A Problem Caused by the Collation of Informations in Dual-System

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In planning a real-time information processing system, it is necessary to consider reliability and cost/performance. For obtaining high reliability, dual system is often used and this not only assure improved availability but also prevent unfavorable cost induced by wrong massages if informations are collated between two subsystems. On the other hand, utility of system, that is percent of time system is busy, decreases by the increase of idle time caused by the collation and this influences on cost/performance.

1. *Examples of system loss caused by collation*

We assume for the first time that each information is collated succeeding to its end of process. Hereinafter, for convenience, two subsystems are identified as SA and SB and let process time distributions of them be $f_A(x)$ and $f_B(x)$ with means m_A and m_B respectively. If collation time that is program execution time of collation program, is so short that can be neglected, idle time of each subsystem comes from the difference of process time of one information between two subsystems. Then, probability density that SA's process time is less than that of SB as much as y expressed $g_A(y)$ is,

$$g_A(y)=\int_0^\infty f_A(x)f_B(x+y)\,dx \tag{1}$$

and its average that is mean idle time of SA is

$$\bar{y}_A=\int_0^\infty yg_A(y)dy=m_B-\int_0^\infty F_A(x)F_B(x)\,dx,$$

where

$$F_A(x)=\int_x^\infty f_A(x)\,dx$$

and

$$F_B(x)=\int_x^\infty f_B(x)\,dx \tag{2}$$

Then utility of SA and similarly that of SB are,

$$\eta_A=m_A/(m_A+\bar{y}_A), \qquad \eta_B=m_B/(m_B+\bar{y}_B), \tag{3}$$

respectively where $1/2\leq\eta_i\leq 1$, (i=A or B).

This paper first appeared in Japanese in Joho Shori (the Journal of the Information Processing Society of Japan), Vol. 5, No. 4 (1964), pp. 185–194.

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Examples. Suppose SA and SB are identical and so denote $f_A(x)=f_B(x)\equiv f(x)$ and $m_A=m_B\equiv m$.

1) Constant process time; $f(x)=\delta(x-m)$, $\eta=1$.
2) Uniform distribution; $f(x)=[u(x)-u(x-2m)]/2m$, $\eta=3/4$.
3) Negative exponential distribution; $f(x)=(1/m)e^{-x/m}$, $\eta=2/3$.

As shown, there are considerable degradations of utilities when any randomness is included in the process time. This can be improved by preparing buffer and in this case utility must be obtained from single server queue problem where one subsystem, for example SA, corresponds to server of which service time is SA's process time and the other, SB, to input source of which inter-arrival time is SB's process time, and queue size corresponds to the difference of number of informations processed in each subsystem. This queue size is limited from $-n$ to n if buffer size is limited to n. For example, let us suppose that SA and SB have processed to the amount of k and l informations respectively ($|k-l|\leq n$) and if $k>l$, then l informations have been collated and gotten out of the system and $k-l$ informations are in the buffer of SA. In the following queue problem, this condition is thought as that queue size is $-(k-l)$. (See Fig. 1.)

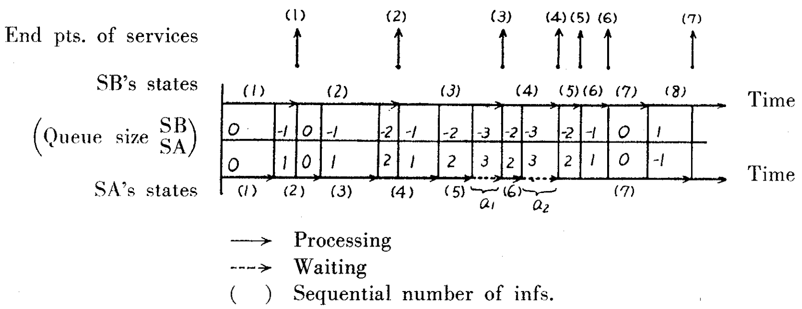

Fig. 1 Timing chart of dual-system with zero collation time (M=3).

Hereinafter, for simplicity, we assume that process time of each subsystem is of negative exponential distribution with parameters λ(SA) and μ(SB) respectively.

Let us denote by p_ν the probability that queue size is ν at steady state, then following transition equations can be obtained.

$$\begin{aligned}
&-(\lambda+\mu)p_0+\lambda p_{-1}+\mu p_1=0,\\
&-(\lambda+\mu)p_\nu+\lambda p_{\nu-1}+\mu p_{\nu+1}=0,\ (1\leq|\nu|\leq n-1)\\
&-\mu p_n+\lambda p_{n-1}=0,\\
&-\lambda p_{-n}+\mu p_{-n+1}=0.
\end{aligned}\tag{4}$$

If SA and SB are identical and so $\lambda=\mu$,

$$p_\nu=1/(2n+1),\ (\nu=-n\sim+n),\tag{5}$$

and utility is obtained as

$$\eta=1-p_n=2n/(2n+1).\tag{6}$$

2. *The influence of collation time*

When collation time is comparatively large, this influences on queue size distribution and consequently on the utility. Collation is executed by whether SA or SB for one information according to conditions of system, that is, whether by late subsystem or by preceding subsystem. When collation is to be executed by preceding subsystem, a number of informations must be always kept in the buffer and particular consideration on program is required regarding to how and when to collate.

