm of t. We denote $s \trianglelefteq t$ that a term s is a subterm of t.

A *substitution* σ is a mapping from X to $\mathcal{T}(F, X)$ whose domain $\mathrm{Dom}(\sigma) = \{x \in X \mid x \neq \sigma(x)\}$ is finite. The term obtained by applying a substitution σ to a term t is written as $t\sigma$. The term $t\sigma$ is an *instance* of t.

A *rewrite rule* is an ordered pair of terms in $\mathcal{T}(F, X)$, written as $l \to r$, such that $l \notin X$ and $\mathrm{Var}(l) \supseteq \mathrm{Var}(r)$. We say that variables in $\mathrm{Var}(l) \setminus \mathrm{Var}(r)$ are *erasing*. A *term rewriting system (over F)* (TRS) is a finite set of rewrite rules. *Rewrite relation* $\xrightarrow[R]{}$ induced by a TRS R is as follows: $s \xrightarrow[R]{} t$ if and only if $s = s[l\sigma]_p$, and $t = s[r\sigma]_p$ for some rule $l \to r \in R$, with substitution σ and position $p \in \mathrm{Pos}(s)$. We call $l\sigma$ a *redex*. We sometimes write $\xrightarrow[R]{}^p$ by presenting the position p explicitly.

A rewrite rule $l \to r$ is *left-linear* (resp. *right-linear*, *linear*, *right-shallow*) if l is linear (resp. r is linear, l and r are linear, r is shallow). A TRS R is *left-linear* (resp. *right-linear*, *linear*, *right-shallow*) if every rule in R is left-linear (resp. right-linear, linear, right-shallow).

Let $\to$ be a binary relation on a set $\mathcal{T}(F)$. We say $s \in \mathcal{T}(F)$ is a *normal form* (with respect to $\to$) if there exists no term $t \in \mathcal{T}(F)$ such that $s \to t$. We use $\circ$ to denote the composition of two relations. We write $\xrightarrow{*}$ for the reflexive and transitive closure of $\to$. We also write $\xrightarrow{n}$ for the relation $\to \circ \cdots \circ \to$ composed of n $\to$'s. The set of *reachable terms* from a term in T is defined by $\to[T] = \{t \mid s \in T,\ s \xrightarrow{*} t\}$. The *reachability problem* (resp. *joinability problem*) with respect to $\to$ is a problem that decides whether $s \xrightarrow{*} s'$ (resp. $s \xrightarrow{*} \circ \xleftarrow{*} s'$) or not, for given terms s and s'.

A *tree automaton* (TA) is a quadruple $\mathcal{A} = (F, Q, Q^f, \Delta)$ where Q is a finite set of states, $Q^f (\subseteq Q)$ is a set of final states, and Δ is a finite set of transition rules of the forms $f(q_1, \ldots, q_n) \to q$ or $q_1 \to q$ where $f \in F$ with $\mathrm{ar}(f) = n$, and $q_1, \ldots, q_n, q \in Q$. We can regard Δ as a (ground) TRS over $F \cup Q$. The rewrite relation induced by Δ is called a *transition relation* denoted by $\xrightarrow[\Delta]{}$. We denote $|\alpha|$ as the length of a transition sequence α (if α is $s \xrightarrow{n} t$, then $|\alpha| = n$). We say that a term s ($\in \mathcal{T}(F)$) is *accepted* by $\mathcal{A}$ if $s \xrightarrow[\Delta]{*} q \in Q^f$. The set of all terms accepted by $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$. We say $\mathcal{A}$ *recognizes* $\mathcal{L}(\mathcal{A})$. A set of terms T is *regular* if there exists a TA that recognizes T. We use a notation $\mathcal{L}(\mathcal{A}, q)$ or $\mathcal{L}(\Delta, q)$ to represent the set $\{s \mid s \xrightarrow[\Delta]{*} q\}$. A TA $\mathcal{A}$ is *deterministic* if $s \xrightarrow[\Delta]{*} q$

and $s \xrightarrow[\Delta]{*} q'$ implies $q = q'$ for any $s \in \mathcal{T}(F)$. A TA $\mathcal{A}$ is *complete* if there exists $q \in Q$ such that $s \xrightarrow[\Delta]{*} q$ for any $s \in \mathcal{T}(F)$.

The following properties on TA are known[2)].

Theorem 2 (1) For a given regular set $T \in \mathcal{T}(F)$, there exists a deterministic complete TA $\mathcal{A} = (F, Q, Q^f, \Delta)$ that recognizes T and has no needless states:

$$\forall q \in Q.\mathcal{L}(\Delta, q) \neq \emptyset.$$

(2) The class of regular sets on $\mathcal{T}(F)$ is closed under union, intersection, and complementation.

(3) The membership problem and the emptiness problem are decidable.

3. Context-sensitive Innermost Reduction

We follow usual definitions of context-sensitivity[9)]. A context-sensitive rewrite relation is a subrelation of the ordinary rewrite relation in which rewritable positions are indicated by specifying arguments of function symbols. A mapping $\mu : F \to \mathcal{P}(\mathbb{N})$ is said to be a *replacement map* (or F-map) if $\mu(f) \subseteq \{1, \ldots, \mathrm{ar}(f)\}$ for all $f \in F$. A *context-sensitive term rewriting system* (CS-TRS) is a pair $\mathcal{R} = (R, \mu)$ of a TRS and a replacement map. The set of *μ-replacing positions* $\mathrm{Pos}^\mu(t)$ ($\subseteq \mathrm{Pos}(t)$) is recursively defined: $\mathrm{Pos}^\mu(t) = \{\varepsilon\}$ if t is a constant or a variable, otherwise $\mathrm{Pos}^\mu(f(t_1, \ldots, t_n)) = \{\varepsilon\} \cup \{ip \mid i \in \mu(f), p \in \mathrm{Pos}^\mu(t_i)\}$. The rewrite relation induced by a CS-TRS $\mathcal{R}$ is defined: $s \underset{\mathcal{R}}{\hookrightarrow} t$ if and only if $s \xrightarrow[R]{p} t$ for some $p \in \mathrm{Pos}^\mu(t)$. We call a normal form for CS-TRS $\mathcal{R}$ *context-sensitive normal form* and we denote the set of context-sensitive normal forms by CS-NF$_\mathcal{R}$.

A rewrite step $s \xrightarrow[R]{p} t$ is *innermost* if all proper subterms of $s|_p$ are normal forms. If every proper subterm is context-sensitive normal forms or not in $pos^\mu(s)$, we say that the step is *context-sensitive innermost*, and we denote context-sensitive innermost reduction by $\underset{R}{\hookrightarrow}_{\mathrm{in}}$. A context-sensitive innermost rewrite relation is a subrelation of a context-sensitive rewrite relation, but it is not a subrelation of a innermost rewrite relation as shown in Example 1.

4. Tree Automaton Accepting Context-sensitive Normal Forms

In this section, we give a procedure to construct a deterministic complete TA $\mathcal{A}_{\mathrm{NF}}$ that recognizes the set of context-sensitive normal forms over F for left-linear CS-TRS $\mathcal{R}$. This procedure is similar to ones for TRSs[2)].

The steps of the procedure to construct TA $\mathcal{A}_{\mathrm{NF}}$ is as follows:

(1) Construct TA $\mathcal{A}_l$ that recognizes the set of terms having a redex $l\sigma$ at μ-replacing position for each $l \to r \in R$.

(2) Construct the union of all $\mathcal{A}_l$'s and take the complementation.

(3) Convert the TA constructed in the step (2) into a deterministic and complete TA having no needless states. We output the TA as $\mathcal{A}_{\mathrm{NF}}$.

The step (2) and (3) are obviously possible from Theorem 2. Now we show the detail of the step (1). We use $s^\perp$ to denote the term obtained from a term s by replacing every variable with $\perp$.

Each component of $\mathcal{A}_l$ is as follows:

- $Q_l = \{u^\circ\} \cup \{u_{t^\perp} \mid t \trianglelefteq l\}$
- $Q_l^f = \{u^\circ\}$
- Δ_l consists of following transition rules:

(i) $f(u_\perp, \ldots, u_\perp) \to u_\perp$ for each $f \in F$,

(ii) $f(u_{t_1^\perp}, \ldots, u_{t_n^\perp}) \to u_{f(t_1,\ldots,t_n)^\perp}$ for each $f \in F$ and state $u_{f(t_1,\ldots,t_n)^\perp}$,

(iii) $u_{l^\perp} \to u^\circ$,

(iv) $f(u_1, \ldots, u_n) \to u^\circ$ for each $f \in F$ if exactly one u_j such that $j \in \mu(f)$ is u° and the other u_i's are $u_\perp$.

Note that transition rules (i), (ii), (iii) are the same in the case of TRSs.

We obtain the following lemmas for $\mathcal{A}_l$.

Lemma 3 If l is linear then $\mathcal{L}(\mathcal{A}_l, u_{l^\perp})$ is equal to the set of all ground instances of l, (that is, $\mathcal{L}(\mathcal{A}_l, u_{l^\perp}) = \{l\sigma \in \mathcal{T}(F)\}$.)

Proof:

($\supseteq$) By induction on height $|t|$ of t, we show the claim that $t\sigma \xrightarrow[\Delta_l]{*} u_{t^\perp}$ for every substitution σ and subterm t of l such that $t\sigma$ is ground.

(1) In the case that t is a variable, we have $t^\perp = \perp$. Since transition rules

(i) guarantee that $s \xrightarrow[\Delta_l]{*} u_\perp$ for any ground term s, the claim follows.

(2) Otherwise, we can write $t = f(t_1, \ldots, t_n)$ for $n = \mathrm{ar}(f) \geq 0$. Since $t_i\sigma \xrightarrow[\Delta_l]{*} u_{t_i^\perp}$ by the induction hypothesis, we have $t\sigma = f(t_1, \ldots, t_n)\sigma \xrightarrow[\Delta_l]{*} f(u_{t_1^\perp}, \ldots, u_{t_n^\perp}) \xrightarrow[\Delta_l]{} u_{t^\perp}$ from (ii) of the construction of Δ_l.

