Notes on the Recurrence Techniques for the Calculation of Bessel Functions $J_\nu(x)$ and $I_\nu(x)$

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1. *Introduction*

J.C.P. Miller [1] originally devised the recurrence techniques for the calculation of modified Bessel functions $I_n(x)$. Since then his algorithm has been applied to the computation of Bessel functions $J_\nu(x)$, $I_\nu(x)$ [2–8] and some other functions with a similar behavior (for example, [9]).

F.W.J. Olver obtained the strict upper bounds for the errors in the values yielded by the algorithm [10].

The purpose of this paper is to describe a method to determine the economical points M_E for given $x>0$, from which the downward recursion processes start in order to generate $J_\nu(x)$ and $I_\nu(x)$ accurate to p significant digits respectively. As a consequence of them, the values of M_E are shown for $p\leqq 30$ and $0.1\leqq x\leqq 100$.

In this paper, we denote hereafter the order by $(\nu+n)$, where $0\leqq\nu<1$ and n is a non-negative integer.

2. *Economical Points M_E for Generating $J_{\nu+n}(x)$*

The process to find $J_{\nu+n}(x)$ by the Miller's algorithm is known as follows:

First, we choose moderately a positive integer M larger than x and n. Secondly, we construct a sequence of trial values $F_{\nu+m-1}(x)$, $m=M, M-1, \cdots, 1$, by the use of the recurrence relation

$$F_{\nu+m-1}(x)=\frac{2(\nu+m)}{x}F_{\nu+m}(x)-F_{\nu+m+1}(x) \quad (1)$$

starting with the initial values

$$F_{\nu+M+1}(x)=0, \quad F_{\nu+M}(x)=a, \quad (2)$$

where a is an arbitrary non-zero constant.

If M is taken sufficiently large, the trial value $F_{\nu+n}(x)$ so obtained turns out to be approximately proportional to $J_{\nu+n}(x)$. Therefore, the approximation for $J_{\nu+n}(x)$ is obtained by the relation

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$$J_{\nu+n}(x)\approx F_{\nu+n}(x)/\alpha_0, \tag{3}$$

where the normalizing factor α_0 is given, for example, in the form

$$\left(\frac{2}{x}\right)^{\nu}\sum_{m=0}^{L/2}\frac{(\nu+2m)}{m!}\Gamma(\nu+m)F_{\nu+2m}(x)=\alpha_0, \tag{4.1}$$

by making use of a positive even integer $L\leqq M$. For $\nu=0$, (4.1) reduces to

$$F_0(x)+2\sum_{m=1}^{L/2}F_{2m}(x)=2\sum_{m=0}^{L/2}{}' F_{2m}(x)=\alpha_0. \tag{4.2}$$

Now, let us first consider the minimum value of M, $M_{\min}$, by which we can guarantee a predetermined accuracy of the normalizing factor α_0 for given x. Later, we shall discuss a method to determine an appropriate value $M_E(\geqq M_{\min})$ as easily as possible.

If we can obtain M_E, we may take

$$L=2[M_E/2], \tag{5}$$

where $[M_E/2]$ is the largest integer not greater than $M_E/2$.

Since $F_{\nu+M+1}(x)$ can be treated as the linear combination of the regular and irregular Bessel functions, we have

$$\begin{aligned} F_{\nu+M+1}(x)&=\alpha J_{\nu+M+1}(x)+\beta Y_{\nu+M+1}(x)\\ F_{\nu+n}(x)&=\alpha J_{\nu+n}(x)+\beta Y_{\nu+n}(x). \end{aligned} \tag{6}$$

Assuming $J_{\nu+n}(x)\neq 0$, we write

$$\frac{J_{\nu+M+1}(x)}{J_{\nu+n}(x)}\frac{Y_{\nu+n}(x)}{Y_{\nu+M+1}(x)}=\varepsilon_{\nu+M,\,\nu+n}. \tag{7}$$

Using the first of (2), (6) and (7), we obtain

$$\left[J_{\nu+n}(x)-\frac{F_{\nu+n}(x)}{\alpha}\right]\Big/J_{\nu+n}(x)=\varepsilon_{\nu+M,\,\nu+n}. \tag{8}$$

Since $\varepsilon_{\nu+M,\,\nu+n}\to 0$ as $M\to\infty$, we have

$$F_{\nu+n}(x)=\alpha J_{\nu+n}(x). \tag{9}$$

By the aid of an addition theorem, we obtain

$$\left(\frac{2}{x}\right)^{\nu}\sum_{m=0}^{\infty}\frac{(\nu+2m)}{m!}\Gamma(\nu+m)(\alpha J_{\nu+2m}(x))=\alpha. \tag{10}$$

