Investigation of Error Accumulation in Runge-Kutta Integration Process by Circle Test

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The circle test is a method of testing accuracy of integrator of an analog type differential analyzer. If the differential equation $\ddot{y}=-y$ is solved and y is plotted against $\dot{y}$, the result should be a circle. In practice slight spiralling effect is produced due to inaccuracy of the integrator. The authors applied this test to the investigation of error accumulation in the process of integration by the fourth-order Runge-Kutta method.

The equation $\ddot{y}=-y$ were integrated from 0 to 50 through 100 radians at several step sizes $h=0.25$ through 0.001 with the initial condition

$$x=0, \quad y=0, \quad \dot{y}=0.1,$$

and the amplitude and phase angle error, ε_r and ε_θ were obtained utilizing the errors of solution, ε_y and ε_z by the formulae,

$$\varepsilon_r \simeq \varepsilon_y \sin x + \varepsilon_z \cos x,$$

$$r\varepsilon_\theta \simeq \varepsilon_y \cos x - \varepsilon_z \sin x,$$

where

$$\varepsilon_y = y - 0.1 \sin x, \qquad \varepsilon_z = \dot{y} - 0.1 \cos x,$$

$$\varepsilon_r = r - 0.1, \qquad r = \sqrt{y^2 + \dot{y}^2},$$

$$\varepsilon_\theta = \theta - x, \qquad \theta = \text{arc tan} \, (y/\dot{y}).$$

Several computational procedures were tried, i.e., combinations of single precision computation (fixed point seven decimal digits), double precision computation for intermediate ones, ordinary rounding-off, and rounding-off using random numbers.

The experimental data are presented in graphical forms for the computational procedures tried. The results obtained are as follows.

  (a) When $h<0.1$, theoretical absolute values of truncation errors in y and $\dot{y}$ per integration step are comparable with or larger than round-off errors (smaller than 0.5 unit, one unit is 10^{-7}). In this case the experimental values of accumulated errors, ε_r and $r\varepsilon_\theta$, by any computational procedure agree fairly well with the theoretical values.

  (b) The error curve of $r\varepsilon_\theta$ versus x is generally composed of a lineary increasing or decreasing component and perturbation. However, when h is smaller than 0.01, the regular oscillatory component of especially