($\subseteq$) We show by induction on length $|\alpha|$ that for subterm t of l, if $\alpha: s \xrightarrow[\Delta_l]{*} u_{t^\perp}$ then there exists a substitution σ such that $s = t\sigma$. If t is a variable x, we can take σ so that $x\sigma = s$. Otherwise, since the transition rule used at the last step of α is constructed by (ii), we can write α as $s = f(s_1, \ldots, s_n) \xrightarrow[\Delta_l]{*} f(u_{t_1^\perp}, \ldots, u_{t_n^\perp}) \xrightarrow[\Delta_l]{} u_{t^\perp}$. Since $s_i \xrightarrow[\Delta_l]{*} u_{t_i^\perp}$, there exists σ_i such that $s_i = t_i\sigma_i$ by the induction hypothesis. Since l is linear, there exists a σ such that $f(t_1, \ldots, t_n)\sigma = f(t_1\sigma_1, \ldots, t_n\sigma_n)$ and hence we have $s = f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)\sigma = t\sigma$. $\square$

Lemma 4 If l is linear then $\mathcal{L}(\mathcal{A}_l) = \{t[s]_p \mid t \in \mathcal{T}(F),\ s \text{ is a ground instance of } l,\ p \in \mathrm{Pos}^\mu(t)\}$.

Proof:

($\supseteq$) From Lemma 3 and (iii) of the construction of Δ_l, we have $t[s]_p \xrightarrow[\Delta_l]{*} t[u_{l^\perp}]_p \xrightarrow[\Delta_l]{} t[u^\circ]_p$. Since $p \in \mathrm{Pos}^\mu(t)$, we have $t[u^\circ]_p \xrightarrow[\Delta_l]{*} u^\circ$ from the definition of Pos^μ and (i) and (iv) of the construction of Δ_l.

($\subseteq$) Let $t \xrightarrow[\Delta_l]{*} u^\circ$, then we have $t \xrightarrow[\Delta_l]{*} t[u_{l^\perp}]_p \xrightarrow[\Delta_l]{} t[u^\circ]_p \xrightarrow[\Delta_l]{*} u^\circ$ for some $p \in \mathrm{Pos}^\mu(t)$ from (iii) and (iv) of the construction of Δ_l. Hence we have $t = t[s]_p$ for some ground instance s of l from Lemma 3. $\square$

As shown by the Lemma 4, the TA $\mathcal{A}_l$ recognizes the set of terms having a redex $l\sigma$ at a μ-replacing position. Now we obtain the following lemma.

Lemma 5 For a left-linear CS-TRS $\mathcal{R}$, we can construct a deterministic complete TA $\mathcal{A}_{\mathrm{NF}}$ that recognizes CS-NF$_\mathcal{R}$ and has no needless states.

Proof: By the step (1) and (2) of the proceeding procedure, we obtain a TA $\mathcal{A}'$ that recognizes the complementation of the following set:

$$\bigcup_{l \to r \in R} \mathcal{L}(\mathcal{A}_l).$$

From Lemma 4 and (2) of Theorem 2, $\mathcal{L}(\mathcal{A}')$ is the set of context-sensitive normal forms. The TA $\mathcal{A}_{\mathrm{NF}}$ obtained by the step (3) of the procedure is a deterministic and complete TA that recognizes CS-NF$_\mathcal{R}$ and having no needless states from (1) of Theorem 2. $\square$

We show an example of $\mathcal{A}_{\mathrm{NF}}$.

Example 6 Consider the following $\mathcal{R} = (R, \mu)$:

$$R = \left\{ \begin{array}{rcl} f(g(x)) & \to & x \\ g(b) & \to & f(e) \\ g(c) & \to & e \\ a & \to & b \\ c & \to & e \end{array} \right\} \quad \mu(f) = \{1\},\ \mu(g) = \emptyset.$$

Final states and transition rules of a deterministic complete TA $\mathcal{A}_{\mathrm{NF}}$ that is minimized are as follows:

$Q^f_{\mathrm{NF}} = \{\overline{u_b}, \overline{u_g}, \overline{u_\perp}\}$

$\Delta_{\mathrm{NF}} =$

$$\left\{ \begin{array}{ccc} \multicolumn{3}{c}{a \to u_\perp,\ b \to \overline{u_b},\ c \to u_c,\ e \to \overline{u_\perp},} \\ g(u_\perp) \to \overline{u_g}, & g(\overline{u_\perp}) \to \overline{u_g}, & g(\overline{u_b}) \to u_g, \\ g(u_c) \to u_g, & g(u_g) \to \overline{u_g}, & h(\overline{u_g}) \to \overline{u_g}, \\ f(u_\perp) \to u_\perp, & f(\overline{u_\perp}) \to \overline{u_\perp}, & f(\overline{u_b}) \to \overline{u_\perp}, \\ f(u_c) \to u_\perp, & f(u_g) \to u_\perp, & f(\overline{u_g}) \to u_\perp \end{array} \right\}$$

$\square$

For the constructed TA $\mathcal{A}_{\mathrm{NF}}$, the following proposition holds from Lemma 5.

Proposition 7 If $f(u_1, \ldots, u_n) \to u \in \Delta_{\mathrm{NF}}$ and $u \in Q^f_{\mathrm{NF}}$, then $i \in \mu(f)$ implies $u_i \in Q^f_{\mathrm{NF}}$.

Proof: Let $f(u_1, \ldots, u_n) \to u \in \Delta_{\mathrm{NF}}$, $u \in Q^f_{\mathrm{NF}}$, and assume $u_i \notin Q^f_{\mathrm{NF}}$ for some $i \in \mu(f)$. Since $\mathcal{A}_{\mathrm{NF}}$ has no needless states by Lemma 5, there exists a terms $t_1, \ldots, t_n$ such that $t_j \xrightarrow[\Delta_{\mathrm{NF}}]{*} u_j$ for each j $(1 \leq j \leq n)$. Hence $f(t_1, \ldots, t_n) \xrightarrow[\Delta_{\mathrm{NF}}]{*} f(u_1, \ldots, u_n) \xrightarrow[\Delta_{\mathrm{NF}}]{} u$. Here that $f(t_1, \ldots, t_n) \in$ CS-NF$_\mathcal{R}$ and $t_i \notin$ CS-NF$_\mathcal{R}$ from Lemma 5. Since t_i is not a context-sensitive normal form and $i \in \mu(f)$, the term $f(t_1, \ldots, t_n)$ is not a context-sensitive normal form, which contradicts

$f(t_1, \ldots, t_n) \in \text{CS-NF}_{\mathcal{R}}$. □

5. Procedure to Construct the Set of Reachable Terms by Context-sensitive Innermost Reduction

In this section, we give a procedure to construct a tree automaton that recognizes the set of reachable terms by context-sensitive innermost reduction of a linear right-shallow TRS. Procedures to construct a tree automaton that recognizes the set of reachable terms from a given set of terms by innermost (resp. context-sensitive) reduction have already shown for linear right-shallow TRSs[8]. One may think that we can construct a tree automaton that recognizes the set of reachable terms by context-sensitive innermost reduction, by natural merger of these two procedures and by introducing context-sensitive normal form. However, it appears that this is not true and we need some more modification.

Before giving the procedure, we define $\mathrm{RS}(R)$ and $\mathcal{A}_{\mathrm{RS}}$, which are necessary to describe the procedure.

Definition 8 $\mathrm{RS}(R)$ is the set of all *non-variable direct subterms of the right-hand sides* of rules in TRS R: $\mathrm{RS}(R) = \{r_i \notin X \mid l \to f(r_1, \ldots, r_m) \in R\}$.

Note that $\mathrm{RS}(R)$ is a set of ground terms if R is right-shallow.

Definition 9 $\mathcal{A}_{\mathrm{RS}} = \langle F, Q_{\mathrm{RS}}, Q^f_{\mathrm{RS}}, \Delta_{\mathrm{RS}}\rangle$ is a TA that accepts $\mathrm{RS}(R)$ and satisfied with the following conditions:
- $Q^f_{\mathrm{RS}} = \{q^t \mid t \in \mathrm{RS}(R)\}$ and
- $\mathcal{L}(\mathcal{A}_{\mathrm{RS}}, q^t) = \{t\}$.

Now we give the procedure $\mathrm{P}_{\mathrm{csin}}$ that constructs a TA recognizing a set of reachable terms by context-sensitive innermost reduction. The procedure design is based on merging the procedures for innermost case and context-sensitive case. More precisely, each state of resulting TA consists of three components; the first component originates in the input automata, the second one indicates whether we can add the transition rules in order to transit to this state or not, The third one remembers whether the corresponding terms are context-sensitive normal forms or not.

Procedure $\mathbf{P}_{\mathrm{csin}}$:

Input A TA $\mathcal{A} = \langle F, Q, Q^f, \Delta\rangle$ such that $w_1 \to q \in \Delta$ and $w_2 \to q \in \Delta$ imply $w_1 = w_2$, and a left-linear right-shallow CS-TRS $\mathcal{R}$ over F of $\mathcal{A}$.

Output A TA $\mathcal{A}_* = \langle F, Q_*, Q^f_*, \Delta_*\rangle$ such that $\mathcal{L}(\mathcal{A}_*) = \underset{\mathcal{R}}{\hookrightarrow}_{\mathrm{in}}[\mathcal{L}(\mathcal{A})]$ if $\mathcal{R}$ is right-linear.

Step 1 (initialize) (1) Prepare $\mathcal{A}_{\mathrm{RS}}$ and $\mathcal{A}_{\mathrm{NF}}$.