Let us consider the absolute error of α_0 in the form:

$$\Delta\alpha=\alpha-\alpha_0.$$

The relative error of α_0 can be written as follows:

$$\begin{aligned} \frac{\Delta\alpha}{\alpha}=&\left(\frac{2}{x}\right)^{\nu}\sum_{m=0}^{[M/2]}\frac{(\nu+2m)}{m!}\Gamma(\nu+m)\frac{J_{\nu+M+1}(x)}{Y_{\nu+M+1}(x)}Y_{\nu+2m}(x)\\ &+\left(\frac{2}{x}\right)^{\nu}\sum_{m=[M/2]+1}^{\infty}\frac{(\nu+2m)}{m!}\Gamma(\nu+m)J_{\nu+2m}(x). \end{aligned} \tag{11.1}$$

For $\nu=0$, we write $\dfrac{\Delta\alpha}{\alpha}=\left(\dfrac{\Delta\alpha}{\alpha}\right)_{\nu=0}$. Then, Eq. (11.1) reduces to

$$\left(\frac{\Delta\alpha}{\alpha}\right)_{\nu=0}=2J_{M+1}(x)\left[\sum_{m=0}^{[M/2]}{}' \frac{Y_{2m}(x)}{Y_{M+1}(x)}+\sum_{m=[M/2]+1}^{\infty}\frac{J_{2m}(x)}{J_{M+1}(x)}\right]. \tag{11. 2}$$

If $\mu\gg x$, then

$$J_\mu(x)\approx\frac{1}{\sqrt{2\pi\mu}}\left(\frac{ex}{2\mu}\right)^{\mu},$$
$$Y_\mu(x)\approx-\sqrt{\frac{2}{\pi\mu}}\left(\frac{ex}{2\mu}\right)^{-\mu}, \tag{12}$$

and if m is a large enough positive integer, we get

$$\Gamma(\nu+m)\approx m^{\nu}\Gamma(m) \tag{13}$$

for $0<\nu<1$.

From (11. 1) with the aids of (12) and (13), it is clear that $\Delta\alpha/\alpha>0$, and that $\Delta\alpha/\alpha$ is a monotone decreasing function of ν, since $M\gg x$ (see, Example 2). Therefore, $M_{\min}$ which satisfies the condition

$$\Delta\alpha/\alpha<0.5\times10^{-p} \tag{14}$$

for any ν must be determined so as to satisfy the inequality

$$\left(\frac{\Delta\alpha}{\alpha}\right)_{\nu=0}<0.5\times10^{-p}. \tag{15}$$

In order to find about three correct significant digits in $(\Delta\alpha/\alpha)_{\nu=0}$, we may take several members of each term in the brackets on the right hand side of (11. 2). That is to say, we have only to compute the last several members of the first term and the first several members of the second term. For the present purpose it is only necessary to obtain about three correct significant digits in $J_{M+1}(x)$ by the use of a mathematical table or any suitable approximation.

Next, let us consider how to determine an approximation M_E for $M_{\min}$, using a function $J_{M+1}(x)$ alone and not using other functions $J_{2m}(x)$, $Y_{2m}(x)$ and $Y_{M+1}(x)$.

Since $M\gg x$, by writing

$$x/2=u \quad\text{and}\quad M-u=v,$$

we obtain

$$\frac{Y_M(x)}{Y_{M+1}(x)}<\frac{u}{v},\quad \frac{Y_{M-1}(x)}{Y_M(x)}<\frac{u}{u-1},\quad \cdots$$
$$\frac{J_{M+1}(x)}{J_M(x)}<\frac{u}{v+1},\quad \frac{J_{M+2}(x)}{J_{M+1}(x)}<\frac{u}{v+2},\quad \cdots.$$

With the aids of these inequalities and (11. 2), we find that

(i) if M is even,

$$\left(\frac{\Delta\alpha}{\alpha}\right)_{\nu=0}<2J_{M+1}(x)\left[\frac{u}{v}+\frac{u}{v+2}+\frac{u^3}{v(v-1)(v-2)}+\frac{u^3}{(v+2)(v+3)(v+4)}+\cdots\right] \tag{16. 1}$$

and

(ii) if M is odd,

$$\left(\frac{\Delta\alpha}{\alpha}\right)_{\nu=0}<2J_{M+1}(x)\left[1+\frac{u^2}{v(v+1)}+\frac{u^2}{(v+2)(v+3)}+\cdots\right]. \tag{16. 2}$$

