The Method for Testing Randomness and Transforming Distribution of Random Number Utilizing Random Walk Simulation*

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1. *Introduction*

The three-dimensional random walk simulator, which is a sort of computer for Monte Carlo methods, has been constructed at Osaka City University by the author and others.[1] Many digits of a equi-distributed random number are used in a random walk simulation, and we used to carry out only the frequency test for the random numbers used for the simulation. This is the simplest test, so that it gives us only one side information of randomness.

In order to obtain much information about randomness, a new method for testing randomness of random numbers utilizing the random walk simulation is devised by the author. Moreover, the method for transforming distributions of random numbers is proposed.

2. *Random Walk Simulation*

2.1. *Probability function* $U(P\to Q_i)$

An outline of the random walk simulation is as follows:

Let Φ be a simply connected domain with a boundary Γ of an arbitrary shape.

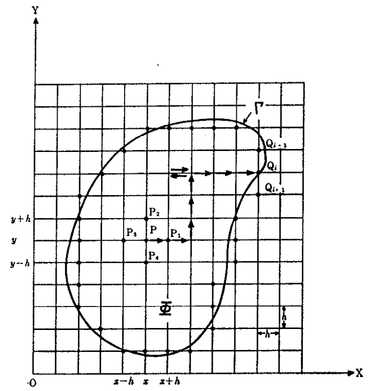

Fig. 1. An example of a random trip for case of two-dimensions.

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As shown in *Fig. 1*, the domain includes small squares with a side h, and the boundary Γ consists of a set of lattice points Q_i $(i=1, 2, \cdots, n)$.

A random walk point (r.w.p.) being at one point of the lattice can go to any one of the four neighboring points, of the lattice with equal probability (1/4) among four directions for case of two-dimensions, according to the instruction (with a form of a random number or a random signal, such as $+X$, $-X$, $+Y$, and $-Y$, which defines one direction of four neighbores) sent from the random number generator. A r.w.p. strats from a fixed lattice point $P(x, y)$ in the domain Φ and walks along the lattice lines. When a r.w.p. arrives at a lattice point Q_i on the boundary Γ, we terminate the random trip. An example of a random trip is shown in *Fig. 1*.

Let P_j $(j=1, 2, 3, 4)$ be the four neighboring points of $P(x, y)$ and $U(P\to Q_i)$ be the function representing the probability that a r.w.p. starting from a point $P(x, y)$ arrives at a lattice point Q_i on the boundary Γ. The function $U(P\to Q_i)$ is one-fourth of the sum of each probability that the r.w.p. starting from each one of the four neighboring points of P arrives at the point Q_i.
Then we have

$$\frac{1}{4}\sum_{j=1}^{4} U(P_j\to Q_i) = U(P\to Q_i). \tag{1}$$

The coordininates of point P_j are following.

$$P_1: (x+h,\ y), \quad P_2: (x,\ y+h)$$
$$P_3: (x-h,\ y), \quad P_4: (x,\ y-h)$$

We have the following equation of the function U provided that the cell length of the lattice, h, is sufficently small:

$$\begin{aligned}&\frac{1}{h^2}\cdot\frac{1}{4}\{U[P_1(x+h, y)\to Q_i]-2U[P(x, y)\to Q_i]\\&+U[P_3(x-h, y)\to Q_i]+U[P_2(x, y+h)\to Q_i]\\&-2U[P(x, y)\to Q_i]+U[P_4(x, y-h)\to Q_i]\}\\&\simeq\frac{1}{4}\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)\equiv\frac{1}{4}\Delta U=0.\end{aligned} \tag{2}$$

This Eq. (2) is nothing but the Laplace's differential equation. There are several methods of numerical calculations to evaluate the function U, such as:

(i) Liebmann's method,[(2)]
(ii) Mass division method[(3)] (Explosive method),[(4)]
(iii) Monte Carlo method.[(3)]

As the methods (i) and (ii) are both deterministic, we call them the analytical method. On the contrary, the method (iii) is stochastic, so we call it the experimental method. From now, we denote values by the analytical method such as $U(P\to Q_i)$ and those by the experimental method such as $\hat{U}(P\to Q_i)$ (estimate).