(2) Let
- $k := 0$,
- $Q_* = (Q \uplus Q_{\mathrm{RS}}) \times \{\mathtt{a}, \mathtt{i}\} \times Q_{\mathrm{NF}}$,
- $Q^f_* = Q^f \times \{\mathtt{a}\} \times Q_{\mathrm{NF}}$, and
- Δ_0 as follows:
  (a) $\langle q', \mathtt{x}, u\rangle \to \langle q, \mathtt{x}, u\rangle \in \Delta_0$ for $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$ where $q' \to q \in \Delta$ and $u \in Q_{\mathrm{NF}}$,
  (b) $f(\langle q_1, \mathtt{i}, u_1\rangle, \ldots, \langle q_n, \mathtt{i}, u_n\rangle) \to \langle q, \mathtt{i}, u\rangle \in \Delta_0$ where $f(q_1, \ldots, q_n) \to q \in \Delta \cup \Delta_{\mathrm{RS}}$ and $f(u_1, \ldots, u_n) \to u \in \Delta_{\mathrm{NF}}$, and
  (c) $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \to \langle q, \mathtt{a}, u\rangle \in \Delta_0$ where $f(q_1, \ldots, q_n) \to q \in \Delta \cup \Delta_{\mathrm{RS}}$, $f(u_1, \ldots, u_n) \to u \in \Delta_{\mathrm{NF}}$, and if $i \in \mu(f)$ then $\mathtt{x}_i = \mathtt{a}$, otherwise $\mathtt{x}_i = \mathtt{i}$.

Step 2 Let Δ_{k+1} be the set of transition rules produced by augmenting transition rules of Δ_k by the following inference rules:

(1) Produce the transition rule $g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle) \to \langle q, \mathtt{a}, u'\rangle \in \Delta_{k+1}$ from $f(l_1, ..., l_n) \to g(r_1, ..., r_m) \in R$ and $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \to \langle q, \mathtt{a}, u\rangle \in \Delta_k$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \le i \le n$ if there exists $\theta : X \to (Q \uplus Q_{\mathrm{RS}}) \times \{\mathtt{a}, \mathtt{i}\} \times Q^f_{\mathrm{NF}}$ such that
- $\mathcal{L}(\Delta_k, x\theta) \neq \emptyset$ for all $x \in \mathrm{Var}(f(l_1, \ldots, l_n))$,
- $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, \mathtt{x}_i, u_i\rangle$,

where
- let $r_j\theta = \langle q''_j, x''_j, u''_j\rangle$ for each $r_j \in X$ then
  - $q'_j = \begin{cases} q''_j & \cdots \text{ if } r_j \in X \\ q^{r_j} & \cdots \text{ if } r_j \notin X \end{cases}$
  - $\mathtt{x}'_j = \begin{cases} \mathtt{x}''_j & \cdots \text{ if } j \notin \mu(g), r_j \in X \\ \mathtt{i} & \cdots \text{ if } j \notin \mu(g), r_j \notin X \\ \mathtt{a} & \cdots \text{ otherwise} \end{cases}$

– $u'_j = \begin{cases} u''_j & \cdots \text{ if } r_j \in X, j \notin \mu(g) \vee \ (j \in \mu(g) \wedge x''_j = \mathtt{a}) \\ v \in Q_{\mathrm{NF}} & \cdots \text{ otherwise} \end{cases}$

for all $1 \leq j \leq m$, and

- $g(u'_1, \ldots, u'_m) \xrightarrow[\Delta_{\mathrm{NF}}]{} u'$.

(2) Produce $\langle q', \mathtt{a}, u' \rangle \to \langle q, \mathtt{a}, u' \rangle \in \Delta_{k+1}$ from $f(l_1, ..., l_n) \to x \in R$ and $f(\langle q_1, \mathtt{x}_1, u_1 \rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n \rangle) \to \langle q, \mathtt{a}, u \rangle \in \Delta_k$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \leq i \leq n$ if there exists $\theta : X \to (Q \uplus Q_{\mathrm{RS}}) \times \{\mathtt{a}, \mathtt{i}\} \times Q^f_{\mathrm{NF}}$ such that

- $\mathcal{L}(\Delta_k, x\theta) \neq \emptyset$ for all $x \in \mathrm{Var}(f(l_1, \ldots, l_n))$,
- $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, \mathtt{x}_i, u_i \rangle$,

where let $\langle q'', \mathtt{x}'', u'' \rangle = x\theta$ then

- $q' = q''$ and
- $u' = \begin{cases} u'' & \cdots \text{ if } \mathtt{x}'' = \mathtt{a} \\ v \in Q_{\mathrm{NF}} & \cdots \text{ otherwise} \end{cases}$

Step 3 If $\Delta_{k+1} = \Delta_k$ then stop and set $\Delta_* = \Delta_k$. Otherwise, $k := k+1$, and go to step 2.

□

Example 10 Let us follow how procedure $\mathrm{P}_{\mathrm{csin}}$ works. Consider $\mathcal{R}$ in Example 6, and let $\mathcal{A}$ be a TA that recognizes $\{f(g(a)), f(g(c))\}$ defined by a set $\{q_{fa}, q_{fc}\}$ of final states and a set $\{a \to q_a, c \to q_c, g(q_a) \to q_{ga}, g(q_c) \to q_{gc}, f(q_{ga}) \to q_{fa}, f(q_{gc}) \to q_{fc}\}$ of transition rules. At the initializing, we have $\mathcal{A}_{\mathrm{RS}}$ and $\mathcal{A}_{\mathrm{NF}}$ at (1). $\mathcal{A}_{\mathrm{RS}}$ is defined by a set $\{q^e\}$ of final state and a set of $\{e \to q^e\}$ transition rules, and $\mathcal{A}_{\mathrm{NF}}$ is defined as Example 6. At (2) of initializing step, we have the following:

- $Q_* = \{\langle q_a, \mathtt{x}, u \rangle, \langle q_c, \mathtt{x}, u \rangle, \langle q_{ga}, \mathtt{x}, u \rangle, \langle q_{gc}, \mathtt{x}, u \rangle, \langle q_{fa}, \mathtt{x}, u \rangle, \langle q_{fc}, \mathtt{x}, u \rangle, \langle q^e, \mathtt{x}, u \rangle\}$ where $u \in Q_{\mathrm{NF}}$ and $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$.
- $Q^f_* = \{\langle q_{fa}, \mathtt{a}, u \rangle, \langle q_{fc}, \mathtt{a}, u \rangle\}$ where $u \in Q_{\mathrm{NF}}$.
- $\Delta_0 =$

$$\left\{ \begin{array}{cc} \multicolumn{2}{c}{a \to \langle q_a, \mathtt{x}, u_\perp \rangle,\ c \to \langle q_c, \mathtt{x}, u_c \rangle,\ e \to \langle q^e, \mathtt{x}, \overline{u_\perp} \rangle,} \\ g(\langle q_a, \mathtt{i}, u_1 \rangle) \to \langle q_{ga}, \mathtt{x}, u_g \rangle, & g(\langle q_a, \mathtt{i}, u_2 \rangle) \to \langle q_{ga}, \mathtt{x}, \overline{u_g} \rangle, \\ g(\langle q_c, \mathtt{i}, u_1 \rangle) \to \langle q_{gc}, \mathtt{x}, u_g \rangle, & g(\langle q_c, \mathtt{i}, u_2 \rangle) \to \langle q_{gc}, \mathtt{x}, \overline{u_g} \rangle, \\ f(\langle q_{ga}, \mathtt{x}, u_3 \rangle) \to \langle q_{fa}, \mathtt{x}, u_\perp \rangle, & f(\langle q_{ga}, \mathtt{x}, u_4 \rangle) \to \langle q_{fa}, \mathtt{x}, \overline{u_\perp} \rangle, \\ f(\langle q_{gc}, \mathtt{x}, u_3 \rangle) \to \langle q_{fc}, \mathtt{x}, u_\perp \rangle, & f(\langle q_{gc}, \mathtt{x}, u_4 \rangle) \to \langle q_{fc}, \mathtt{x}, \overline{u_\perp} \rangle \end{array} \right\}$$

where $\mathtt{x} \in \{\mathtt{a}, \mathtt{i}\}$, $u_1 \in \{\overline{u_b}, u_c\}$, $u_2 \in \{u_\perp, \overline{u_\perp}, u_g, \overline{u_g}\}$, $u_3 \in \{u_\perp, u_c, u_g, \overline{u_g}\}$, and $u_4 \in \{\overline{u_b}, \overline{u_\perp}\}$.

The saturation step stops at $k = 1$, and we have

- $\Delta_1 = \Delta_0 \cup \{b \to \langle q_a, \mathtt{a}, \overline{u_b} \rangle, e \to \langle q_c, \mathtt{a}, \overline{u_\perp} \rangle, e \to \langle q_{gc}, \mathtt{a}, \overline{u_\perp} \rangle, \langle q_a, \mathtt{a}, u \rangle \to \langle q_{fa}, \mathtt{a}, u \rangle\}$ where $u \in Q_{\mathrm{NF}}$,

$\Delta_2 = \Delta_1$, and $\Delta_* = \Delta_1$.

The output TA $\mathcal{A}_*$ recognizes terms in $\{f(g(a)), f(g(c)), a, b, f(e)\} = \xhookrightarrow[\mathcal{R}]{}_{\mathrm{in}} [f(g(a)), f(g(c))]$. □

This procedure $\mathrm{P}_{\mathrm{csin}}$ eventually terminates at some k, because the procedure add transition rules in the repetition steps but it does not change states Q_*, which guarantees that possible transitions rules are finite. Apparently $\Delta_0 \subset \cdots \subset \Delta_k = \Delta_{k+1} = \cdots$.

Differences from natural merger of existing procedures are the following:

(i) a restriction that the input TA has no rules such that $s \to q$ and $t \to q$ where $s \neq t$,

(ii) the condition $\mathcal{L}(\Delta_k, x\theta) \neq \emptyset$ at Step 2 is added, and

(iii) in the third item of Step 2-(1), u'_j is an arbitrary state $v \in Q_{\mathrm{NF}}$ even if $j \in \mu(g) \wedge \mathtt{x}''_j = \mathtt{i}$, while $u'_j = u''_j$ in the naturally merged procedure.