Table 1. M_E and N_E for generating $J_{\nu+n}(x)$.
(E.g., $M_E=6$, $N_E=3$ for $p=10$ and $x=0.3$)
(With respect to N_E, see paragraph 3)

p \ x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
10	5 (3)	6 (4)	6 (3)	8 (5)	8 (5)	8 (5)	8 (5)	10 (7)	10 (7)
20	10 (6)	11 (6)	12 (7)	13 (8)	14 (8)	14 (8)	15 (9)	16 (10)	16 (10)
30	14 (8)	16 (9)	17 (10)	18 (10)	20 (12)	20 (12)	20 (11)	22 (13)	22 (13)

p \ x	1	2	3	4	5	6	7	8	9
10	10 (6)	13 (9)	16 (11)	18 (13)	20 (14)	21 (15)	23 (17)	24 (17)	26 (19)
20	16 (9)	20 (12)	24 (16)	26 (17)	30 (20)	32 (22)	34 (23)	36 (25)	38 (27)
30	22 (13)	28 (17)	32 (20)	35 (22)	38 (25)	40 (26)	44 (30)	46 (31)	48 (33)

p \ x	10	20	30	40	50	60	70	80	90	100
10	28 (21)	42 (33)	55 (45)	68 (57)	80 (68)	92 (79)	103 (90)	115 (101)	126 (111)	138 (123)
20	40 (28)	56 (41)	72 (56)	86 (68)	99 (80)	112 (92)	124 (103)	137 (115)	149 (126)	161 (137)
30	50 (34)	69 (50)	86 (64)	101 (77)	115 (90)	129 (102)	142 (114)	156 (127)	168 (137)	181 (150)

Hence, using only $J_{M+1}(x)$ accurate to a few significant digits, we can find the minimum value of M, i.e. M_E, for given x which makes the right hand side of (16. 1) or (16. 2) smaller than 0.5×10^{-p}. In this procedure, we may take several terms in the blackets on the right hand side of (16. 1) or (16. 2). Consequently, one can obtain easily an approximation M_E for $M_{\min}$ by the methods described above.

We have obtained the values of M_E for $p=9$, 10, 18, 20, 30 and $0.01\leqq x\leqq 100$. Some of them are shown in Table 1. The values of M_E for $p=10$ agree entirely with the optimum values of M obtained experimentally by I. Uoki [6].

3. *The Error in the Approximation of* $J_{\nu+n}(x)$

If $M\geqq M_E$, $\Delta\alpha/\alpha<0.5\times 10^{-p}$. With the aid of (8), neglecting the terms of higher order than $(\Delta\alpha/\alpha)^2$, we have

$$\frac{F_{\nu+n}(x)}{\alpha_0}\approx J_{\nu+n}(x)(1-\varepsilon_{\nu+M,\,\nu+n})\left(1+\frac{\Delta\alpha}{\alpha}\right). \tag{17}$$

For given x and M, let us consider the largest value of n, N, which satisfies

$$|\varepsilon_{\nu+M,\,\nu+n}|<0.5\times 10^{-p}. \tag{18}$$

If $n\leqq N$, $(\Delta\alpha/\alpha)\varepsilon_{\nu+M,\,\nu+n}$ is negligible small because of the above definitions. Therefore, we may write the absolute error $\Delta_{\nu+n}$ in an approximation of $J_{\nu+n}(x)$ as follows:

$$\Delta_{\nu+n}=J_{\nu+n}(x)-\frac{F_{\nu+n}(x)}{\alpha_0}$$

$$\approx J_{\nu+n}(x)\left(\varepsilon_{\nu+M,\,\nu+n}-\frac{\Delta\alpha}{\alpha}\right). \tag{19}$$

Case 1: $J_{\nu+n}(x)\neq 0$.

The relative error $\delta_{\nu+n}$ in $F_{\nu+n}(x)/\alpha_0$ is written in the form:

$$\delta_{\nu+n}\approx\varepsilon_{\nu+M,\,\nu+n}-\frac{\Delta\alpha}{\alpha}. \tag{20}$$

It is obvious that

$$0<\varepsilon_{\nu+M,\,\nu+n}<0.5\times 10^{-p} \quad \text{for} \quad x\leqq n\leqq N,$$

$$\left|\varepsilon_{\nu+M,\,\nu+n}\right|\ll\left|\frac{\Delta\alpha}{\alpha}\right| \quad \text{for} \quad n<x.$$

When $M\geqq M_E(\geqq M_{\min})$, therefore, we find from (20) that

$$|\delta_{\nu+n}|<0.5\times 10^{-p} \quad \text{for} \quad n\leqq N.$$

In other words, we can obtain approximation of $J_{\nu+n}(x)$ accurate to p significant digits for $0\leqq n\leqq N$ by making use of $M\geqq M_E$.

Using (12), we can find that $\varepsilon_{\nu+M,\,\nu+n}$ is a monotone decreasing function of ν. Hence we shall denote again by N the largest value of n which satisfies the inequality

$$\frac{J_{M+1}(x)}{J_n(x)}\,\frac{Y_n(x)}{Y_{M+1}(x)}<0.5\times 10^{-p} \tag{21}$$

for given p, x and M. We write $N=N_E$ when $M=M_E$, and show also the values of N_E in Table 1.

Case 2: $J_{\nu+n}(x)\approx 0$.