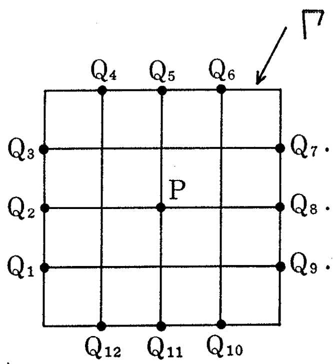
2.2. *Example of calculation*

We chose, as an example, a square boundary for Γ and set the starting point P on the center of the domain, for making easier to treat analytically and to keep high symmetry; and then calculated the value of $U(P\rightarrow Q_i)$. On the other hand, we got experimentally the value of $\hat{U}(P\rightarrow Q_i)$ from results 10,000 random trips. In *Figs. 2* and *3* we show the value of $U(P\rightarrow Q_i)$ and $\hat{U}(P\rightarrow Q_i)$. The confidence limit of the values $\hat{U}(P\rightarrow Q_i)$ are estimated by the central limit theorem.

3. *Method for Testing Randomness*

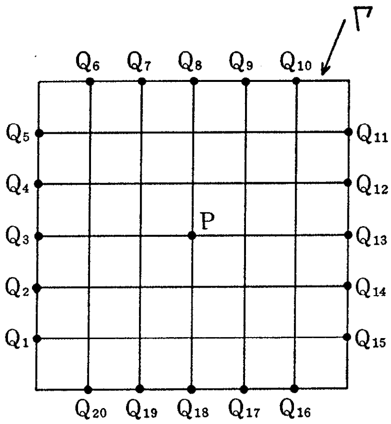
3.1. *Principle*

The experimental value $\hat{U}(P\rightarrow Q_i)$ which is obtained from the above mentioned simulation of random walk process using complete random numbers or equi-distributed random numbers should coincide the analytically calculated value $U(P\rightarrow Q_i)$. Accordingly, if we simulate the random walk process in the same way using random numbers to be tested and we carry out χ^2 test between the value of $U(P\rightarrow Q_i)$ and that of $\hat{U}(P\rightarrow Q_i)$ obtained from the simulation, then we may



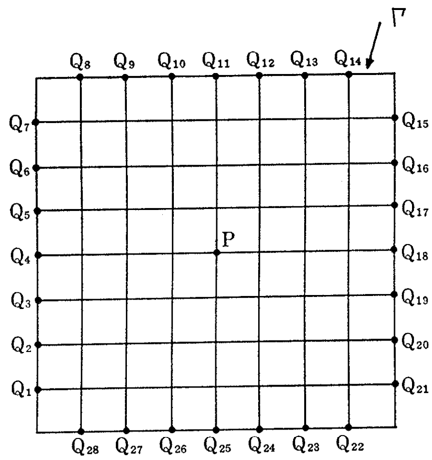
method	U(P→Qi) (i)	U(P→Qi) (ii)	Û(P→Qi) (iii)
Q7	0.0625	0.06250	0.062
Q8	0.1250	0.12500	0.126
Q9	0.0625	0.06250	0.062

(a)



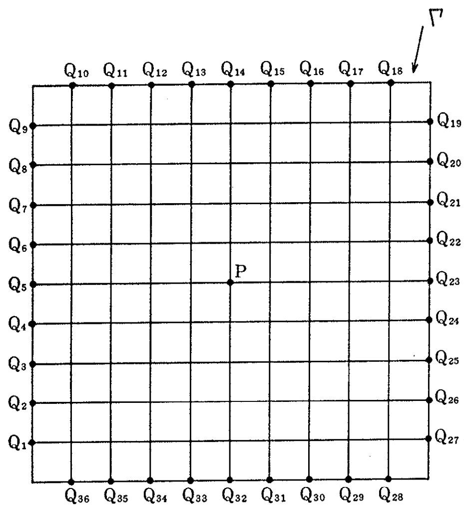
method	U(P→Qi) (i)	U(P→Qi) (ii)	Û(P→Qi) (iii)
Q11	0.0289	0.02885	0.029
Q12	0.0577	0.05769	0.057
Q13	0.0769	0.07692	0.077
Q14	0.0577	0.05769	0.057
Q15	0.0289	0.02885	0.029

