Inversion of Tridiagonal Matrices and the Stability of Tridiagonal Systems of Linear Equations

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1. *Introduction*

It is important to know whether the numerical solution of a linear system is stable or unstable with the increase of its order. We first invert a tridiagonal matrix A analytically, and then discuss the stability and instability of the tridiagonal linear system using the norm $\|A^{-1}\|$ of A^{-1}. For a certain tridiagonal system we find a simple criterion of its stability.

2. *Inversion of Tridiagonal Matrices*

A tridiagonal matrix A of order n is written in the form:

$$A=\begin{pmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix}.$$

For convenience we denote

$$A=[a_i,\ b_i,\ c_i]\,.$$

From this we consider such a homogeneous linear difference equation of second order that

$$a_i x_{i-1}+b_i x_i+c_i x_{i+1}=0, \quad i=1,\ 2, \ldots\ldots, n \tag{1}$$

a_1, c_n; arbitrary non-zero constants

and we denote two fundamental solutions of (1) by

$$\left.\begin{array}{l} \boldsymbol{x}^{(1)}=(0,\ x_1^{(1)}, \ldots\ldots, x_n^{(1)},\ x_{n+1}^{(1)}) \\ \boldsymbol{x}^{(2)}=(x_0^{(2)},\ x_1^{(2)}, \ldots\ldots, x_n^{(2)},\ 0) \end{array}\right\} \tag{2}$$

where both $x_{n+1}^{(1)}$ and $x_0^{(2)}$ are always not zero, if and only if A is non-singular matrix. When $a_i c_i=0$ for a particular value of i, we can reduce nth order tridiagonal system to the lower, so we may consider $a_i c_i \neq 0$ for any i.

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In matrix notation, we rewrite equations (1) and (2)

$$P_i X_i = X_{i-1} \tag{3}$$

where

$$P_i = \begin{pmatrix} -\dfrac{b_i}{a_i} & -\dfrac{c_i}{a_i} \\ 1 & 0 \end{pmatrix}, \quad X_i = \begin{pmatrix} x_i^{(1)} & x_i^{(2)} \\ x_{i+1}^{(1)} & x_{i+1}^{(2)} \end{pmatrix}.$$

Letting α_{ij} be the element of A^{-1}, we write $A^{-1} = (\alpha_{ij})$. Suppose that, using the suitable numbers $y_j^{(1)}$ and $y_j^{(2)}$, we are able to set

$$\alpha_{ij} = \begin{cases} x_i^{(1)} \ y_j^{(2)}, & j \geq i \\ x_i^{(2)} \ y_j^{(1)}, & j \leq i. \end{cases} \tag{4}$$

In order to satisfy the identity $AA^{-1} = I$ we must have a relation

$$\begin{pmatrix} x_i^{(1)} & x_i^{(2)} \\ x_{i+1}^{(1)} & x_{i+1}^{(2)} \end{pmatrix} \begin{pmatrix} y_i^{(2)} \\ -y_i^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ -\dfrac{1}{c_i} \end{pmatrix} \tag{5a}$$

from (3) and (4). Applying (3) to (5a) we can obtain another expressions such that

$$\begin{pmatrix} x_i^{(1)} & x_i^{(2)} \\ x_{i+1}^{(1)} & x_{i+1}^{(2)} \end{pmatrix} \begin{pmatrix} y_{i-1}^{(2)} \\ -y_{i-1}^{(1)} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{c_{i-1}} \\ \dfrac{b_i}{c_{i-1}c_i} \end{pmatrix}, \tag{5b}$$

$$\begin{pmatrix} x_i^{(1)} & x_i^{(2)} \\ x_{i+1}^{(1)} & x_{i+1}^{(2)} \end{pmatrix} \begin{pmatrix} y_{i+1}^{(2)} \\ -y_{i+1}^{(1)} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{a_{i+1}} \\ 0 \end{pmatrix}. \tag{5c}$$