Unless (ii), we produce a transition rule $\langle q_c, \mathtt{a}, \overline{u_\perp} \rangle \to \langle q_{fc}, \mathtt{a}, \overline{u_\perp} \rangle$ from $f(g(x)) \to x \in R$, $f(\langle q_{gc}, \mathtt{a}, \overline{u_g} \rangle) \to \langle q_{fc}, \mathtt{a}, u_\perp \rangle \in \Delta_0$, and $x\theta = \langle q_c, \mathtt{i}, \overline{u_\perp} \rangle$. In this case, the output TA accepts e, which is unreachable from $f(g(a))$ nor $f(g(c))$.

If we do not have the modification (iii), then u'_j is u''_j of $r_j\theta$. Then a transition rule $b \to \langle q_a, \mathtt{a}, \overline{u_b} \rangle$ is not produced and hence the output TA does not accept b which is reachable from $f(g(a))$. However, the extension (iii) causes a problem that if the input TA $\mathcal{A}$ in Example 10 has set $\{a \to q_{ac},\ c \to q_{ac},\ g(q_{ac}) \to q_g,\ f(q_g) \to q_f\}$ of transition rules, and the output TA accepts c and e. To avoid this problem, we introduce the restriction (i). Even if we have the restriction (i), we can construct a TA accepting only one term and hence it suffices for the reachability problem and joinability problem.

We show several propositions and lemmas to show $\mathcal{L}(\mathcal{A}_*) = \xhookrightarrow[R]{}_{\mathrm{in}}[\mathcal{L}(\mathcal{A})]$ for the input TA $\mathcal{A}$ and the output TA $\mathcal{A}_*$ of the procedure $\mathrm{P}_{\mathrm{csin}}$.

Proposition 11 (a) Let $s \in \mathcal{T}(F)$, then $s \xrightarrow[\Delta]{*} q \in Q$ or $s \xrightarrow[\Delta_{\mathrm{RS}}]{*} q \in Q_{\mathrm{RS}}$ if and only if $s \xrightarrow[\Delta_0]{*} \langle q, \mathtt{i}, u\rangle$ for some $u \in Q_{\mathrm{NF}}$.
(b) $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \to \langle q, \mathtt{x},\ u\rangle \in \Delta_*$ implies $f(u_1, \ldots, u_n) \to u \in \Delta_{\mathrm{NF}}$.

Proof: The claim is a direct consequence of the construction of Δ_0 and the completeness of $\mathcal{A}_{\mathrm{NF}}$. □

Proposition 12 If a rule $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \to \langle q, \mathtt{i}, u\rangle$ is in Δ_*, then it is also in Δ_0. Moreover, $\mathtt{x}_i = \mathtt{i}$ for all $1 \leq i \leq n$.

Proof: Such rules are introduced at Step 1 and hence the claim follows from construction of Δ_0. □

Proposition 13 Let $t \in \mathcal{T}(F)$, $q \in Q \uplus Q_{\mathrm{RS}}$, and $u \in Q_{\mathrm{NF}}$, then $t \xrightarrow[\Delta_k]{*} \langle q, \mathtt{i}, u\rangle$ implies $t \xrightarrow[\Delta_0]{*} \langle q, \mathtt{i}, u\rangle$.

Proof: The proposition follows from Proposition 12. □

Proposition 14 Let $t \in \mathcal{T}(F)$, $q \in Q \uplus Q_{\mathrm{RS}}$, and $u \in Q_{\mathrm{NF}}$. Then, $t \xrightarrow[\Delta_0]{*} \langle q, \mathtt{a}, u\rangle$ if and only if $t \xrightarrow[\Delta_0]{*} \langle q, \mathtt{i}, u\rangle$.

Proof: We present a proof for "only if" part, since the proof for the other part is not only similar to this part but also simpler. We use induction on the length of the former sequence. In the case of $t \xrightarrow[\Delta_0]{*} \langle q', \mathtt{x}, u'\rangle \xrightarrow[\Delta_0]{} \langle q, \mathtt{a}, u\rangle$, the rule $\langle q', \mathtt{x}, u'\rangle \xrightarrow[\Delta_0]{} \langle q, \mathtt{a}, u\rangle$ is introduced at (a) of Step 1 (2), and $\mathtt{x} = \mathtt{a}$. Thus $\langle q', \mathtt{i}, u'\rangle \to \langle q, \mathtt{i}, u\rangle$ is also in Δ_0. Since $t \xrightarrow[\Delta_0]{*} \langle q', \mathtt{i}, u'\rangle$ by the induction hypothesis, the claim follows.

Otherwise, $t \xrightarrow[\Delta_0]{*} \langle q, \mathtt{a}, u\rangle$ can be represented as $t = f(t_1, \ldots, t_n) \xrightarrow[\Delta_0]{*} f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \xrightarrow[\Delta_0]{} \langle q, \mathtt{a}, u\rangle$. The rule $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \xrightarrow[\Delta_0]{} \langle q, \mathtt{a}, u\rangle$ is introduced at (c) of Step 1 (2), where $\mathtt{x}_i \in \{\mathtt{i}, \mathtt{a}\}$, $f(q_1, \ldots, q_n) \to q \in \Delta \cup \Delta_{\mathrm{RS}}$, and $f(u_1, \ldots, u_n) \to u \in \Delta_{\mathrm{NF}}$. Thus we also have $f(\langle q_1, \mathtt{i}, u_1\rangle, \ldots, \langle q_n, \mathtt{i}, u_n\rangle) \to \langle q, \mathtt{i}, u\rangle \in \Delta_0$ (by (b) of Step 1 (2)). Since we have $t_i \xrightarrow[\Delta_0]{*} \langle q_i, \mathtt{i}, u_i\rangle$ for each t_i by the induction hypothesis, the claim follows. □

Proposition 15 Let $t \in \mathcal{T}(F)$, $q \in Q \uplus Q_{\mathrm{RS}}$, and $u \in Q_{\mathrm{NF}}$. Then, $t \xrightarrow[\Delta_k]{*} \langle q, \mathtt{i}, u\rangle$ implies $t \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u\rangle$.

Proof: Let $t \xrightarrow[\Delta_k]{*} \langle q, \mathtt{i}, u\rangle$, then $t \xrightarrow[\Delta_0]{*} \langle q, \mathtt{i}, u\rangle$ by Proposition 13. The proposition follows from Proposition 14 and $\Delta_0 \subseteq \Delta_k$. □

Lemma 16 Let $s, t \in \mathcal{T}(F)$, $s \xrightarrow[\Delta_0]{*} \langle q, \mathtt{x}, u\rangle$, and $t \xrightarrow[\Delta_0]{*} \langle q', \mathtt{x}', u'\rangle$. Then, $q = q'$ implies $s = t$.

Proof: First we have s $\xrightarrow[\Delta_0]{*} \langle q, \mathtt{i}, u\rangle$ and $t \xrightarrow[\Delta_0]{*} \langle q', \mathtt{i}, u'\rangle$ from Proposition 14. If $q = q'$, then we have either $s \xrightarrow[\Delta]{*} q \wedge t \xrightarrow[\Delta]{*} q$ or $s \xrightarrow[\Delta_{\mathrm{RS}}]{*} q^s = q \wedge t \xrightarrow[\Delta_{\mathrm{RS}}]{*} q^t = q^s = q$ from Proposition 11 (a) and the construction of $\mathcal{A}_{\mathrm{RS}}$. In the case of $s \xrightarrow[\Delta_{\mathrm{RS}}]{*} q^s$ and $t \xrightarrow[\Delta_{\mathrm{RS}}]{*} q^t$, $q^s = q^t$ implies $s = t$ from the construction of $\mathcal{A}_{\mathrm{RS}}$. We show the case of $\alpha :\ s \xrightarrow[\Delta]{*} q$ and $t \xrightarrow[\Delta]{*} q$ by induction on $|\alpha| (> 0)$.
(1) In the case of $|\alpha| = 1$, we have $s \xrightarrow[\Delta]{} q$. In this case, s is the only term to transit to q from the restriction for Δ. Hence $t \xrightarrow[\Delta]{} q$ implies $s = t$.
(2) (a) In the case of $|\alpha| > 1$, we consider that the last transition rule applied in α is (in the form of) $f(q_1, \ldots, q_n) \xrightarrow[\Delta]{} q$. Let $s = f(s_1, \ldots, s_n)$, then we have the transition sequence $s = f(s_1, \ldots, s_n) \xrightarrow[\Delta]{*} f(q_1, \ldots, q_n) \xrightarrow[\Delta]{} q$. From the restriction for Δ, $f(q_1, \ldots, q_n)$ is the only term to transit to q, and hence we have $t = f(t_1, \ldots, t_n) \xrightarrow[\Delta]{*} f(q_1, \ldots, q_n) \xrightarrow[\Delta]{} q$ by letting $t = f(t_1, \ldots, t_n)$. Since we have $s_i = t_i$ for each i from the induction hypothesis, we also have $s = t$.
(b) In the case where $|\alpha| > 0$ and the last transition rule applied in α is (in the form of) $q'' \xrightarrow[\Delta]{} q$, we can show the lemma similarly to the previous case (a).

□

Lemma 17 Let $\alpha :\ s[\langle q, \mathtt{a}, u\rangle]_p \xrightarrow[\Delta_*]{*} \langle q', \mathtt{a}, u'\rangle$ and $p \in \mathrm{Pos}^{\mu}(s)$. Then $u' \in Q^f_{\mathrm{NF}}$ implies $u \in Q^f_{\mathrm{NF}}$.