(b)



	U(P→Qi)		Û(P→Qi)
method	(i)	(ii)	(iii)
Q15	0.0165	0.01654	0.017
Q16	0.0331	0.03309	0.033
Q17	0.0478	0.04779	0.047
Q18	0.0552	0.05515	0.055
Q19	0.0478	0.04779	0.047
Q20	0.0331	0.03309	0.033
Q21	0.0165	0.01654	0.017

(c)



	U(P→Qi)		Û(P→Qi)
method	(i)	(ii)	(iii)
Q19	0.0107	0.01070	0.010
Q20	0.0214	0.02140	0.020
Q21	0.0316	0.03159	0.032
Q22	0.0397	0.03973	0.040
Q23	0.0432	0.04317	0.044
Q24	0.0397	0.03973	0.040
Q25	0.0316	0.03159	0.032
Q26	0.0214	0.02140	0.020
Q27	0.0107	0.01070	0.010

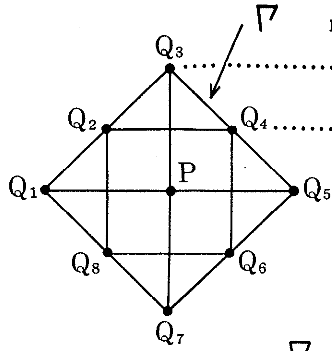
(d)

Fig. 2. Values of $U(P\rightarrow Q_i)$ and $\hat{U}(P\rightarrow Q_i)$.

evaluate another side of randomness of random numbers to be tested. This is the principle of the method for testing randomness utilizing random walk simulation.

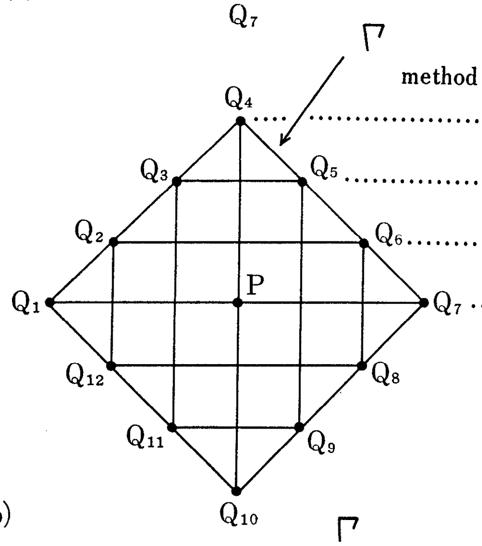
3. 2. *Results of experiment*

We made an experiment to test randomness using a square boundary shown in *Fig. 2 (a)*. In the experiment, we performed 1,600 random trips and used 7, 217 digits of random numbers. In *Table 1* we show the values of $U(P\rightarrow Q_i)$, the values of $\hat{U}(P\rightarrow Q_i)$, the absolute values of difference between $U(P\rightarrow Q_i)$ and



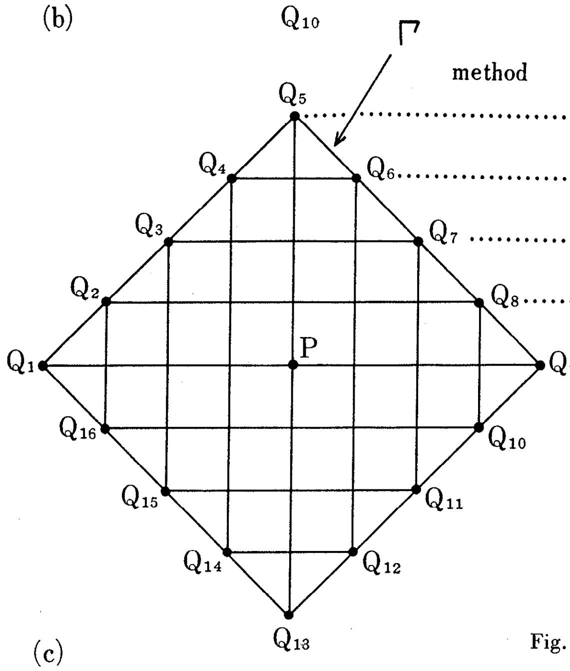
	Γ method	$U(P \to Q_i)$ (i)	$U(P \to Q_i)$ (ii)	$\hat{U}(P \to Q_i)$ (iii)
Q_3		0.0833	0.08333	0.0835
Q_4		0.1667	0.16667	0.1662
Q_5		0.0833	0.08333	0.0835