Eliminating the non-singular matrix X_i from the last three relations, we have

$$a_{i+1} \begin{pmatrix} y_{i+1}^{(2)} \\ -y_{i+1}^{(2)} \end{pmatrix} + b_i \begin{pmatrix} y_i^{(2)} \\ -y_i^{(2)} \end{pmatrix} + c_{i-1} \begin{pmatrix} y_{i-1}^{(2)} \\ -y_{i-1}^{(1)} \end{pmatrix} = 0.$$

That is, both numbers $y_i^{(1)}$ and $y_i^{(2)}$ satisfy the second order linear difference equation

$$a_{i+1} y_{i+1} + b_i y_i + c_{i-1} y_{i-1} = 0, \tag{6}$$

a_{n+1}, c_0; arbitrary non-zero constants.

Therefore, in order to know all elements of A^{-1}, we must determine either boundary conditions or initial conditions of (1) and (6).

So that, putting $i=0$, n in equations (5a) and (5c), we get

$$y_0^{(1)} = 0, \qquad y_{n+1}^{(2)} = 0, \tag{7a}$$

$$y_0^{(2)} = -\frac{1}{c_0 x_1^{(1)}}, \quad y_{n+1}^{(1)} = -\frac{1}{a_{n+1} x_n^{(2)}}. \tag{7b}$$

Hence, if we decide two parameters, for example, $x_1^{(1)}$, $x_n^{(2)}$, the rests $y_0^{(2)}$, $y_{n+1}^{(1)}$ are uniquely determined. This means that the boundary conditions of (6) are determined.

Similarly, putting $i=n$, 0 in (5a) and (5c), we have

$$y_1^{(1)}=-\frac{1}{a_1x_0^{(2)}},\quad y_n^{(2)}=-\frac{1}{c_nx_{n+1}^{(1)}}, \tag{8a}$$

$$\frac{y_1^{(2)}}{y_1^{(1)}}=\frac{x_1^{(2)}}{x_1^{(1)}},\qquad \frac{y_n^{(2)}}{y_n^{(1)}}=\frac{x_n^{(2)}}{x_n^{(1)}}. \tag{8b}$$

From (7a) and (8a) we have two independent initial conditions of (6)

$$y_0^{(1)}=0,\quad y_1^{(1)}=-\frac{1}{a_1x_0^{(2)}}$$

and

$$y_{n+1}^{(2)}=0,\quad y_n^{(2)}=-\frac{1}{c_nx_{n+1}^{(1)}}.$$

Thus, when initial conditions of difference equations (1) and (6) are given, their fundamental solutions can be easily computed.

As we stated above, all elements of A^{-1} are determined by substituting $x_i^{(1)}$, $x_i^{(2)}$, $y_j^{(1)}$, $y_j^{(2)}$ into formula (4).

For convenience, let us designate the two fundamental solutions of difference equation (1) anew by

$$\left.\begin{aligned}\boldsymbol{x}^{(1)}&=(0,\ 1,\ x_2^{(1)},\ldots\ldots,\ x_n^{(1)},\ x_{n+1}^{(1)})\\ \boldsymbol{x}^{(2)}&=(x_0^{(2)},\ x_1^{(2)},\ldots\ldots,\ x_{n-1}^{(2)},\ 1,\ 0)\end{aligned}\right\}, \tag{9}$$

and similarly for equation (6)

$$\left.\begin{aligned}\boldsymbol{y}^{(1)}&=(0,\ 1,\ y_2^{(1)},\ldots\ldots,\ y_n^{(1)},\ y_{n+1}^{(1)})\\ \boldsymbol{y}^{(2)}&=(y_0^{(2)},\ y_1^{(2)},\ldots\ldots,\ y_{n-1}^{(2)},\ 1,\ 0)\end{aligned}\right\}. \tag{10}$$