In this case, the absolute error of $F_{\nu+n}(x)/\alpha_0$ must be considered. From (19) we get

$$\Delta_{\nu+n}\approx\frac{J_{\nu+M+1}(x)}{Y_{\nu+M+1}(x)}\,Y_{\nu+n}(x)-J_{\nu+n}(x)\frac{\Delta\alpha}{\alpha}.$$

If x is a zero of $J_{\nu+n}(x)$ which is accurate to p significant digits, then

$$|J_{\nu+n}(x)|\approx 10^{-p}.$$

On the other hand, $\Delta\alpha/\alpha<0.5\times 10^{-p}$ and $\left|\frac{J_{\nu+M+1}(x)}{Y_{\nu+M+1}(x)}Y_{\nu+n}(x)\right|$ is smaller than or nearly equal to 10^{-2p} for $M\geqq M_E$. Therefore, $|\Delta_{\nu+n}|$ is nearly equal to 10^{-2p}. Moreover, we can easily find that

$$\frac{J_{\nu+n}(x)}{J_{\nu+n+1}(x)}-\frac{F_{\nu+n}(x)}{F_{\nu+n+1}(x)}\approx\frac{\Delta_{\nu+n}}{J_{\nu+n+1}(x)}.$$

Then, the magnitude of the difference between $J_{\nu+n}(x)/J_{\nu+n+1}(x)$ and $F_{\nu+n}(x)/F_{\nu+n+1}(x)$ is also as small as the order of 10^{-2p}.

It is interesting to note that a zero of $J_{\nu+n}(x)$ accurate to about $2p$ significant digits can be obtained by using $M=M_E$ which guarantees the values of functions $J_{\nu+n\pm r}(x)$, $r=1, 2, \cdots$, accurate only to p significant digits.

4. *A Method for Evaluating the Zeros of* $J_{\nu+n}(x)$

Applying the recurrence technique, we consider now to evaluate the zeros of $J_{\nu+n}(x)$ by the Newton's iteration [11].

If x_i is an approximation for a zero of $J_{\nu+n}(x)$, a better approximation x_{i+1} can be obtained in the form*:

$$x_{i+1}=x_i-\frac{J_{\nu+n}(x_i)}{J'_{\nu+n}(x_i)},\quad i=0,\ 1,\ 2,\ \cdots. \tag{22}$$

As is known,

$$J'_{\nu+n}(x)=\frac{1}{2}(J_{\nu+n-1}(x)-J_{\nu+n+1}(x)),$$

$$J_{\nu+n-1}(x_i)\approx -J_{\nu+n+1}(x_i).$$

By these reasons, Eq. (22) may be written approximately in the form:

$$x_{i+1}=x_i+\frac{F_{\nu+n}(x_i)}{F_{\nu+n+1}(x_i)}. \tag{23}$$

The procedure of computing the second term on the right hand side of (23) is as follows:

(i) Choose M slightly larger than M_E for given p and $x\geqq x_i$, when we want to evaluate a zero accurate to $2p$ significant digits.

(ii) Construct a sequence of trial values

$$F_{\nu+M+1}(x_i),\ F_{\nu+M}(x_i),\ F_{\nu+M-1}(x_i),\ \cdots,\ F_{\nu+n+1}(x_i),\ F_{\nu+n}(x_i) \tag{24}$$

discarding the values corresponding to the functions of smaller order than $(\nu+n)$.

(iii) Divide $F_{\nu+n}(x_i)$ by $F_{\nu+n+1}(x_i)$.

Thus there is no need to compute the function values themselves. This is very convenient for saving the computing time and simplifying the program especially when $\nu\neq 0$. Moreover, it must be noted that we can easily determine the number of positive zeros ($\neq 0$) of $J_{\nu+n}(x)$ on a given interval by making use of such a sequence as (24), since the sequence of functions $\{J_{\nu+n}(x)\}$ constructs a Sturm's chain.

5. *Economical Points* M_E *for Generating* $I_{\nu+n}(x)$

The economical points M_E for generating $I_{\nu+n}(x)$ can be determined essentially in the same way that is described in paragraph 2. The details of the process will be shown below.

Now, let us consider the process to find $I_{\nu+n}(x)$ by the Miller's algorithm. For given x and n, we first choose moderately a positive integer $M>n$. Next, we construct a sequence of trial values $G_{\nu+m-1}(x)$, $m=M,\ M-1,\ \cdots,\ 1$, by the use of the recurrence relation

$$G_{\nu+m-1}(x)=\frac{2(\nu+m)}{x}G_{\nu+m}(x)+G_{\nu+m+1}(x), \tag{25}$$

* A starting value x_0 is easily obtained by the McMahon series.