(a)



	Γ method	$U(P \to Q_i)$ (i)	$U(P \to Q_i)$ (ii)	$\hat{U}(P \to Q_i)$ (iii)
Q_4		0.0357	0.03571	0.0359
Q_5		0.1071	0.10714	0.1070
Q_6		0.1071	0.10714	0.1070
Q_7		0.0357	0.03571	0.0359

(b)



	Γ method	$U(P \to Q_i)$ (i)	$U(P \to Q_i)$ (ii)	$\hat{U}(P \to Q_i)$ (iii)
Q_5		0.0189	0.01891	0.0187
Q_6		0.0672	0.06723	0.0674
Q_7		0.0966	0.09664	0.0962
Q_8		0.0672	0.06723	0.0674
Q_9		0.0189	0.01891	0.0187

(c)

Fig. 3. Values of $U(P \to Q_i)$ and $\hat{U}(P \to Q_i)$.

$\hat{U}(P \to Q_i)$ and the values of χ^2. In *Table 2* we show the results of the frequency test which we carried out at the same time. We denote the expected value of probability of occurrence as P, and the experimental one as $\hat{P}$. We carried out χ^2 test under the hypothesis that if the values of $\hat{U}(P \to Q_i)$ coincide the values of $U(P \to Q_i)$, random numbers are equi-distributed.

As shown in *Table 1*, the degree of freedom is 11 and the value of χ^2 is 15.24 in this example, so we can see from χ^2 table that the probability is between 0.20 and 0.10. We choose the value of significance to be 0.05 in general, so the above hypothesis is not rejected.

Simulally, we carried out χ^2 test under the hypothsis that if the values of $\hat{P}$ coincide the values of P, random numbers are equi-distributed. In this case, the degree of freedom is 3 and the value of χ^2 is 3.22, then we see that the probability is between 0.50 and 0.30. Therefore this hypothsis also is not rejected. Consequently, we conclude that these random numbers are equi-distributed.

3. 3. *Consideration*

This method for testing randomness has the following characteristics.

Table 1.
Results of random walk simulation.

Boundary Point	$U(P \to Q_i)$	$\hat{U}(P \to Q_i)$	$\lvert U \to \hat{U} \rvert$	χ^2
Q_1	0.0625	0.0569	0.0056	0.81
Q_2	0.1250	0.1219	0.0031	0.13
Q_3	0.0625	0.0700	0.0075	1.44
Q_4	0.0625	0.0494	0.0131	4.41
Q_5	0.1250	0.1219	0.0031	0.13
Q_6	0.0625	0.0663	0.0038	0.36
Q_7	0.0625	0.0588	0.0037	0.36
Q_8	0.1250	0.1244	0.0006	0.01
Q_9	0.0625	0.0631	0.0006	0.01
Q_{10}	0.0625	0.0719	0.0094	2.25
Q_{11}	0.1250	0.1419	0.0164	3.37
Q_{12}	0.0625	0.0538	0.0087	1.96
Total	1.0000	1.0003		15.24

Table 2.
Results of frequency test.

Random Number	P	$\hat{P}$	$\lvert P-\hat{P} \rvert$	χ^2
$+X$	0.2500	0.2504	0.0004	0.00
$-X$	0.2500	0.2439	0.0061	1.07
$+Y$	0.2500	0.2476	0.0024	0.16
$-Y$	0.2500	0.2581	0.0081	1.99
Total	1.0000	1.0000		3.22

(1) If any periodicity appears in some part of random numbers, either will happen that the r.w.p. does not arrive for a while at a lattice point Q_i on the boundary Γ, or that the r.w.p. arrives at a particular lattice point for some times. We can find out periodicity by checking the above situation; whereas we cannot check such a situation in the frequency test.