Then, conditions (7b) and (8b) are written such that

$$c_0y_0^{(2)}=c_nx_{n+1}^{(1)},\quad a_{n+1}y_{n+1}^{(1)}=a_1x_0^{(2)} \tag{11}$$

and

$$\frac{y_1^{(2)}}{y_0^{(2)}}=\frac{c_0x_1^{(2)}}{a_1x_0^{(2)}},\quad \frac{y_{n+1}^{(1)}}{y_n^{(1)}}=\frac{c_nx_{n+1}^{(1)}}{a_{n+1}x_n^{(1)}}. \tag{12}$$

Consequently, elements of A^{-1} are written by

$$\alpha_{ij}=\begin{cases}-\dfrac{x_i^{(1)}y_j^{(2)}}{c_nx_{n+1}^{(1)}}=-\dfrac{x_i^{(1)}y_j^{(2)}}{c_0y_0^{(2)}}, & j\geq i\\[2ex] -\dfrac{x_i^{(2)}y_j^{(1)}}{a_1x_0^{(2)}}=-\dfrac{x_i^{(2)}y_j^{(1)}}{a_{n+1}y_{n+1}^{(1)}}, & j\leq i.\end{cases} \tag{13}$$

For example, we now consider the inversion of a particular matrix $A=[a, b, c]$. The characteristic equation (6) is $az^2+bz+c=0$, and let α, β be its two roots.

Using the formula (13) elements of A^{-1} are given by

$$a_{ij}=\begin{cases} -c^{-1}\cdot\dfrac{\beta^{-i}-\alpha^{-i}}{\beta^{-1}-\alpha^{-1}}\cdot\dfrac{\beta^{-n+j-1}-\alpha^{-n+j-1}}{\beta^{-(n+1)}-\alpha^{(n+1)}}, & j\geq i \\ -a^{-1}\cdot\dfrac{\alpha^{n-i+1}-\beta^{n-i+1}}{\alpha^{n+1}-\beta^{n+1}}\cdot\dfrac{\alpha^{j}-\beta^{j}}{\alpha-\beta}, & j\leq i,\ \alpha\neq\beta. \end{cases} \tag{14}$$

When $\alpha=\beta$ we first set $\beta\neq\alpha$ and next approach β to α in above formula. It follows that

$$\alpha_{ij}=\begin{cases} -c^{-1}\cdot i\left(1-\dfrac{j}{n+1}\right)\alpha^{j-i+1}, & j\geq i \\ -a^{-1}\cdot j\left(1-\dfrac{i}{n+1}\right)\alpha^{j-i-1}, & j\leq i. \end{cases} \tag{15}$$

If α and β are conjugate complex, $\alpha=\overline{\beta}=re^{i\theta}$, then

$$\alpha_{ij}=\begin{cases} -c^{-1}\cdot r^{j-i+1}\cdot\dfrac{\sin i\theta\sin(n-j+1)\theta}{\sin\theta\sin(n+1)\theta}, & j\geq i \\ -a^{-1}\cdot r^{j-i-1}\cdot\dfrac{\sin(n-i+1)\theta\sin j\theta}{\sin\theta\sin(n+1)\theta}, & j\leq i, \end{cases} \tag{16}$$

where $(n+1)\theta$ is not an integer multiple of π.

3. *Stability of Tridiagonal Systems of Linear Equations*

Let us consider the constant $\|A\|\ \|A^{-1}\|$ named condition number of a matrix A, from which we can obtain several condition numbers[1]. If we take the matrix norm $\|A\|=\max\lambda_i$ for a positive definite matrix A whose proper numbers are λ's, we get a well-known P-condition number

$$P=\frac{\max\lambda_i}{\min\lambda_i}.$$

If A is a tridiagonal matrix whose elements are all finite for any order n, we can define $\|A\|$ so as to be bounded. Therefore, in this case it may be considered that the condition number $\|A\|\ \|A^{-1}\|$ depends mainly upon $\|A^{-1}\|$.

Now, we define the stability of tridiagonal systems as follows.