Proof: We show this lemma by induction on $|\alpha|$. In the case of $|\alpha| = 0$, this lemma trivially holds from $s[\langle q, \mathtt{a}, u\rangle]_p = \langle q, \mathtt{a}, u\rangle = \langle q', \mathtt{a}, u'\rangle$. Hence we suppose $|\alpha| > 0$.

(1) Consider the case where the last transition rule applied in α is (in the form of) $g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle) \to \langle q', \mathtt{a}, u'\rangle \in \Delta_k$. Then α can be represented as $s[\langle q, \mathtt{a}, u\rangle]_p \xrightarrow[\Delta_*]{*} g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow[\Delta_k]{} \langle q', \mathtt{a}, u'\rangle$. In this case, the position p can be represented as jp' for $1 \leq j \leq m$. From $j \in \mu(g)$, Proposition 11 (b), and Proposition 7, we have $u'_j \in Q^f_{\mathrm{NF}}$ and hence we also have $u \in Q^f_{\mathrm{NF}}$ by the induction hypothesis.

(2) In the case where the last transition rule applied in α is (in the form of) $\langle q'', \mathtt{a}, u''\rangle \to \langle q', \mathtt{a}, u'\rangle \in \Delta_k$, we have $u'' = u'$ from the construction of Δ_0 or the second inference rule of Step 2. Hence this lemma holds by the induction hypothesis.

□

Lemma 18 If $j \notin \mu(g)$ and $g(\ldots, \langle q'_j, \mathtt{x}'_j, u'_j\rangle, \ldots) \to \langle q, \mathtt{x}', u'\rangle \in \Delta_k$, then $u'_j \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}'_j = \mathtt{i}$.

Proof:

(1) If $k = 0$, then $\mathtt{x}'_j = \mathtt{i}$ from the construction of Δ_0

(2) Consider the case of $k > 0$. We can assume $g(\ldots, \langle q'_j, \mathtt{x}'_j, u'_j\rangle, \ldots) \to \langle q, \mathtt{x}', u'\rangle \in \Delta_k \backslash \Delta_{k-1}$ without loss of generality. Since this rule is introduced by the first inference rule of Step 2, we have $\mathtt{x}' = \mathtt{a}$, and there exist $f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m) \in R$ and $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \to \langle q, \mathtt{a}, u\rangle \in \Delta_{k-1}$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \leq i \leq n$, and $\theta : X \to Q_*$ such that $l_i\theta \xrightarrow[\Delta_{k-1}]{*} \langle q_i, \mathtt{x}_i, u_i\rangle$. From $j \notin \mu(g)$, we have either $\langle q'_j, \mathtt{x}'_j, u'_j\rangle = r_j\theta$ if $r_j \in X$, or $\mathtt{x}'_j = \mathtt{i}$ if $r_j \notin X$. In the case $r_j \in X$, since we have $l_i|_p = r_j$ for some i and p, we have $l_i\theta = l_i[\langle q'_j, \mathtt{x}'_j, u'_j\rangle]_p \xrightarrow[\Delta_{k-1}]{*} \langle q_i, \mathtt{x}_i, u_i\rangle$. If $\mathtt{x}'_j = \mathtt{a}$, then we have $\mathtt{x}_i = \mathtt{a}$ by Proposition 12 and hence $u_i \in Q^f_{\mathrm{NF}}$. Therefore, $u'_j \in Q^f_{\mathrm{NF}}$ follows from Lemma 17.

□

Lemma 19 Let $\alpha :\ s[\langle q, \mathtt{x}, u\rangle]_p \xrightarrow[\Delta_k]{*} \langle q', \mathtt{x}', u'\rangle$. If $k = 0$ or $u \in Q_{\mathrm{NF}} \setminus Q^f_{\mathrm{NF}} \wedge \mathtt{x} = \mathtt{x}' = \mathtt{a} \wedge p \in \mathrm{Pos}^\mu(s)$, then there exists $v' \in Q_{\mathrm{NF}}$ such that $s[\langle q, \mathtt{x}, v\rangle]_p \xrightarrow[\Delta_k]{*} \langle q', \mathtt{x}', v'\rangle$ for $v \in Q_{\mathrm{NF}}$.

Proof: We prove the lemma by induction on $|\alpha|$. In the case of $|\alpha| = 0$, we have $s[\langle q, \mathtt{x}, u\rangle]_p = \langle q, \mathtt{x}, u\rangle = \langle q', \mathtt{x}, u'\rangle$. Hence this lemma holds from $s[\langle q, \mathtt{x}, v\rangle]_p = \langle q, \mathtt{x}, v\rangle = \langle q', \mathtt{x}, v'\rangle$.

Following we suppose $|\alpha| > 0$.

(1) Consider the case where the last transition rule applied in α is (in the form of) $g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle) \to \langle q', \mathtt{x}', u'\rangle \in \Delta_k$. Then α can be represented as $s[\langle q, \mathtt{x}, u\rangle]_p \xrightarrow[\Delta_k]{*} g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle) \xrightarrow[\Delta_k]{} \langle q', \mathtt{x}', u'\rangle$. Let $p = jp'$ for some $j \in \mathbb{N}$.

If $k = 0$, then the rule $g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle) \to \langle q, \mathtt{x}', u\rangle$ is introduced at (b) or (c) of Step 1 (2). Thus for any $u''_j \in Q_{\mathrm{NF}}$, there exists $u'' \in Q_{\mathrm{NF}}$ such that $g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_j, \mathtt{x}'_j, u''_j\rangle, \ldots, \langle q'_n, \mathtt{x}'_n, u'_n\rangle) \to \langle q, \mathtt{x}, u''\rangle \in \Delta_0$ by the completeness of $\mathcal{A}_{\mathrm{NF}}$. Hence this lemma holds by the induction hypothesis.

Consider the case of $k \neq 0$ and $u \in Q_{\mathrm{NF}} \setminus Q^f_{\mathrm{NF}} \wedge \mathtt{x} = \mathtt{x}' = \mathtt{a} \wedge p \in \mathrm{Pos}^\mu(s)$. In this case, j is in $\mu(g)$ from $p \in \mathrm{Pos}^\mu(s)$, and we have $\mathtt{x}'_j = \mathtt{a}$ from $\mathtt{x}' = \mathtt{a}$ and Proposition 12. From $\alpha_j : (s|_j)[\langle q, \mathtt{a}, u\rangle]_{p'} \xrightarrow[\Delta_k]{*} \langle q'_j, \mathtt{x}'_j, u'_j\rangle$, we have $(s|_j)[\langle q, \mathtt{a}, v\rangle]_{p'} \xrightarrow[\Delta_k]{*} \langle q'_j, \mathtt{a}, v'_j\rangle$ for some $v'_j \in Q_{\mathrm{NF}}$ by the induction hypothesis. Note that $u'_j \notin Q^f_{\mathrm{NF}}$ from $u \notin Q^f_{\mathrm{NF}}$ and Lemma 17. Thus we have $s[\langle q, \mathtt{a}, v\rangle]_p \xrightarrow[\Delta_k]{*} g(\ldots, \langle q'_j, \mathtt{a}, v'_j\rangle, \ldots)$.

(a) If the transition rule in the last step of α $\langle q'_j, \mathtt{a}, v'_j\rangle, \ldots) \to \langle q', \mathtt{a}, v'\rangle \in \Delta_0$ from the construction, where v' is determined by $g(\ldots, u'_{i-1}, v'_j, u'_{i+1}, \ldots) \to v' \in \Delta_{\mathrm{NF}}$.

(b) Otherwise we assume that the transition rule in the last step of α is in $\Delta_k \setminus \Delta_{k-1}$ without loss of generality. It is known that the rule is produced by the first inference rule of Step 2. Hence there exist $f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m) \in R$ and $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \to \langle q', \mathtt{a}, u''\rangle \in \Delta_{k-1}$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \leq i \leq n$, and $\theta : X \to Q_*$ such that $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, \mathtt{x}_i, u_i\rangle$ for all $1 \leq i \leq n$ where $r_j\theta = \langle q''_j, x''_j, u''_j\rangle$ for each $r_j \in X$.

In the subcase $r_j \notin X$, we have $g(\ldots, \langle q'_j, \mathtt{a}, v'_j\rangle, \ldots) \to \langle q', \mathtt{a}, v'\rangle \in \Delta_k \setminus \Delta_{k-1}$ where $q'_j = q^{r_j}$ for some $v' \in Q_{\mathrm{NF}}$.

In the remaining subcase $r_j \in X$, we have $l_i|_{p'} = r_j$ for some i and p'. In this case, we have $\mathtt{x}''_j = \mathtt{i}$; otherwise we have $u'_j = u''_j$ from $j \in \mu(g)$ and $\mathtt{x}_i = \mathtt{a}$ from $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, \mathtt{x}_i, u_i\rangle$ and Proposition 12. Hence we have $u_i \in Q^f_{\mathrm{NF}}$. Then we have $u''_j \in Q^f_{\mathrm{NF}}$ from Lemma 17. This contradicts $u'_j = u''_j$ and

$u'_j \notin Q^f_{\mathrm{NF}}$. Since $r_j \in X$, $j \in \mu(g)$, and $\mathtt{x}''_j = \mathtt{i}$, we have $g(\ldots, \langle q'_j, \mathtt{a}, v'_j\rangle, \ldots) \to \langle q', \mathtt{a}, v'\rangle \in \Delta_k \setminus \Delta_{k-1}$ for some $v' \in Q_{\mathrm{NF}}$ from the construction.

(2) In the case where the last transition rule applied in α is (in the form of) $\langle q', \mathtt{x}', u'\rangle \to \langle q, \mathtt{x}, u\rangle \in \Delta_k$, we have $u' = u$ from the construction of Δ_0 or the second inference rule of Step 2. Hence this lemma holds from the induction hypothesis.