(2) If the probability for a particular random number (for instance $+X$) to be generated is higher than that for others, the probability of the r.w.p. going to the direction which is dertermined by the number (for instance the direction to go to $+X$) becomes high. Consequently, a deviation of probability for a random number to be generated affects the value of $\hat{U}(P\rightarrow Q_i)$, and the order of the deviation may be estimated from the value.

We may conclude that this method for testing randomness has a function of the frequency test. This is clear from the results of experiments; for example, in *Table 2* we can see that the probability $\hat{P}$ for the generated random number $-Y$ is higher than that for others and that it affects the value of $\hat{U}(P\rightarrow Q_{11})$.

Here, we consider the relation between the function $U(P\rightarrow Q_i)$ and the probability P of generating random number. Let the function $U(P\rightarrow Q_i)$ have the variables α, β, γ and δ which correspond the respective probability of generating random numbers $+X$, $-X$, $+Y$ and $-Y$ and let us formulate the function as $U(P\rightarrow Q;\ \alpha,\ \beta,\ \gamma,\ \delta)$. For the case of equi-distribution, variables α, β, γ and δ should be all 1/4 theoretically, but really, there are deviations $d\alpha$, $d\beta$, $d\gamma$ and $d\delta$ respectively from all 1/4 in a case of random numbers to be tested. Therefore, in this case, a function $U(P\rightarrow Q_i;\ \alpha,\ \beta,\ \gamma,\ \delta)$ is rewritten as $U(P\rightarrow Q_i;\ 1/4+d\alpha,\ 1/4+d\beta,\ 1/4+d\gamma,\ 1/4+d\delta)$. When $d\alpha$, $d\beta$, $d\gamma$ and $d\delta$ are sufficiently small, the function U can be expanded as follows: (Taylor expansion).

$$\begin{aligned} &U(P\rightarrow Q_i;\ 1/4+d\alpha,\ 1/4+d\beta,\ 1/4+d\gamma,\ 1/4+d\delta) \\ &\quad -U(P\rightarrow Q_i;\ 1/4,\ 1/4,\ 1/4,\ 1/4) \\ &=d\alpha\frac{\partial U}{\partial\alpha}+d\beta\frac{\partial U}{\partial\beta}+d\gamma\frac{\partial U}{\partial\gamma}+d\delta\frac{\partial U}{\partial\delta} \\ &\quad +\frac{1}{2!}\left(d\alpha\frac{\partial}{\partial\alpha}+\cdots\cdots+d\delta\frac{\partial}{\partial\delta}\right)^2 U+\cdots\cdots. \end{aligned} \tag{3}$$

The left hand side of Eq. (3) is difference between the value of U which is obtained from the experiment and that obtaitend analytically from the numerical calculation.

On the othere hand, in the right hand side of Eq. (3) if we may neglect the terms with second order of differentials, we can obtain the magnitude of $d\alpha\frac{\partial U}{\partial\alpha}+\cdots\cdots+d\delta\frac{\partial U}{\partial\delta}$. If the first order of derivatives are more than unity, the deviations $d\alpha$, $d\beta$, $d\gamma$ and $d\delta$ (which may be fund by the frequency test) reflect remarkably. Therefore, this method of testing randomness in which the information of the left hand side Eq. (3) is utilized is more sensitive than the frequency test. But, in general, it is not clear whether the first derivatives are more than unity or not.