starting with the initial values

$$G_{\nu+M+1}(x)=0, \quad G_{\nu+M}(x)=a. \tag{26}$$

Since M is taken appropriately large, we can write

$$I_{\nu+n}(x)\approx G_{\nu+n}(x)/\alpha_0. \tag{27}$$

In this case, we assume that the normalizing factor α_0 is given by the relation

$$2\left(\frac{2}{x}\right)^\nu \frac{\Gamma(1+\nu)}{\Gamma(1+2\nu)}\sum_{m=0}^{L}\frac{(\nu+m)}{m!}\Gamma(2\nu+m)(e^{-x}G_{\nu+m}(x))=\alpha_0, \tag{28.1}$$

using a positive integer $L\leqq M$. For $\nu=0$, (28.1) reduces to

$$e^{-x}\left(G_0(x)+2\sum_{m=1}^{L}G_m(x)\right)=2\sum_{m=0}^{L}{}'(e^{-x}G_m(x))=\alpha_0. \tag{28.2}$$

Let us consider to determine an optimum value M_E not smaller than the minimum value of M, which guarantees the normalizing factor α_0 accurate to p significant digits for given x. If M_E is determined, we may take

$$L=M_E. \tag{29}$$

Denoting $\overline{K}_{\nu+n}(x)=(-)^n K_{\nu+n}(x)$, we may write

$$\begin{aligned} G_{\nu+M+1}(x)&=\alpha I_{\nu+M+1}(x)+\beta\overline{K}_{\nu+M+1}(x)\\ G_{\nu+n}(x)&=\alpha I_{\nu+n}(x)+\beta\overline{K}_{\nu+n}(x)\,. \end{aligned} \tag{30}$$

Assuming $I_{\nu+n}(x)\neq 0$, we introduce again the notation

$$\begin{aligned} \varepsilon_{\nu+M,\,\nu+n}&=\frac{I_{\nu+M+1}(x)}{I_{\nu+n}(x)}\,\frac{\overline{K}_{\nu+n}(x)}{\overline{K}_{\nu+M+1}(x)}\\ &=(-)^{M-n+1}\frac{I_{\nu+M+1}(x)}{I_{\nu+n}(x)}\,\frac{K_{\nu+n}(x)}{K_{\nu+M+1}(x)}. \end{aligned} \tag{31}$$

With the aids of the first of (26), (30) and (31), we obtain

$$\left[I_{\nu+n}(x)-\frac{G_{\nu+n}(x)}{\alpha}\right]\Big/ I_{\nu+n}(x)=\varepsilon_{\nu+M,\,\nu+n}\,. \tag{32}$$

Since $\varepsilon_{\nu+M,\,\nu+n}\to 0$ as $M\to\infty$, we have

$$G_{\nu+n}(x)=\alpha\ I_{\nu+n}(x). \tag{33}$$

Following an addition theorem, we have

$$2\left(\frac{2}{x}\right)^\nu \frac{\Gamma(1+\nu)}{\Gamma(1+2\nu)}\sum_{m=0}^{\infty}\frac{(\nu+m)}{m!}\Gamma(2\nu+m)(\alpha e^{-x}I_{\nu+m}(x))=\alpha. \tag{34}$$

Denoting by $\Delta\alpha$ the absolute error in the approximation of α as before, we can write

$$\frac{\Delta\alpha}{\alpha}=\frac{\Delta\alpha_1}{\alpha}+\frac{\Delta\alpha_2}{\alpha}, \tag{35}$$

where

$$\frac{\Delta\alpha_1}{\alpha}=2\left(\frac{2}{x}\right)^\nu \frac{\Gamma(1+\nu)}{\Gamma(1+2\nu)}\sum_{m=0}^{M}\frac{(\nu+m)}{m!}\Gamma(2\nu+m)e^{-x}\frac{I_{\nu+M+1}(x)}{\overline{K}_{\nu+M+1}(x)}K_{\nu+m}(x), \tag{36.1}$$

$$\frac{\Delta\alpha_2}{\alpha}=2\left(\frac{2}{x}\right)^{\nu}\frac{\Gamma(1+\nu)}{\Gamma(1+2\nu)}\sum_{m=M+1}^{\infty}\frac{(\nu+m)}{m!}\Gamma(2\nu+m)(e^{-x}I_{\nu+m}(x)). \tag{36. 2}$$

Then it is clear that $\Delta\alpha_1/\alpha<0$, $\Delta\alpha_2/\alpha>0$ and $|\Delta\alpha_1/\alpha|<|\Delta\alpha_2/\alpha|$. Therefore, we see that $\Delta\alpha/\alpha>0$.

If $\mu\gg x$, then

$$I_\mu(x)\approx\frac{1}{\sqrt{2\pi\mu}}\left(\frac{ex}{2\mu}\right)^{\mu},\qquad K_\mu(x)\approx\sqrt{\frac{\pi}{2\mu}}\left(\frac{ex}{2\mu}\right)^{-\mu}. \tag{37}$$

From (35) with the aids of (13) and (37), it is clear that $\Delta\alpha/\alpha$ is a monotone increasing function of ν when $M\gg x$. In some cases when M is not large enough compared with x, $\Delta\alpha/\alpha$ is also a monotone increasing function of ν (see, Example 3). Accordingly, we see that $\Delta\alpha/\alpha$ takes the maximum value as $\nu\to1$ for given x and M.