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$|\varepsilon|_{max}$ per 70 radians versus $1/h$
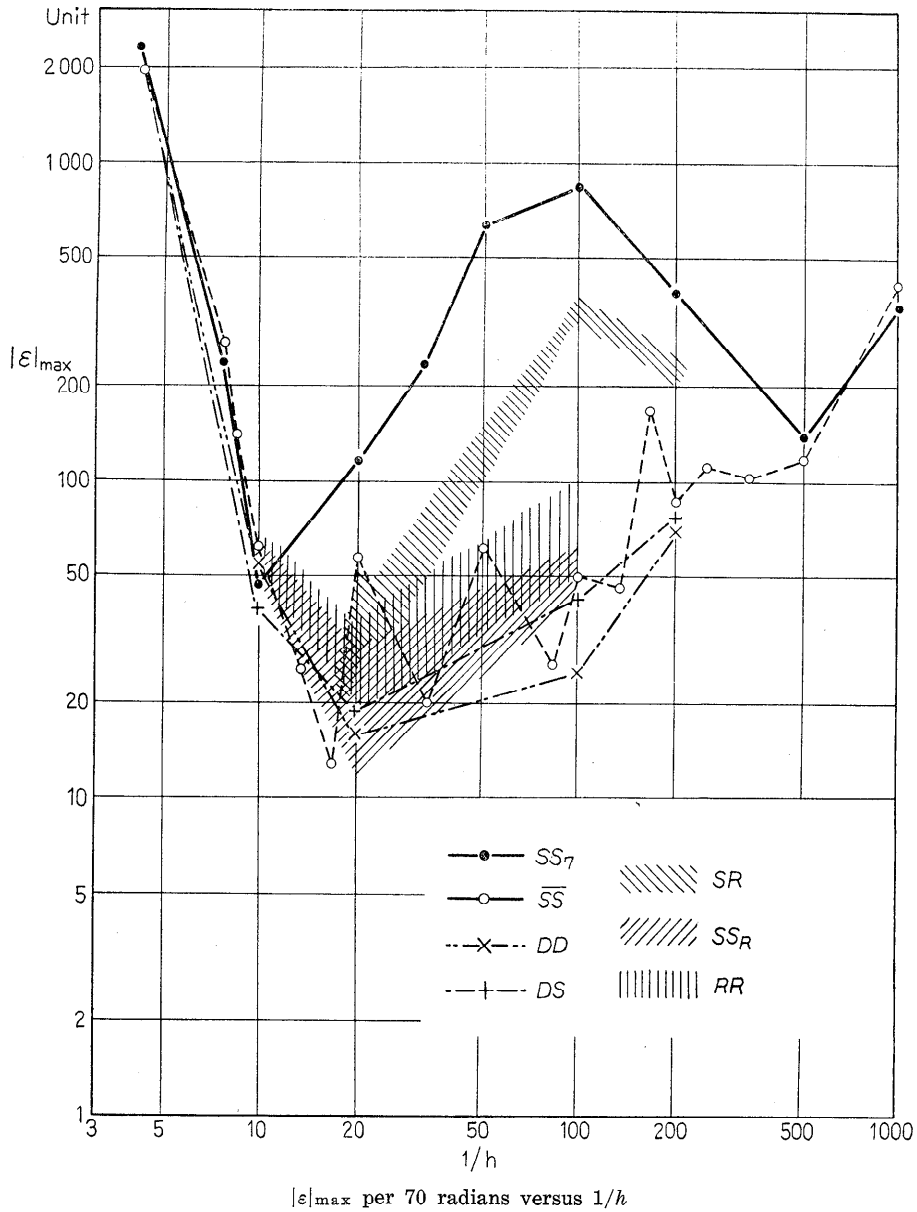
subharmonic nature with respect to the period of solution appears in the error curve especially for the amplitude.

(c) $|\varepsilon| = \sqrt{\varepsilon_r^2 + r^2\varepsilon_\theta^2}$ was adopted as a measure of errors for the comparison of the procedures. The figure illustrates maximtm value of $|\varepsilon|$ per 70 radians versus $1/h$ for several computational procedures. The procedure according to the formulae given below using single precision

arithmetic and ordinary rounding-off, $\overline{SS}$, yielded good results. The speed of integration of the procedure is the fastest and the programming is simple, and the procedure may generally be recommended.

$$dy/dx = z , \qquad\qquad dz/dx = -y ,$$

$$\bar{k}_1 = z_0 , \qquad\qquad \bar{l}_1 = -y_0 ,$$

$$\bar{k}_2 = z_0 + \left(\frac{h}{2}\right)l_1 , \qquad\qquad \bar{l}_2 = -\left[y_0 + \left(\frac{h}{2}\right)\bar{k}_1 \right] ,$$

$$\bar{k}_3 = z_0 + \left(\frac{h}{2}\right)l_2 , \qquad\qquad \bar{l}_3 = -\left[y_0 + \left(\frac{h}{2}\right)\bar{k}_2 \right] ,$$

$$\bar{k}_4 = z_0 + h\bar{l}_2 , \qquad\qquad \bar{l}_4 = -\left[y_0 + h\bar{k}_3 \right] ,$$

$$\Delta y = h \cdot \left(\frac{1}{6}\right) \cdot (\bar{k}_1 + 2\bar{k}_2 + 2\bar{k}_3 + \bar{k}_4) ,$$

$$\Delta z = h \cdot \left(\frac{1}{6}\right) \cdot (\bar{l}_1 + 2\bar{l}_2 + 2\bar{l}_3 + \bar{l}_4) .$$