4. *Method for Transforming Distribution of Random Number*

4. 1. *Principle*

If we fix a boundary Γ, a closed domain Φ, a starting point P and a set of Q_i, the values of $U(P\rightarrow Q_i)$ are uniquely given. In other words, the r.w.p. arrives at a lattice point Q_i with probability $U(P\rightarrow Q_i)$. Thus we may get a signal with probability $U(P\rightarrow Q_i)$ on every time the r.w.p. arrives at the lattice points Q_i on the boundary Γ, and we may consider the signal as a kind of random number. Moreover, if we add the signal with probability $U(P\rightarrow Q_i)$ to another signal with probability $U(P\rightarrow Q_j)(i\neq j)$, we may get the signal or random number with probability of the sum of $U(P\rightarrow Q_i)$ and $U(P\rightarrow Q_j)$. Therefore, we may produce random numbers whose distribution f is any type (say, normal type or Poisson type etc.) by means of making adequate combinations of values of $U(P\rightarrow Q_i)$ and fitting the distribution of combined values to the distribution f. Thus we can transform the equi-distributed random number into the random number with any other distribution. This is the principle of the method for transforming distribution of random numbers utilizing the random walk simulation. Several examples of transformation are shown below.

Table 3.
Combination of $U(P\rightarrow Q_i)$ for transformation of equi-distribution to normal distribution.

Range of x	Prob $\left[\frac{1}{2}n<x<\frac{1}{2}(n+1)\right]$	Combination of $U(P\rightarrow Q_i)$	Combined Value of $U(P\rightarrow Q_i)$	Setting Error
$-\infty\sim-3.0$	0.001	——	0.000	0.001
$-3.0\sim-2.5$	0.005	——	0.000	0.005
$-2.5\sim-2.0$	0.017	$U(P\rightarrow Q_{28})$	0.017	0.000
$-2.0\sim-1.5$	0.044	$U(P\rightarrow Q_{26})$	0.047	0.003
$-1.5\sim-1.0$	0.092	$U(P\rightarrow Q_{19})+U(P\rightarrow Q_{23})$	0.094	0.002
$-1.0\sim-0.5$	0.150	$U(P\rightarrow Q_{16})+U(P\rightarrow Q_{20})+U(P\rightarrow Q_{22})$ $+U(P\rightarrow Q_{23})+U(P\rightarrow Q_{27})$	0.149	0.001
$-0.5\sim0$	0.191	$U(P\rightarrow Q_{15})+U(P\rightarrow Q_{17})+U(P\rightarrow Q_{18})$ $+U(P\rightarrow Q_{21})+U(P\rightarrow Q_{25})$	0.191	0.000
$0\sim0.5$	0.191	$U(P\rightarrow Q_4)+U(P\rightarrow Q_8)+U(P\rightarrow Q_{11})$ $+U(P\rightarrow Q_{12})+U(P\rightarrow Q_{14})$	0.191	0.000
$0.5\sim1.0$	0.150	$U(P\rightarrow Q_2)+U(P\rightarrow Q_6)+U(P\rightarrow Q_7)$ $+U(P\rightarrow Q_9)+U(P\rightarrow Q_{13})$	0.149	0.001
$1.0\sim1.5$	0.092	$U(P\rightarrow Q_5)+U(P\rightarrow Q_{10})$	0.094	0.002
$1.5\sim2.0$	0.044	$U(P\rightarrow Q_3)$	0.047	0.003
$2.0\sim2.5$	0.017	$U(P\rightarrow Q_1)$	0.017	0.000
$2.5\sim3.0$	0.005	——	0.000	0.005
$3.0\sim\infty$	0.001	——	0.000	0.001
Total	1.000		0.996	

(1) Example of transforming the equi-distribution to the normal distribution (mean: 0, standard deviation: 1) using a square boundary in *Fig. 2 (c)*.

The theoretical values of occurence probability Prob is given as:

$$\text{Prob}\left(\frac{1}{2}n<x<\frac{1}{2}(n+1)\right)=\frac{1}{\sqrt{2\pi}}\int_n^{n+1}\exp\left(-\frac{x^2}{2}\right)dx \quad (n=0,\ \pm1,\ \pm2,\ \cdots\cdots).$$

In *Table 3*, we show the range of x, Prob $\left(\frac{1}{2}n<x<\frac{1}{2}(n+1)\right)$, combinations of $U(P\to Q_i)$, combined values of $U(P\to Q_i)$ and setting errors.