Substituting $\nu=1$ into (36. 1) and (36. 2) respectively, we have

$$\frac{\Delta\alpha_1}{\alpha}=e^{-x}I_{M+2}(x)\left[-(M+1)+M\frac{K_{M+1}(x)}{K_{M+2}(x)}+2\frac{K_M(x)}{K_{M+2}(x)}-\cdots\right], \tag{38. 1}$$

$$\frac{\Delta\alpha_2}{\alpha}=e^{-x}[(M+2)I_{M+1}(x)+(M+3)I_{M+2}(x)+2I_{M+3}(x)+\cdots]. \tag{38. 2}$$

As a behavior of modified Bessel functions, we have the inequalities as follows:

$$I_{M+1}(x)>I_{M+2}(x)>I_{M+3}(x)>\cdots,$$
$$K_{M+2}(x)>K_{M+1}(x)>K_M(x)>\cdots.$$

Then, with the aids of (38. 1) and (38. 2) we obtain

$$\frac{\Delta\alpha}{\alpha}<e^{-x}[(M+2)I_{M+1}(x)+(M+3)I_{M+2}(x)]. \tag{39}$$

If we get $I_{M+1}(x)$ and $I_{M+2}(x)$ accurate to a few significant figures respectively by the use of any mathematical table or approximating formula, then we can easily determine the minimum of M which satisfies

$$e^{-x}[(M+2)I_{M+1}(x)+(M+3)I_{M+2}(x)]<0.25\times10^{-p}. \tag{40}$$

The minimum value of M so obtained is the value for M_E described before, and it satisfies the inequality

$$\Delta\alpha/\alpha<0.25\times10^{-p} \tag{41}$$

for any ν. From (39) and (40), it is clear that M_E is somewhat larger than or equal to the minimum of M which satisfies the inequality (41).

We have obtained the values of M_E for $p=9$, 10, 18, 20, 30 and $0.01\leqq x\leqq100$. Some of them are indicated in Table 2. For $x\geqq50$ and comparatively small p, we find that $M_E<x$. It must be noted that there is no such a case for $J_{\nu+n}(x)$.

The reason why the upper bound for $\Delta\alpha/\alpha$ has been chosen to be equal to 0.25×10^{-p} will be discussed in the next paragraph.

Table 2. M_E and N_E for generating $I_{\nu+n}(x)$.
(E.g., $M_E=8$, $N_E=5$ for $p=10$ and $x=0.3$)
(With respect to N_E, see paragraph 6)

p \ x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	
10	6 (4)	7 (5)	8 (5)	8 (5)	9 (6)	9 (6)	10 (7)	10 (7)	10 (6)	
20	10 (6)	12 (7)	13 (8)	14 (9)	15 (9)	15 (9)	16 (10)	17 (11)	17 (11)	
30	15 (9)	17 (10)	18 (11)	19 (11)	20 (12)	21 (13)	22 (14)	23 (14)	23 (14)	
p \ x	1	2	3	4	5	6	7	8	9	
10	11 (7)	14 (10)	16 (11)	17 (11)	19 (13)	20 (14)	22 (16)	23 (16)	24 (17)	
20	18 (12)	22 (14)	24 (15)	27 (18)	29 (19)	31 (21)	33 (22)	34 (23)	36 (24)	
30	24 (15)	29 (18)	32 (20)	35 (22)	38 (25)	40 (26)	42 (27)	44 (29)	46 (30)	
p \ x	10	20	30	40	50	60	70	80	90	100
10	25 (17)	34 (24)	41 (29)	47 (34)	52 (38)	57 (41)	61 (44)	65 (47)	69 (50)	73 (53)
20	38 (26)	50 (35)	59 (41)	67 (47)	74 (52)	80 (56)	86 (61)	92 (65)	97 (69)	102 (73)
30	48 (32)	63 (43)	74 (51)	84 (58)	92 (64)	100 (70)	107 (75)	113 (79)	120 (84)	126 (89)

6. *The Error in the Approximation of $I_{\nu+n}(x)$*

If $M\geqq M_E$, then $\Delta\alpha/\alpha<0.25\times10^{-p}$. Neglecting the terms of higher order than $(\Delta\alpha/\alpha)^2$, we find that

$$\frac{G_{\nu+n}(x)}{\alpha_0}\approx I_{\nu+n}(x)(1-\varepsilon_{\nu+M,\,\nu+n})\left(1+\frac{\Delta\alpha}{\alpha}\right). \tag{42}$$