Generally, queue size probability at time t is devided into two cases as following,

$p_\nu(t)$; Probability that queue size is ν and not under collation,

$\bar{p}_\nu(t)$; Probability that queue size is ν and being collated by one of two subsystems.

At steady state, they are expressed simply by $\bar{p}_\nu$ and p_ν respectively.

Let collation time be constant denoted by α and we introduce new quantities as following,

$$\gamma_{i\lambda}(x)=e^{-\lambda x}(\lambda x)^i/i!,\ \gamma_{i\mu}(x)=e^{-\mu x}(\mu x)^i/i!, \tag{7}$$

$$R_{i\lambda}(\alpha)=\mu\int_0^\alpha \gamma_{i\lambda}(x)\,dx=[1-\sum_{k=0}^{i}\gamma_{k\lambda}(\alpha)]\mu/\lambda,$$

$$R_{i\mu}(\alpha)=\lambda\int_0^\alpha \gamma_{i\mu}(x)\,dx=[1-\sum_{k=0}^{i}\gamma_{k\mu}(\alpha)]\lambda/\mu, \tag{8}$$

where $\gamma_{i\lambda}(x)$ and $\gamma_{i\mu}(x)$ are probabilities that the events of Poisson distribution with parameters λ and μ respectively occur i times in time interval x. When $x=\alpha$, we express $\gamma_{i\lambda}(\alpha)$ and $\gamma_{i\mu}(\alpha)$ simply as $\gamma_{i\lambda}$ and $\gamma_{i\mu}$ respectively and moreover as γ_i when $\lambda=\mu$.

2.1. *The case of collation by late subsystem*

As preceding, let queue size ν be the difference of number of informations processed in SA and SB, then either SA or SB undertake collation according to $\nu<0$ or $\nu>0$ respectively and $\bar{p}_\nu=0$ when $\nu=0$.

As shown in Appendix A, transition equations at steady state are,

$$\begin{aligned}
&-(\lambda+\mu)p_0+\mu p_1\gamma_{0\lambda}+\lambda p_{-1}\gamma_{0\mu}=0,\\
&-(\lambda+\mu)p_\nu+\lambda p_{\nu-1}+\mu\sum_{k=1}^{\nu+1}p_k\gamma_{\nu+1-k\lambda}=0,\ (1\le\nu\le n-1)\\
&-\mu p_n+\lambda p_{n-1}=0,\\
&-(\lambda+\mu)p_{-\nu}+\mu p_{-\nu+1}+\lambda\sum_{k=1}^{\nu+1}p_{-k}\gamma_{\nu+1-k\mu}=0,\ (1\le\nu\le n-1)\\
&-\lambda p_{-n}+\mu p_{-n+1}=0,\\
&\qquad\bar{p}_0=0,\\
&\qquad\bar{p}_\nu=\sum_{k=1}^{\nu}p_k R_{\nu-k\lambda}(\alpha),\ (1\le\nu\le n-1)\\
&\qquad\bar{p}_n=\sum_{j=1}^{n}p_j\sum_{k=n-j}^{\infty}R_{k\lambda}(\alpha),
\end{aligned} \tag{9}$$

$$\bar{p}_{-\nu}=\sum_{k=1}^{\nu} p_{-k} R_{\nu-k\mu}(\alpha),\ (1\leq\nu\leq n-1)$$

$$\bar{p}_{-n}=\sum_{j=1}^{n} p_{-j} \sum_{k=n-j}^{\infty} R_{k\mu}(\alpha). \tag{10}$$

and

$$\sum_{\nu=-n}^{n}(p_\nu+\bar{p}_\nu)=1. \tag{11}$$

From eqs. (9) and (10), p_ν and $\bar{p}_\nu$ can be expressed by p_0 and substituting them into eqs. (11), p_0 can be obtained. It is easily understood that utility of SA is $1-(p_n+\bar{p}_n)$.