(2) Example of transforming the equi-distribution to the Poisson distribution (mean: 2, standard deviation: $\sqrt{2}$) using a square boundary in *Fig. 2 (c)*.

$$\text{Prob}\,[x]=\frac{e^{-2}\,2^x}{x!} \quad (x=0,\ 1,\ \cdots\cdots)$$

In *Table 4*, we show values of x, Prob $[x]$, combinations of $U(P\to Q_i)$, combined values of $U(P\to Q_i)$ and setting errors.

(3) Eyample of transforming the equi-distribution to the binomial distribution (mean: 5, standard deviation: $\sqrt{2.5}$) using a square boundary in *Fig. 2 (d)*.

$$\text{Prob}\,[x]={}_{10}C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x} \quad (x=0,\ 1,\ \cdots\cdots,\ 10)$$

In *Table* 5, we show values of x, Prob $[x]$, combinations of $U(P\to Q_i)$, combined values of $U(P\to Q_i)$ and setting errors.

4.2. *Results of experiment*

We carried out 10,000 random trips for transforming distributions using the

Table 4.

Combination of $U(P\to Q_i)$ for transformation of equi-distribution to Poisson distribution.

x	Prob $[x]$	Combination of $U(P\to Q_i)$	Combined Value of $U(P\to Q_i)$	Setting Error
0	0.135	$U(P\to Q_2)+U(P\to Q_6)+U(P\to Q_9)+U(P\to Q_{13})$	0.132	0.003
1	0.271	$U(P\to Q_3)+U(P\to Q_{10})+U(P\to Q_{16})+U(P\to Q_{17})+\sum_{i=22}^{24}U(P\to Q_i)$	0.271	0.000
2	0.271	$U(P\to Q_5)+U(P\to Q_{12})+U(P\to Q_{19})+U(P\to Q_{20})+\sum_{i=26}^{28}U(P\to Q_i)$	0.271	0.000
3	0.180	$U(P\to Q_{11})+U(P\to Q_{18})+U(P\to Q_{21})+U(P\to Q_{25})$	0.182	0.002
4	0.090	$U(P\to Q_4)+U(P\to Q_{14})+U(P\to Q_{15})$	0.089	0.001
5	0.036	$U(P\to Q_7)+U(P\to Q_8)$	0.034	0.002
6	0.012	$U(P\to Q_1)$	0.017	0.005
7	0.003	—		
8	0.001	—		
Total	0.999		0.996	

Table 5.
Combination of $U(P\to Q_i)$ for transformation of equi-distribution to binominal distribution.

x	Prob $[x]$	Combination of $U(P\to Q_i)$	Combined Value of $U(P\to Q_i)$	Setting Error
0	0.001	——	0.000	0.001
1	0.010	$U(P\to Q_{36})$	0.011	0.001
2	0.044	$U(P\to Q_{32})$	0.043	0.001
3	0.117	$U(P\to Q_{23})+U(P\to Q_{28})+U(P\to Q_{30})+U(P\to Q_{34})$	0.118	0.001
4	0.205	$U(P\to Q_{24})+U(P\to Q_{25})+U(P\to Q_{27})+U(P\to Q_{29})$ $+U(P\to Q_{31})+U(P\to Q_{33})+U(P\to Q_{35})$	0.205	0.000
5	0.245	$U(P\to Q_{11})+\sum_{i=15}^{22} U(P\to Q_i)+U(P\to Q_{26})$	0.250	0.005
6	0.205	$U(P\to Q_2)+U(P\to Q_4)+U(P\to Q_6)+U(P\to Q_8)$ $+U(P\to Q_{10})+U(P\to Q_{12})+U(P\to Q_{13})$	0.205	0.000
7	0.117	$U(P\to Q_3)+U(P\to Q_7)+U(P\to Q_9)+U(P\to Q_{14})$	0.118	0.001
8	0.044	$U(P\to Q_5)$	0.043	0.001
9	0.010	$U(P\to Q_1)$	0.011	0.001
10	0.001	——	0.000	0.001
Total	1.000		1.004	

Table 6.
Results of experiment for case of normal distribution.