Now, we denote by N the largest value of n which holds

$$|\varepsilon_{\nu+M,\,\nu+n}|<0.25\times10^{-p} \tag{43}$$

for given p, x and M. Then, we can discard $(\Delta\alpha/\alpha)\varepsilon_{\nu+M,\,\nu+n}$ because of the above definitions. So that, the absolute error $\Delta_{\nu+n}$ in the approximation $G_{\nu+n}(x)/\alpha_0$ for $I_{\nu+n}(x)$ is written as follows:

$$\begin{aligned}\Delta_{\nu+n}&=I_{\nu+n}(x)-\frac{G_{\nu+n}(x)}{\alpha_0}\\&\approx I_{\nu+n}(x)\left(\varepsilon_{\nu+M,\,\nu+n}-\frac{\Delta\alpha}{\alpha}\right).\end{aligned} \tag{44}$$

Therefore, the relative error in $G_{\nu+n}(x)/\alpha_0$ can be represented in the form:

$$\delta_{\nu+n}\approx\varepsilon_{\nu+M,\,\nu+n}-\frac{\Delta\alpha}{\alpha}. \tag{45}$$

Then, we may consider

$$|\delta_{\nu+n}|\leqq|\varepsilon_{\nu+M,\,\nu+n}|+\left|\frac{\Delta\alpha}{\alpha}\right|. \tag{46}$$

Accordingly, from (46) with the aids of (41) and (43) we find that

$$|\delta_{\nu+n}|<0.5\times 10^{-p} \text{ for } n\leqq N.$$

Namely, if $M\geqq M_E$ we can generate $I_{\nu+n}(x)$ accurate to p significant digits for $0\leqq n\leqq N$.

It is clear that $|\varepsilon_{\nu+M,\, \nu+n}|$ is also a monotone decreasing function of ν. Hence, we denote again by N the largest value of n which holds

$$\frac{I_{M+1}(x)}{I_n(x)}\frac{K_n(x)}{K_{M+1}(x)}<0.25\times 10^{-p}. \tag{47}$$

We write $N=N_E$ when $M=M_E$ for given p and x, and show also the values of N_E in Table 2.

Let us now consider the minimum of M, $M_{\min}$, by which we can obtain $I_{\nu+n}(x)$ accurate to p significant digits. That is to say, suppose that $M_{\min}$ satisfies the condition

$$|\delta_{\nu+n}|<0.5\times 10^{-p} \tag{48}$$

for any ν when p and x are given. We will show the relation $M_{\min}$ and M_E.

If $n\ll N$, then $|\varepsilon_{\nu+M,\, \nu+n}|\ll|\varDelta\alpha/\alpha|$. So that we find from (46)

$$|\delta_{\nu+n}|\approx|\varDelta\alpha/\alpha| \text{ for } n\ll N.$$

Therefore, $M_{\min}$ may hold

$$\varDelta\alpha/\alpha<0.5\times 10^{-p} \tag{49}$$

as well as (48). However, we have determined M_E so as to satisfy the inequality $\varDelta\alpha/\alpha<0.25\times 10^{-p}$. Consequently, it is clear that M_E is nearly equal to $M_{\min}$, and that $M_E\geqq M_{\min}$.

Since $|\delta_{\nu+n}|\leqq|\varepsilon_{\nu+M,\, \nu+n}|+|\varDelta\alpha/\alpha|$, we are not able to set an upper bound for $|\varepsilon_{\nu+M,\, \nu+n}|$ under the conditions (48) and (49). In other words, it is actually difficult in these circumstances to determine the largest value of order n, i.e. N.

Table 3. Computed values of $J_n(30)$, using $M=55$, $L=54$.

n	$F_n(30)/\alpha_0$	$\varDelta_n$	δ_n	$\varepsilon_{M,n}$
55	5.3605 54054 32144 (−11)	5.06 (−12)	8.62 (− 2)	8.62 (− 2)
54	1.9655 36486 58453 (−10)	1.54 (−12)	7.76 (− 3)	7.76 (− 3)
53	6.5398 75946 27216 (−10)	4.78 (−13)	7.31 (− 4)	7.31 (− 4)
…	…… …… ……	……	……	……
46	1.4463 11623 21019 (− 6)	2.13 (−16)	1.47 (−10)	1.88 (−10)
45	3.9157 69889 72646 (− 6)	−5.37 (−17)	−1.37 (−11)	2.66 (−11)
44	1.0300 99804 59692 (− 5)	−3.74 (−16)	−3.63 (−11)	4.00 (−12)
43	2.6300 49104 51165 (− 5)	−1.04 (−15)	−3.97 (−11)	6.42 (−13)
42	6.5093 74295 00315 (− 5)	−2.62 (−15)	−4.02 (−11)	1.10 (−13)
41	1.5596 19892 14972 (− 4)	−6.29 (−15)	−4.03 (−11)	2.02 (−14)
…	…… …… ……	……	……	……
2	7.8451 24607 64292 (− 2)	−3.16 (−12)	−4.03 (−11)	−6.73 (−20)
1	−1.1875 10626 21412 (− 1)	4.79 (−12)	−4.03 (−11)	3.05 (−20)
0	−8.6367 98358 45234 (− 2)	3.48 (−12)	−4.03 (−11)	−5.83 (−20)

By this reason, we have determined M_E and N_E setting the upper bounds for $\Delta\alpha/\alpha$ and $|\varepsilon_{\nu+M,\,\nu+n}|$ to be equal to 0.25×10^{-p} respectively.