For SB, we can get similar results by interchanging λ and μ.
When SA is identical with SB and collation time is less than mean process time, we approximate as $\gamma_1=1-\gamma_0$ and $\gamma_i=0$ for $i\geq 2$ because γ_i decreases so rapidly when i increases, then,

$$\begin{aligned}&-p_0+p_1\gamma_0=0,\\&-2p_\nu+p_{\nu-1}+p_{\nu+1}\gamma_0+p_\nu(1-\gamma_0)=0,\ (1\leq\nu\leq n-1)\\&-p_n+p_{n-1}=0.\end{aligned} \tag{12}$$

can be obtained. By solving these equations,

$$\begin{aligned}&p_\nu=\gamma_0^{-1}p_{\nu-1}=\gamma_0^{-\nu}p_0,\ (1\leq\nu\leq n-1),\\&p_n=p_{n-1}=\gamma_0^{-n+1}p_0.\end{aligned} \tag{13}$$

can be obtained. Meanwhile, as $R_0=1-\gamma_0, R_i=0$ for $i\geq 2$,

$$\begin{aligned}&\bar{p}_\nu=p_\nu(1-\gamma_0)=\gamma_0^{-\nu}(1-\gamma_0)p_0,\ (1\leq\nu\leq n-1)\\&\bar{p}_n=\gamma_0^{-n+1}(1-\gamma_0)p_0\end{aligned} \tag{14}$$

and substituting them into eq. (11),

$$\begin{aligned}&p_0=(1-\gamma_0)\gamma_0^{n-1}/[-3\gamma_0^{n-1}+\gamma_0^n+2(2-\gamma_0)^2],\\&p_n+\bar{p}_n=(2-\gamma_0)p_n=(1-\gamma_0)(2-\gamma_0)/[-3\gamma_0^{n-1}+\gamma_0^n+2(2-\gamma_0)^2].\end{aligned} \tag{15}$$

can be obtained.

Fig. 2 shows the relationship between system's loss and collations time. There, what labeled A shows the case of collation by late subsystem.

2.2. *The case of collation by preceding subsystem*

In this case, whether SA or SB undertake collation according to $\nu>0$ or $\nu<0$ respectively. As in the preceding case, following transition equations can be obtained.

$$-(\lambda+\mu)p_0+\mu p_1+\lambda p_{-1}+\lambda\sum_{k=0}^{n-1}p_k\gamma_{k+1\mu}+\mu\sum_{k=0}^{n-1}p_{-k}\gamma_{k+1\lambda}=0,$$

$$-(\lambda+\mu)p_\nu+\mu p_{\nu+1}+\lambda\sum_{k=0}^{n-\nu}p_{\nu-1+k}\gamma_{k\mu}+\mu\sum_{k=0}^{n-1}p_{-k}\gamma_{\nu+1+k\lambda}=0,\ (1\leq\nu\leq n-1)$$

$$-\mu p_n+\lambda p_{n-1}\gamma_{0\mu}+\mu\sum_{k=0}^{n-1}p_{-k}\gamma_{n+1+k\lambda}=0,$$

$$-(\lambda+\mu)p_{-\nu}+\lambda p_{-\nu-1}+\mu\sum_{k=0}^{n-\nu}p_{-\nu+1-k}\gamma_{k\lambda}+\lambda\sum_{k=0}^{n-1}p_k\gamma_{\nu+1+k\mu}=0,\ (1\leq\nu\leq n-1)$$

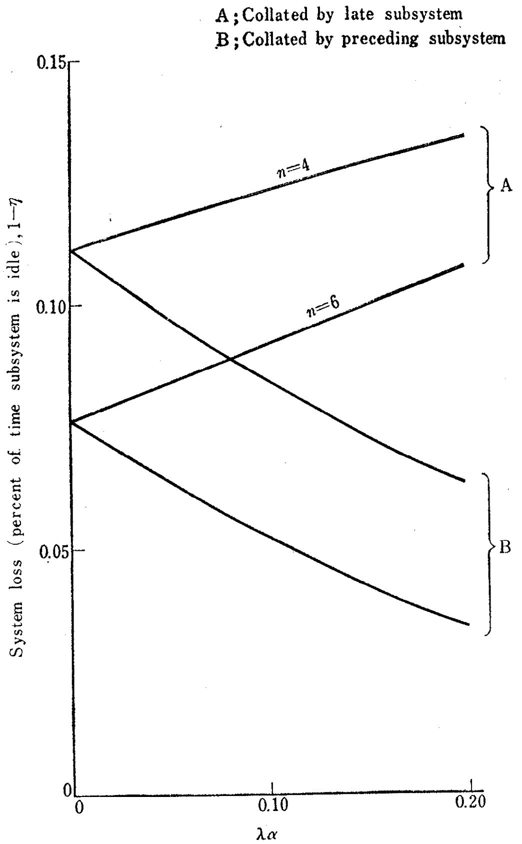

Fig. 2 System loss vs $\lambda\alpha$.