□

Lemma 20 If $\alpha:\ t[t']_p \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u\rangle$ and $p \in \mathrm{Pos}^{\mu}(t)$, then there exists $\langle q', \mathtt{a}, u'\rangle$ such that $t' \xrightarrow[\Delta_k]{*} \langle q', \mathtt{a}, u'\rangle$ and $t[\langle q', \mathtt{a}, u'\rangle]_p \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u\rangle$.

Proof: We show the lemma by induction on $|\alpha|$.

(1) If $|\alpha| = 0$, then we have $p = \varepsilon$ and $t = t'$. Hence $t' \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u\rangle$.

(2) In the case of $|\alpha| > 0$, we have $p = ip'$ for some $i \in \mathbb{N}$. Then α can be represented as $t[t']_p = f(\ldots, t_i[t']_{p'}, \ldots) \xrightarrow[\Delta_k]{|\alpha|-1} f(\ldots,\ \langle q_i, \mathtt{x}_i, u_i\rangle,\ \ldots) \xrightarrow[\Delta_k]{} \langle q, \mathtt{a}, u\rangle$. Since $ip' = p \in \mathrm{Pos}^{\mu}(t)$, we have $i \in \mu(f)$. Hence $\mathtt{x}_i = \mathtt{a}$ follows from the construction of Δ_k.
By the induction hypothesis, there exists $\langle q', \mathtt{a}, u'\rangle$ such that $t' \xrightarrow[\Delta_k]{*} \langle q', \mathtt{a}, u'\rangle$ and $t_i[\langle q', \mathtt{a}, u'\rangle]_{p'} \xrightarrow[\Delta_k]{*} \langle q_i, \mathtt{a}, u_i\rangle$. Thus we have $t[\langle q', \mathtt{a}, u'\rangle]_p = f(\ldots, t_i[\langle q', \mathtt{a}, u'\rangle]_{p'}, \ldots) \xrightarrow[\Delta_k]{*} f(\ldots,\ \langle q_i, \mathtt{a}, u_i\rangle,\ \ldots) \xrightarrow[\Delta_k]{} \langle q, \mathtt{a}, u\rangle$.

□

Lemma 21 Let R be left-linear and right-shallow. Then $s \xrightarrow[\Delta_*]{*} \langle q, \mathtt{a}, u\rangle$ and $s \overset{*}{\underset{\mathcal{R}}{\hookrightarrow}}_{\mathrm{in}} t$ imply $t \xrightarrow[\Delta_*]{*} \langle q, \mathtt{a}, u'\rangle$ for some $u' \in Q_{\mathrm{NF}}$.

Proof: We present the proof in the case where $s \underset{\mathcal{R}}{\hookrightarrow}_{\mathrm{in}} t$, because the one in the case of $s = t$ is trivial, and the one in the case of $s \overset{n}{\underset{\mathcal{R}}{\hookrightarrow}}_{\mathrm{in}} t$ for $n > 1$ is given by applying the case of $n = 1$ repeatedly. Let $s \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u\rangle$ and $s = s[l\sigma]_p \underset{\mathcal{R}}{\hookrightarrow}_{\mathrm{in}} s[r\sigma]_p = t$ for some rewrite rule $l \to r \in R$, where $p \in \mathrm{Pos}^{\mu}(s)$. Then we have the transition sequence $s \xrightarrow[\Delta_k]{*} s[\langle q'', \mathtt{a}, u''\rangle]_p \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u\rangle$ by Lemma 20.

(1) Consider the case where the rewrite rule is in the form $f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m)$. The diagram of this case is shown in **Fig. 2**. Then we have $s = s[f(l_1, \ldots, l_n)\sigma]_p \underset{\mathcal{R}}{\hookrightarrow}_{\mathrm{in}} s[g(r_1, \ldots, r_m)\sigma]_p = t$, and thus $l_i\sigma$ is a context-sensitive normal form for each $i \in \mu(f)$. Since this rewrite rule is left-linear, $s \xrightarrow[\Delta_k]{*}$

$$\begin{tikzcd}
s[f(l_1,\ldots,l_n)\theta]_p \arrow[r, "*", "\Delta_k"'] & s[f(\langle q_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle q_n,\mathtt{x}_n,u_n\rangle)]_p \arrow[r, "*", "\Delta_k"'] & s[\langle q'',\mathtt{a},u''\rangle]_p \arrow[d, "*"', "\Delta_k"] \\
s = s[f(l_1,\ldots,l_n)\sigma]_p \arrow[u, "*", "\Delta_k"'] \arrow[d, hook, "\mathcal{R}", "\mathrm{in}"'] & & \begin{array}{c}\langle q,\mathtt{a},u\rangle \\ \langle q,\mathtt{a},u'\rangle\end{array} \\
t = s[g(r_1,\ldots,r_m)\sigma]_p \arrow[r, "*", "\Delta_k"'] & s[g(\langle q'_1,\mathtt{x}'_1,u'_1\rangle,\ldots,\langle q'_m,\mathtt{x}'_m,u'_m\rangle)]_p \arrow[r, "*", "\Delta_{k+1}"'] & s[\langle q'',\mathtt{a},v'\rangle]_p \arrow[u, "*", "\Delta_k"']
\end{tikzcd}$$

Fig. 2 The diagram of the proof of Lemma 21.

$\langle q, \mathtt{a}, u\rangle$ is represented as $s = s[f(l_1, \ldots, l_n)\sigma]_p \xrightarrow[\Delta_k]{*} s[f(l_1, \ldots, l_n)\theta]_p \xrightarrow[\Delta_k]{*} s[f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle)]_p \xrightarrow[\Delta_k]{} s[\langle q'',\ \mathtt{a}, u''\rangle]_p \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u\rangle$ for some $\theta : X \to Q_*$ such that $l_i\sigma \xrightarrow[\Delta_k]{*} l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, \mathtt{x}_i, u_i\rangle$. Note that $u'' \notin Q^f_{\mathrm{NF}}$ since $s|_p$ is not a context-sensitive normal form.
For $i \in \mu(f)$, $l_i\sigma$ is a context-sensitive normal form and hence we have $u_j \in Q^f_{\mathrm{NF}}$. For i such that $i \notin \mu(f)$, we have $\mathtt{x}_i = \mathtt{i}$ or $u_j \in Q^f_{\mathrm{NF}}$ from Lemma 18.
Let $r_j\theta = \langle q''_j, \mathtt{x}''_j, u''_j\rangle$ for each $r_j \in X$. Then, we have $r_j\sigma \xrightarrow[\Delta_k]{*} r_j\theta$ if $r_j \in X$.
Since $f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m) \in R$, $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \to \langle q'', \mathtt{a}, u''\rangle \in \Delta_k$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$, and θ such that $l_i\theta \xrightarrow[\Delta_k]{*} \langle q_i, \mathtt{x}_i, u_i\rangle$, there exist transition rules $g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle) \to \langle q'', \mathtt{a}, v'\rangle \in \Delta_{k+1}$ where

- $q'_j = \begin{cases} q''_j & \cdots \text{ if } r_j \in X \\ q^{r_j} & \cdots \text{ if } r_j \notin X \end{cases}$
- $\mathtt{x}'_j = \begin{cases} \mathtt{x}''_j & \cdots \text{ if } j \notin \mu(g), r_j \in X \\ \mathtt{i} & \cdots \text{ if } j \notin \mu(g), r_j \notin X \\ \mathtt{a} & \cdots \text{ otherwise} \end{cases}$
- $u'_j = \begin{cases} u''_j & \cdots \text{ if } r_j \in X, j \notin \mu(g) \vee\ (j \in \mu(g) \wedge x''_j = \mathtt{a}) \\ v \in Q_{\mathrm{NF}} & \cdots \text{ otherwise} \end{cases}$
- $g(u'_i, \ldots, u'_m) \xrightarrow[\Delta_{\mathrm{NF}}]{} v'$.

Now we show that $r_j\sigma \xrightarrow[\Delta_k]{*} \langle q'_j, \mathtt{x}'_j, u'_j\rangle$.

(a) For j such that $r_j \in X$, since we have $q'_j = q''_j$ and since we can take $u'_j = u''_j$, we have $r_j\sigma \xrightarrow[\Delta_k]{*} r_j\theta = \langle q'_j, \mathtt{x}''_j, u'_j\rangle$. If $\mathtt{x}'_j \neq \mathtt{x}''_j$ then $\mathtt{x}'_j = \mathtt{a}$ and $\mathtt{x}''_j = \mathtt{i}$. Thus we obtain $r_j\sigma \xrightarrow[\Delta_k]{*} \langle q'_j, \mathtt{x}'_j, u'_j\rangle$ by Proposition 15.

(b) For j such that $r_j \notin X$, we have $q'_j = q^{r_j}$ and $r_j\sigma = r_j$ since R is right-shallow. And then we can take arbitrary state in Q_{NF} as u'_j. Since $r_j \xrightarrow[\Delta_{\mathrm{RS}}]{*} q^{r_j}$,a we have $r_j \xrightarrow[\Delta_0]{*} \langle q^{r_j}, \mathtt{i}, v''\rangle$ for some $v'' \in Q_{\mathrm{NF}}$ from Proposition 11 (a). Since we also have $r_j \xrightarrow[\Delta_0]{*} \langle q^{r_j}, \mathtt{a}, v''\rangle$ by Proposition 15, we obtain $r_j\sigma = r_j \xrightarrow[\Delta_k]{*} \langle q^{r_j}, \mathtt{x}'_j, v''\rangle = \langle q'_j, \mathtt{x}'_j, u'_j\rangle$ where $u'_j = v''$.

It follows from Lemma 19 and $s[\langle q'', \mathtt{a}, u''\rangle]_p \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u\rangle$ with $u'' \notin Q^f_{\mathrm{NF}}$ that $s[\langle q'', \mathtt{a}, v'\rangle]_p \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u'\rangle$ for some $u' \in Q_{\mathrm{NF}}$. Therefore we have $t = s[g(r_1, \ldots, r_m)\sigma]_p \xrightarrow[\Delta_k]{*} s[g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle)]_p \xrightarrow[\Delta_{k+1}]{} s[\langle q'', \mathtt{a}, v'\rangle]_p \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, u'\rangle$.