7. *Numerical Examples*

Example 1. Using $M=55$ and $L=54$, let us find $J_n(30)$. The computed results are shown in Table 3. It is seen that the ten significant digits of $J_n(30)$ are accurate for $0\leqq n\leqq45$. For $n\leqq41$, $|\varepsilon_{M,\,n}|$ are so small that δ_n are nearly equal to a constant -4.03×10^{-11}. Consequently, $\Delta\alpha/\alpha\approx4.03\times10^{-11}$ is obtained in this example.

Example 2. Using again $M=55$ and $L=54$, let us find $J_\nu(30)$ for $\nu=0$, $^1/_4$, $^1/_2$, $^3/_4$ and $^{39}/_{40}$. The results are shown in Table 4. It is clear that the larger the value of ν, the smaller the value of $\Delta\alpha/\alpha$ ($\approx|\delta_\nu|$), and that $J_\nu(30)$ are obtained accurately to ten significant digits.

Example 3. The values of $I_\nu(100)$, which are correct at least to ten significant digits for $\nu=0$, $^1/_4$, $^1/_2$, $^3/_4$, $^{39}/_{40}$ and $^{99}/_{100}$, are shown in Table 5. These computed values are obtained by making use of $M=L=73$ (see, Table 2). It is clear that the value of $\Delta\alpha/\alpha$ ($\approx|\delta_\nu|$) becomes larger with increasing ν, and that all values of $\Delta\alpha/\alpha$ are smaller than 2.5×10^{-11}.

8. *Conclusion*

To determine M_E and N_E for given p and x is very convenient for the purpose of obtaining the predetermined accuracy for $J_{\nu+n}(x)$ and $I_{\nu+n}(x)$ yielded by the Miller's algorithm, and making these function tables by a computer.

Using $M=M_E$ and N_E for $p=30$ shown in Tables 1 and 2, we have obtained the mathematical tables of $J_n(x)$, $J_\nu(x)$, $I_n(x)$ and $I_\nu(x)$ for the ranges

Table 4. Computed values of $J_\nu(30)$, using $M=55$, $L=54$.

ν	$F_\nu(30)/\alpha_0$	$\lvert\Delta_\nu\rvert$	$\lvert\delta_\nu\rvert$
0	−8.6367 98358 45234 (−2)	3.48 (−12)	4.03 (−11)
1/4	−1.2460 44300 13096 (−1)	4.29 (−12)	3.44 (−11)
1/2	−1.4392 96533 74639 (−1)	4.24 (−12)	2.95 (−11)
3/4	−1.4176 16910 44798 (−1)	3.57 (−12)	2.52 (−11)
39/40	−1.2190 67728 79151 (−1)	2.67 (−12)	2.19 (−11)

Table 5. Computed values of $I_\nu(100)$, using $M=L=73$.

ν	$G_\nu(100)/\alpha_0$	$\lvert\Delta_\nu\rvert$	$\lvert\delta_\nu\rvert$
0	1.0737 51707 13156 (42)	4.82 (29)	4.48 (−13)
1/4	1.0734 14516 64668 (42)	1.35 (30)	1.26 (−12)
1/2	1.0724 03582 54554 (42)	3.23 (30)	3.01 (−12)
3/4	1.0707 20814 87736 (42)	6.94 (30)	6.48 (−12)
39/40	1.0686 34505 80441 (42)	1.30 (31)	1.21 (−11)
99/100	1.0684 76234 00686 (42)	1.35 (31)	1.26 (−11)

$n=0(1)N_E \qquad \nu=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}\left(\frac{1}{5}\right)\frac{4}{5}$

$x=0.01(0.01)0.1(0.1)1(1)10(10)100.$

The calculations of these functions were carried out on the NEAC-2206 at the Computing Center of Osaka University. All computations were performed with triple-length floating point arithmetic, i.e. thirty-three decimal digits. Accordingly, the computed results are accurate to twenty-nine significant digits at least.

Although no details are shown in this paper, we have also obtained the mathematical tables of $Y_{\nu+n}(x)$ and $K_{\nu+n}(x)$ accurate to twenty-nine places respectively. The tables of $Y_{\nu+n}(x)$ are designed in just the same way as in the case of the tables of $J_{\nu+n}(x)$. The values of $K_{\nu+n}(x)$ are tabulated for the same ranges of $(\nu+n)$ and $x(\leqq 2)$ as those for which the values of $I_{\nu+n}(x)$ are covered.

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