$$-\lambda p_{-n}+\mu p_{-n+1}\gamma_{0\lambda}+\lambda\sum_{k=0}^{n-1}p_k\gamma_{n+k+1\mu}=0. \tag{16}$$

$$\bar{p}_0=\sum_{k=0}^{n-1}[p_kR_{k\mu}(\alpha)+p_{-k}R_{k\lambda}(\alpha)],$$

$$\bar{p}_\nu=\sum_{k=0}^{n-\nu-1}p_{\nu+k}R_{k\mu}(\alpha),\ (1\leq\nu\leq n-1)$$

$$\bar{p}_{-\nu}=\sum_{k=0}^{n-\nu-1}p_{-\nu-k}R_{k\lambda}(\alpha),\ (1\leq\nu\leq n-1)$$

$$\bar{p}_n=\bar{p}_{-n}=0. \tag{17}$$

If $\lambda=\mu$ and permitted to approximate as $\gamma_1=1-\gamma_0$, $\gamma_i=0$ for $i\geq 2$,

$$p_\nu=\gamma_0^\nu p_0,\ (1\leq\nu\leq n)$$

$$\bar{p}_0=2(1-\gamma_0)p_0,$$

$$\bar{p}_\nu=\gamma_0^\nu(1-\gamma_0)p_0,\ (1\leq\nu\leq n-1)$$

$$\bar{p}_n=0, \tag{18}$$

can be obtained and substituting them into eq. (11),

$$p_0=(1-\gamma_0)/(3-\gamma_0-2\gamma_0^n),\ p_\nu=(1-\gamma_0)\gamma_0^\nu/(3-\gamma_0-2\gamma_0^n)\ (1\le\nu\le n)$$
$$p_\nu+\bar{p}_\nu=(1-\gamma_0)(2-\gamma_0)\gamma_0^\nu/(3-\gamma_0-2\gamma_0^n)\ (1\le\nu\le n-1). \qquad (19)$$

can be obtained. As $\bar{p}_n=0$, system's loss is p_n.

The relationship between system's loss and collation time in the case of collation by preceding subsystem is shown in Fig. 2, labeled B.

References

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Appendix A.

A queue problem where input inter-arrival time or service time varies according to queue size.

We consider finite length queue problem where queue size is limited to $2n+1$ from $-n$ to n. Input inter-arrival time and service time are principally of negative exponential distributions with parameters λ and μ respectively. However, there need additional constant time for input inter-arrival time when $\upsilon<0$ where υ is queue size and for service time when $\upsilon>0$. This additional part is thought as particular state and each queue size probability is devided into 2 phases called A and B and denoted by $p_\nu(t)$ and $\bar{p}_\nu(t)$ respectively as described in Sec. 2.

The queue size probability in phase A for $\upsilon>0$ at time $t+h$ denoted by $p_\nu(t+h)$ is derived by summing up following probabilities,

(1) queue size is υ and in phase A at time t and no event occur in small time interval h,

(2) queue size is $\upsilon-1$ and in phase A at time t and one input arrives in time interval h and

(3) queue size is $\upsilon+1$ being in phase B at time t and changes the phase from B to A decreasing queue size by one in time interval h.

The last case is further devided into $\upsilon+1$ cases, that is, queue size was $\upsilon+1-k$ at time $t-\alpha$ being in phase A ($k=0\sim\upsilon$) and changed to phase B in time interval h and then k inputs arrived in succeeding time interval α.

Then,

$$p_\nu(t+h)=p_\nu(t)(1-\lambda h-\mu h)+p_{\nu-1}(t)\lambda h+\mu\sum_{k=1}^{\nu+1}p_k(t-\alpha)\gamma_{\nu+1-k\lambda}h,$$

or making h tend to 0,

$$p_\nu'(t)=-(\lambda+\mu)p_\nu(t)+\lambda p_{\nu-1}(t)+\mu\sum_{k=1}^{\nu+1}p_k(t-\alpha)\gamma_{\nu+1-k\lambda}$$

can be obtained. In steady state, this reduce to,

$$-(\lambda+\mu)p_\nu+\lambda p_{\nu-1}+\mu\sum_{k=1}^{\nu+1}p_k\gamma_{\nu+1-k\lambda}=0.$$

Meanwhile, queue size probability of being in phase B at t is also sum of υ probabilities, that is, queue size was $k(k=1\sim\upsilon)$ at $t-x$ where x is smaller than α and changed the phase from A to B in small time interval and then $\upsilon-k$ inputs arrived in succeeding time interval x. This is integrated with regard to x from 0 to α.

Then,

$$\bar{p}_\nu(t)=\mu\sum_{k=1}^{\nu}\int_0^\alpha p_k(t-x)\gamma_{\nu-k\lambda}(x)\,dx$$

and in steady state,

$$\bar{p}_\nu=\sum_{k=1}^{\nu}p_k R_{\nu-k\lambda}(\alpha)$$

can be obtained. With similar processes for any other queue size, we can obtain eq. (9) and (10).