(2) In the case that the rewrite rule is in the form $f(l_1, \ldots, l_n) \to x$, we can show the lemma similarly to the previous case (1). □

Lemma 22 Let R be left-linear and right-shallow. Then $\mathcal{L}(\mathcal{A}_*) \supseteq \xhookrightarrow[R]{}_{\mathrm{in}}[\mathcal{L}(\mathcal{A})]$.

Proof: Let $s \xhookrightarrow[R]{*}_{\mathrm{in}} t$ and $s \xrightarrow[\Delta]{*} q \in Q^f$. Since $s \xrightarrow[\Delta_0]{*} \langle q, \mathtt{i}, u\rangle \in Q^f_*$ from Proposition 11 (a), we have $s \xrightarrow[\Delta_0]{*} \langle q, \mathtt{a}, u\rangle \in Q^f_*$ by Proposition 15. Hence $t \xrightarrow[\Delta_*]{*} \langle q, \mathtt{a}, u'\rangle \in Q^f_*$ for some $u' \in Q_{\mathrm{NF}}$ by Lemma 21. □

Before proving soundness of the procedure $\mathrm{P}_{\mathrm{csin}}$, we define a measurement of transitions.

A measurement of transitions of Δ_* is defined as $||s \xrightarrow[\Delta_0]{} t|| = 0$ and $||s \xrightarrow[\Delta_{i+1}\backslash\Delta_i]{} t|| = i+1$ for $i \geq 0$. This is extended on transition sequences as a multiset:

$$||s_0 \xrightarrow[\Delta_*]{} s_1 \xrightarrow[\Delta_*]{} \cdots \xrightarrow[\Delta_*]{} s_{n+1}|| = \{||s_i \xrightarrow[\Delta_*]{} s_{i+1}|| \mid 0 \leq i < n\}.$$

Now we can define the following order $\sqsupset$ on transition sequences by Δ_*, which is necessary in proofs:

$$\alpha \sqsupset \beta \overset{\mathrm{def}}{\Leftrightarrow} ||\alpha|| >_{\mathrm{mul}} ||\beta||$$

where $>_{\mathrm{mul}}$ is the multiset extension of $>$ on $\mathbb{N}$.

Lemma 23 Let Δ_* be generated from a linear right-shallow CS-TRS $\mathcal{R}$. Then $\alpha: t \xrightarrow[\Delta_*]{*} \langle q, \mathtt{x}, u'\rangle$ implies $s \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} t$ and $\beta: s \xrightarrow[\Delta_0]{*} \langle q, \mathtt{x}, u\rangle$ for some term s and $u \in Q_{\mathrm{NF}}$.

Proof: Let $t = g(t_1, \ldots, t_m)$. We show this lemma by induction on $||\alpha||$ with respect to $\sqsupset$.

(1) Consider the case where the last transition rule applied in α is in the form of $g(\langle q'_1, \mathtt{x}'_1, u'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, u'_m\rangle) \to \langle q, \mathtt{x}, u'\rangle \in \Delta_k$. Let $\alpha_j: t_j \xrightarrow[\Delta_*]{*} \langle q'_j, \mathtt{x}'_j, u'_j\rangle$.

$$\begin{tikzcd}
t \arrow[d, equal] & & & \\
g(t_1,\ldots,t_m) \arrow[rr, "*", "\Delta_*"'] & & g(\langle q'_1,\mathtt{x}'_1,u'_1\rangle,\ldots,\langle q'_m,\mathtt{x}'_m,u'_m\rangle) \arrow[r, "\Delta_k"'] & \langle q,\mathtt{a},u'\rangle \\
g(s_1,\ldots,s_m) \arrow[u, hook, "*", "\mathcal{R}"', "\mathrm{in}" near end] \arrow[r, "*", "\Delta_*"'] \arrow[d, equal] & f(r_1,\ldots,r_m)\theta \arrow[r, "*", "\Delta_*"'] \arrow[u, phantom, "\mathrm{I.H.}"] & g(\langle q'_1,\mathtt{x}'_1,v'_1\rangle,\ldots,\langle q'_m,\mathtt{x}'_m,v'_m\rangle) \arrow[r, "\Delta_k"'] & \langle q,\mathtt{a},v'\rangle \\
g(r_1,\ldots,r_m)\sigma & & & \\
f(l_1,\ldots,l_n)\sigma \arrow[u, hook, "\mathcal{R}"', "\mathrm{in}" near end] \arrow[r, "*", "\Delta_*"'] & f(l_1,\ldots,l_n)\theta \arrow[r, "*", "\Delta_{k-1}"'] & f(\langle q_1,\mathtt{x}_1,u_1\rangle,\ldots,\langle q_n,\mathtt{x}_n,u_n\rangle) \arrow[r, "\Delta_{k-1}"'] & \langle q,\mathtt{a},v\rangle \\
s \arrow[u, hook, "*", "\mathcal{R}"', "\mathrm{in}" near end] \arrow[rrr, "*", "\Delta_0"'] & \arrow[u, phantom, "\mathrm{I.H.}"] & & \langle q,\mathtt{a},u\rangle
\end{tikzcd}$$

Fig. 3 The diagram of the proof of Lemma 23.

(a) In the subcase $k = 0$, if $\mathtt{x} = \mathtt{i}$, then we have $t \xrightarrow[\Delta_0]{*} \langle q, \mathtt{x}, u'\rangle$ from Proposition 13. Hence this lemma holds by letting $s = t$. If $\mathtt{x} = \mathtt{a}$, then there exists s_j such that $s_j \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle q'_j, \mathtt{x}'_j, v'_j\rangle$ for each $j \in \mu(g)$ from the induction hypothesis and we take $s_j = t_j$ and $v'_j = u'_j$ for each $j \notin \mu(g)$. Then we have $g(s_1, \ldots, s_m) \xrightarrow[\Delta_0]{*} g(\langle q'_1, \mathtt{x}'_1, v'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, v'_m\rangle)$. We have no j that satisfies both $j \notin \mu(g)$ and $\mathtt{x}'_j = \mathtt{a}$, since $j \in \mu(g)$ coincides with $\mathtt{x}'_j = \mathtt{a}$ from the construction of Δ_0. We also have $s_j \xrightarrow[\Delta_0]{*} \langle q'_j, \mathtt{x}'_j, v'_j\rangle = \langle q'_j, \mathtt{i}, v'_j\rangle$ for $j \notin \mu(g)$ from Proposition 13. Thus we have the transition $g(\langle q'_1, \mathtt{x}'_1, v'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, v'_m\rangle) \xrightarrow[\Delta_0]{} \langle q, \mathtt{a}, v\rangle$ from the construction of Δ_0. Therefore the claim holds by letting $s = g(s_1, \ldots, s_m)$.

(b) In the subcase $k > 0$, we assume that the transition rule in the last step of α is in $\Delta_k \backslash \Delta_{k-1}$ without loss of generality. The diagram of this case is shown in **Fig. 3**. Since this rule is introduced at Step 2, there exist $f(l_1, \ldots, l_n) \to g(r_1, \ldots, r_m) \in R$, $f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \to \langle q, \mathtt{a}, v\rangle \in \Delta_{k-1}$ where $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$ for all $1 \leq i \leq n$, and $\theta: X \to Q_*$ such that

- $\mathcal{L}(\Delta_{k-1}, x\theta) \neq \emptyset$ for all $x \in \mathrm{Var}(f(l_1, \ldots, l_n))$,
- $l_i\theta \xrightarrow[\Delta_{k-1}]{*} \langle q_i, \mathtt{x}_i, u_i\rangle$,
- $\mathcal{L}(\Delta_0, x\theta) \neq \emptyset$ for each erasing variable x,

and $\langle q'_j, \mathtt{x}'_j, u'_j\rangle$ is given as follows:

- $q'_j = \begin{cases} q''_j & \cdots \text{ if } r_j \in X \\ q^{r_j} & \cdots \text{ if } r_j \notin X \end{cases}$
- $\mathtt{x}'_j = \begin{cases} \mathtt{x}''_j & \cdots \text{ if } j \notin \mu(g), r_j \in X \\ \mathtt{i} & \cdots \text{ if } j \notin \mu(g), r_j \notin X \\ \mathtt{a} & \cdots \text{ otherwise} \end{cases}$
- $u'_j = \begin{cases} u''_j & \cdots \text{ if } r_j \in X, j \notin \mu(g) \vee \ (j \in \mu(g) \wedge x''_j = \mathtt{a}) \\ v \in Q_{\mathrm{NF}} & \cdots \text{ otherwise} \end{cases}$

where $r_j\theta = \langle q''_j, x''_j, u''_j\rangle$ for $r_j \in X$.

The following (i) – (iv) show that for each j, there exists s_j such that $s_j \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} t_j$, $s_j \xrightarrow[\Delta_*]{*} r_j\theta$ and $\alpha'_j : s_j \xrightarrow[\Delta_*]{*} \langle q'_j, \mathtt{x}'_j, v'_j\rangle$ with $\alpha_j \sqsupseteq \alpha'_j$ for some $v'_j \in Q_{\mathrm{NF}}$.

(i) For j such that $r_j \in X$ and $j \notin \mu(g) \vee (j \in \mu(g) \wedge \mathtt{x}''_j = \mathtt{a})$, we have $t_j \xrightarrow[\Delta_*]{*} \langle q'_j, \mathtt{x}'_j, u'_j\rangle = \langle q''_j, \mathtt{x}''_j, u''_j\rangle = r_j\theta$. We take t_j as s_j and v'_j as $u'_j = u''_j$.