When an arbitrary vector $\boldsymbol{r}$ is added to the right hand side of nth order tridiagonal linear system $A\boldsymbol{x}=\boldsymbol{f}$, we designate the change of $\boldsymbol{x}$ by $\Delta\boldsymbol{x}$. Let us consider a suitable positive number $\varepsilon>\|\boldsymbol{r}\|$. If the condition

$$\|\Delta\boldsymbol{x}\|<M\varepsilon,\ 0<M<\infty$$

is satisfied for any n, then the solution is stable.

From the reason mentioned above, it is a subject for us to clarify the condition that $\|A^{-1}\|$ is bounded for the bounded matrix A.

We define here the vector norm by

$$\|\boldsymbol{x}\|=\max_i|x_i|, \tag{17}$$

where x_i is a element of vector $\boldsymbol{x}$.

Then we take the norm of the nth order matrix $A=(a_{ij})$, subordinate to this vector norm, such as

$$\|A\|=\max_i \sum_{j=1}^{n}|a_{ij}|. \tag{18}$$

Next, let us consider the norm of inverse matrix A^{-1} of A, which is non-singular and tridiagonal. Since elements α_{ij} of A^{-1} are given by formula (13), we have

$$\|A^{-1}\|=\max_i\left\{\sum_{j=i}^{i}\left|\frac{x_i^{(2)}y_j^{(1)}}{a_{n+1}y_{n+1}^{(1)}}\right|+\sum_{j=i+1}^{n}\left|\frac{x_i^{(1)}y_j^{(2)}}{c_0y_0^{(2)}}\right|\right\}. \tag{19}$$

Therefore, the sequences $\{x_i^{(1)}\}$, $\{x_i^{(2)}\}$ and $\{y_i^{(1)}\}$, $\{y_i^{(2)}\}$ determine whether $\|A^{-1}\|$ is bounded or not. The former are the fundamental solutions of second order linear difference equation (1) and the latter that of (6).

First, we explain the sufficient condition for $A=[a_i, b_i, c_i]$ to be unstable. From the formula (19) and $x_1^{(1)}=y_1^{(1)}=x_n^{(2)}=y_n^{(2)}=1$ it is obvious that

$$\|A^{-1}\|\geq\max\left\{\frac{1}{|a_{n+1}y_{n+1}^{(1)}|},\ \frac{1}{|c_0y_0^{(2)}|}\right\}. \tag{20}$$

Hence, $\|A^{-1}\|$ is unbounded, if one of $y_{n+1}^{(1)}$ and $y_0^{(2)}$ approaches to zero as $n\to\infty$. Between $y_{n+1}^{(1)}$, $y_0^{(2)}$, $x_{n+1}^{(1)}$ and $x_0^{(2)}$ we should note that there exists a relation (11).

Next, we describe a special matrix $A=[a, b, c]$. Using the relations (14), (15) and (16) we get the following simple result:

$\|A^{-1}\|$ is bounded if and only if

$$|\alpha|>1>|\beta|, \tag{21}$$

where α and β denote two roots of quadratic equation $az^2+bz+c=0$.

We can write this stability condition in another expression such as

$$|a+c|<|b|, \tag{22}$$

where a, b, c are all real.

4. *Remarks*

In case of a special matrix $A=[a, b, c]$, Nagasaka[2] stated that when we solve the tridiagonal system $A\boldsymbol{x}=\boldsymbol{f}$ by the elimination method, stable solution is obtained if the following condition is satisfied

$$\left|\frac{a}{\alpha}\right|\leq 1,\quad \left|\frac{c}{\alpha}\right|\leq 1,$$

where α, β ($|\alpha|\geq|\beta|$) are two roots of a quadratic equation $z^2-bz+ac=0$. We can easily find that this stability condition is the same to ours (21), excluding the equalities.

For a symmetric matrix $A=[a, b, a]$, Evans and Forrington[3] showed a condition $b^2\geq 4a^2$ which is clearly a special case of $|b|\geq|a+c|$, $a=c$.

Our fundamental view in respect of a tridiagonal matrix may be extended to a more general band matrix. A detailed description and examples will be seen in Technology Reports of the Osaka University, Vol. 16 (1966).

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