Range of x	Combined Value	Experimental Value	|Comb. Value—Exper. Value|
$-\infty\sim-3.0$	0.000	0.000	0.000
$-3.0\sim-2.5$	0.000	0.000	0.000
$-2.5\sim-2.0$	0.017	0.016	0.001
$-2.0\sim-1.5$	0.047	0.046	0.001
$-1.5\sim-1.0$	0.094	0.104	0.010
$-1.0\sim-0.5$	0.149	0.151	0.002
$-0.5\sim0$	0.191	0.192	0.001
$0\sim0.5$	0.191	0.192	0.001
$0.5\sim1.0$	0.149	0.153	0.004
$1.0\sim1.5$	0.094	0.091	0.003
$1.5\sim2.0$	0.047	0.041	0.006
$2.0\sim2.5$	0.017	0.018	0.001
$2.5\sim3.0$	0.000	0.000	0.000
$3.0\sim\infty$	0.000	0.000	0.000
Total	0.996	1.004	

three-dimensional random walk simulator. In *Tables 6~8*, we show combined values of $U(P\to Q_i)$, experimental values, absolute values of difference between the combined value and the experimental value.

4. 3. *Consideration*

This method for transforming distribution has the following characteristics.

Table 7.
Results of experiment for case of Poisson distribution.

x	Combined Value	Experimental Value	|Comb. Value—Exper. Value|
0	0.132	0.135	0.003
1	0.271	0.270	0.001
2	0.271	0.273	0.002
3	0.182	0.183	0.001
4	0.089	0.086	0.003
5	0.034	0.034	0.000
6	0.017	0.018	0.001
7	0.000	0.000	0.000
8	0.000	0.000	0.000
Total	0.996	0.999	

Table 8.
Results of experiment for case of binomial distribution.

x	Combined Value	Experimental Value	|Comb. Value—Exper. Value|
0	0.000	0.000	0.000
1	0.011	0.013	0.002
2	0.043	0.049	0.006
3	0.118	0.122	0.004
4	0.205	0.223	0.018
5	0.250	0.231	0.019
6	0.205	0.195	0.010
7	0.118	0.116	0.002
8	0.043	0.040	0.003
9	0.011	0.011	0.000
10	0.000	0.000	0.000
Total	1.004	1.000	

(1) Because the value of $U(P \to Q_i)$ decreases with inerease the number of partition of a side of a squate boundary, we can fit the distribution of the combined value of $U(P \to Q_i)$ to a given distribution f more exactly.

(2) There is not one to one correspondence between the number of equi-distributed random numbers comsumed in the random walk simulation and the number of random numbers with transformed distribation. The efficiency of transformation depends upon the mean number of equi-distributed random numbers per a trip, that is the means duration of random walks.(6) It has been already known that the approximate value of $\tilde{n}$, a mean number of steps n in a trip, is given in the following formular.(7)

$$\tilde{n} = k\left(\frac{R}{h}\right)^2 \quad (k:\ \text{constant})$$

Table 9.
Numbers of steps per a trip.

Number of Partition of a Side	$\tilde{n}$	n	Number of Random Trips
4	4	4.51	1,600
6	9	10.62	800 [5]
10	25	29.34	5,000 [5]
20	100	117.64	10,000
28	196	236.50	2,000
⋮	⋮	⋮	⋮

Where R is a radius of circle which surrounds completely the closed domain Φ. In *Table 9*, we show the number of partition of a side in a square boundary, the approximate value: $\tilde{n}$, the experimental value of n: $\hat{n}$ and the total numbers of random trips in the experiment.

Further, together with the results of computing the auto-correlation function of random numbers after transformation, we concluded that these random numbers after transforming had sufficient randomness.

5. *Conclusion*

These methods are originally devised on the standpoint that we utilize the three-dimensional random walk simulator, but we can perform these testing and transforming of random numbers according to these methods using the digital computer.

There are a few unsolved problems in these methods, those are:

(1) The relation between the randomness of random numbers to the function U is not analytically clear.

(2) The efficiency of transformation falls in proportion to the number of partitions of a side of a square boundary.

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