(ii) For $j \in \mu(g)$ such that $r_j \in X$ and $\mathtt{x}''_j = \mathtt{i}$, we have $s_j \xrightarrow[\Delta_0]{*} \langle q'_j, \mathtt{x}'_j, v'_j\rangle = \langle q''_j, \mathtt{a}, v'_j\rangle$ and $s_j \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} t_j$ for some $v'_j \in Q_{\mathrm{NF}}$ from the induction hypothesis. Since $\mathcal{L}(\Delta_{k-1}, r_j\theta) \neq \emptyset$, we have a term s''_j such that $s''_j \xrightarrow[\Delta_*]{*} \langle q''_j, \mathtt{i}, u''_j\rangle = r_j\theta$. Then we have $s''_j \xrightarrow[\Delta_0]{*} \langle q''_j, \mathtt{i}, u''_j\rangle$ from Proposition 13 and $s_j \xrightarrow[\Delta_0]{*} \langle q''_j, \mathtt{i}, v'_j\rangle$ from Proposition 14. Hence we have $s_j = s''_j$ from Lemma 16 and $u''_j = v'_j$ from determinacy of Δ_{NF}. Thus we have $s_j \xrightarrow[\Delta_*]{*} r_j\theta = \langle q'_j, \mathtt{x}'_j, v'_j\rangle$.

(iii) For $j \notin \mu(g)$ such that $r_j \notin X$, we have $q'_j = q^{r_j}$ and $x'_j = \mathtt{i}$, and u'_j is an arbitrary state in Q_{NF}. Since $t_j \xrightarrow[\Delta_0]{*} \langle q^{r_j}, \mathtt{i}, u'_j\rangle$ by Proposition 13, we have $t_j = r_j$ from Proposition 11 (a) and the construction of Δ_{RS}. Therefore $t_j = r_j\theta$ follows from right-shallowness. We take s_j as t_j and v'_j as u'_j.

(iv) For $j \in \mu(g)$ such that $r_j \notin X$, we have $s_j \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} t_j$ and $s_j \xrightarrow[\Delta_0]{*} \langle q'_j, \mathtt{x}'_j, v'_j\rangle = \langle q^{r_j}, \mathtt{a}, v'_j\rangle$ for some $v'_j \in Q_{\mathrm{NF}}$ from the induction hypothesis. Since $s_j \xrightarrow[\Delta_0]{*} \langle q^{r_j}, \mathtt{i}, v'_j\rangle$ by Proposition 14, we have $s_j = r_j$ from Proposition 11 (a) and the construction of Δ_{RS}. Therefore $s_j = r_j\theta$ follows from right-shallowness.

Thus we have $g(s_1, \ldots, s_m) \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} g(t_1, \ldots, t_m)$, $\alpha' : g(s_1, \ldots, s_m) \xrightarrow[\Delta_*]{*} g(r_1\theta, \ldots, r_m\theta) \xrightarrow[\Delta_*]{*} g(\langle q'_1, \mathtt{x}'_1, v'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, v'_m\rangle)$ such that $\alpha \sqsupseteq \alpha'$. From the construction of Δ_*, we also have $g(\langle q'_1, \mathtt{x}'_1, v'_1\rangle, \ldots, \langle q'_m, \mathtt{x}'_m, v'_m\rangle) \xrightarrow[\Delta_k]{*} \langle q, \mathtt{a}, v'\rangle$.

We define a substitution $\sigma : \mathrm{Var}(f(l_1, \ldots, l_n)) \to \mathcal{T}(F)$ as follows:

$$x\sigma = \begin{cases} s_j \cdots & \text{if there exists } j \text{ such that } r_j = x \\ s' \cdots & \text{otherwise, choose an arbitrary } t' \text{ such that } t' \xrightarrow[\Delta_0]{*} x\theta \end{cases}$$

where σ is well-defined from the right-linearity of rewrite rules. We can construct $\beta : f(l_1, \ldots, l_n)\sigma \xrightarrow[\Delta_*]{*} f(l_1, \ldots, l_n)\theta \xrightarrow[\Delta_{k-1}]{*} f(\langle q_1, \mathtt{x}_1, u_1\rangle, \ldots, \langle q_n, \mathtt{x}_n, u_n\rangle) \xrightarrow[\Delta_{k-1}]{} \langle q, \mathtt{a}, v\rangle$.

Since $u_i \in Q^f_{\mathrm{NF}}$ or $\mathtt{x}_i = \mathtt{i}$, $l_i\sigma$ is a normal form or $i \notin \mu(f)$ for each i. Hence we have $f(l_1, \ldots, l_n)\sigma \xhookrightarrow[\mathcal{R}]{}_{\mathrm{in}} g(r_1, \ldots, r_m)\sigma = g(s_1, \ldots, s_m)$. Here $\alpha \sqsupseteq \alpha' \sqsupset \beta$ follows from the left-linearity of rewrite rules. Hence we have $s \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} f(l_1, \ldots, l_n)\sigma \xhookrightarrow[\mathcal{R}]{}_{\mathrm{in}} g(s_1, \ldots, s_m) \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} g(t_1, \ldots, t_m) = t$ and $s \xrightarrow[\Delta_0]{*} \langle q, \mathtt{a}, u\rangle$ for some u by the induction hypothesis.

(2) For the case where the transition rule applied last in α is (in the form of) $\langle q', \mathtt{x}, u''\rangle \to \langle q, \mathtt{x}, u'\rangle \in \Delta_k \setminus \Delta_{k-1}$, the lemma can be shown similarly to the previous case (1).

□

Lemma 24 If $\mathcal{R}$ be linear and right-shallow, then $\mathcal{L}(\mathcal{A}_*) \subseteq \xhookrightarrow[\mathcal{R}]{}_{\mathrm{in}}[\mathcal{L}(\mathcal{A})]$.

Proof: Let $t \xrightarrow[\Delta_*]{*} \langle q, \mathtt{x}, u'\rangle \in Q^f_*$ then we have $s \xhookrightarrow[\mathcal{R}]{*}_{\mathrm{in}} t$ and $s \xrightarrow[\Delta_0]{*} \langle q, \mathtt{x}, u\rangle \in Q^f_*$ by Lemma 23. Since $s \xrightarrow[\Delta_0]{*} \langle q, \mathtt{i}, u\rangle$ by Proposition 14, we have $s \xrightarrow[\Delta]{*} q \in Q^f$ by Proposition 11 (a) and the construction of Δ_0. □

Lemma 25 If $\mathcal{R}$ is linear and right-shallow, then $\mathcal{L}(\mathcal{A}_*) = \xhookrightarrow[\mathcal{R}]{}_{\mathrm{in}}[\mathcal{A}]$

Proof: By Lemma 22 and Lemma 24. □

Finally we obtain the following theorems.

Theorem 26 Context-sensitive innermost reachability is decidable for linear right-shallow TRSs.

Proof: Let ground terms t_1, t_2 and CS-TRS $\mathcal{R}$ be an instance of reachability problem. We can construct a TA $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = \{t_1\}$ where t_1 is ground term and if $t \to q \in \Delta$ and $t' \to q \in \Delta$ then $t = t'$. By $\mathrm{P_{csin}}$, we can construct a TA $\mathcal{A}_*$ such that $\mathcal{L}(\mathcal{A}_*) = \xhookrightarrow[\mathcal{R}]{}_{\text{in}}[\{t_1\}]$ from Lemma 25. Since the problem whether $t_2 \in \mathcal{L}(\mathcal{A}_*)$ or not is decidable from Theorem 2, whether t_1 is reachable to t_2 by $\mathcal{R}$ or not is also decidable. □

Theorem 27 Context-sensitive innermost joinability is decidable for linear right-shallow TRSs.

Proof: Let ground terms t_1, t_2 and CS-TRS $\mathcal{R}$ be an instance of joinability problem. We can construct TAs $\mathcal{A}^1$ and $\mathcal{A}^2$ such that $\mathcal{L}(\mathcal{A}^1) = \{t_1\}$ and $\mathcal{L}(\mathcal{A}^2) = \{t_2\}$, and output $\mathcal{A}^1_*$ and $\mathcal{A}^2_*$ by P_{csin}, such that $\mathcal{L}(\mathcal{A}^1_*) = \xhookrightarrow[\mathcal{R}]{}_{\text{in}}[\{t_1\}]$, $\mathcal{L}(\mathcal{A}^2_*) = \xhookrightarrow[\mathcal{R}]{}_{\text{in}}[\{t_2\}]$ from Lemma 25. We can construct the TA $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}^1_*) \cap \mathcal{L}(\mathcal{A}^2_*)$, and it is decidable whether $\mathcal{L}(\mathcal{A})$ is empty or not from Theorem 2 (3). □

6. Conclusion

In this paper, we proved that context-sensitive innermost reachability and context-sensitive innermost joinability are decidable for linear right-shallow TRSs.

One of the future works is to show regularity preservation that is more powerful property than reachability and joinability. We say the relation $\to$ *preserves regularity* if we can construct a TA $\mathcal{A}_*$ such that $\mathcal{L}(\mathcal{A}_*) = \to[\mathcal{L}(\mathcal{A})]$ for any TA $\mathcal{A}$. To adjust P_{csin} for regularity preservation, one of possible way is modifying the third component of the states that occur in $r_j \in X$ such that $r_j \in \mathrm{Var}(l)$, $r_j \notin \mathrm{Pos}^{\mu}(l)$, and $r_j \in \mathrm{Pos}^{\mu}(g(r_1, \dots, r_m))$ for $l \to g(r_1, \dots, r_m) \in R$. In this paper, the third component of such states is arbitrary $u \in Q_{\mathrm{NF}}$, but we must have some restriction to u. We think it may be complex.

Another future work is to find other subclasses that context-sensitive innermost reachability is decidable. The class of non-linear shallow TRSs is one of candidates, because reachability and joinability of that class are undecidable for normal TRS[7),10)], while they are decidable in innermost case[5)].

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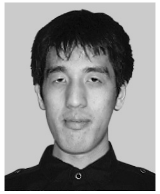

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