An Efficient Algorithm for Chebyshev Expansion

TATSUO TORII* and SABURO MAKINOUCHI*

1. *Introduction*

We expand a given function $f(x)$, "well-behaved" over the range $-1 \leq x \leq 1$, by a series of Chebyshev polynomials:

$$f(x)=\sum_{k=0}^{\infty}{}' a_k T_k(x), \tag{1 a}$$

where

$$a_k=\frac{2}{\pi}\int_0^{\pi}(f \cos\theta)\cos k\theta d\theta, \quad T_k(x)\equiv \cos(k\cos^{-1}x). \tag{1 b}$$

A prime indicates that the first term of the series needs to be halved.

Let us consider to evaluate all the coefficients $\{a_k\}$ by numerical quadrature with preassigned accuracy and to reduce the number of operations.

The following two summation rules are known.

$$\frac{2}{n}\sum_{k=1}^{n} T_r(x_k)T_s(x_k)=\begin{cases} 0, & s=2mn\pm r,\ r=n \text{ and } s\neq 2mn\pm r \\ (-1)^m, & s=2mn\pm r,\ r\neq 0,\ n \\ (-1)^m\times 2, & s=2mn\pm r,\ r=0 \end{cases} \tag{2}$$

$$r=0, 1, \cdots\cdots, n;\ m=0, 1, 2, \cdots\cdots$$

$$x_k=\cos((2k-1)/2n)\pi$$

$$\frac{2}{n}\sum_{k=0}^{n}{}'' T_r(x_k')T_s(x_k')=\begin{cases} 0, & s\neq 2mn\pm r \\ 1, & s=2mn\pm r,\ r\neq 0, n \\ 2, & s=2mn\pm r,\ r=0, n \end{cases} \tag{3}$$

$$r=0, 1, \cdots\cdots, n;\ m=0, 1, 2, \cdots\cdots$$

$$x_k'=\cos(k/n)\pi.$$

A double prime in the above equality means that the first and the last terms are halved.

Following the summation rule (2) and (3), we have classical Chebyshev expansion

$$f(x)\simeq\sum_{r=0}^{n-1}{}' b_r T_r(x), \tag{4 a}$$

$$b_r=\frac{2}{n}\sum_{k=1}^{n} T_r(x_k)f(x_k), \tag{4 b}$$

and the other Chebyshev expansion

$$f(x)\simeq\sum_{r=0}^{n}{}'' c_r T_r(x), \tag{5 a}$$

This paper first appeared in Japanese in Joho Shori (the Journal of the Information Processing Society of Japan), Vol. 9, No. 3 (1968), pp. 143-147.

* Faculty of Engineering, Osaka University.

$$c_r=\frac{2}{n}\sum_{k=0}^{n}{}'' T_r(x_k')f(x_k') \tag{5 b}$$

respectively.

It is easily seen that formulas (4 b) and (5 b) are the approximations of integral (1 b) by midpoint rule and by trapezoidal rule, in respect of argument θ. Each coefficient b_r and c_r will be abbreviately called midpoint coefficient and trapezoidal coefficient, respectively.

On the errors of coefficients b_r and c_r, following relations

$$b_r-a_r=-(a_{2n-r}+a_{2n+r})+(a_{4n-r}+a_{4n+r})-\cdots\cdots, \tag{6 a}$$

$$c_r-a_r=(a_{2n-r}+a_{2n+r})+(a_{4n-r}+a_{4n+r})+\cdots\cdots \tag{6 b}$$

are known from the two summation rules mentioned above.

Conbining these two relations (6 a) and (6 b), the more accurate coefficients have been found as follows [1]:

$$\frac{c_r+b_r}{2}-a_r=(a_{4n-r}+a_{4n+r})+\cdots\cdots,$$

$$\frac{c_r-b_r}{2}-a_{2n-r}=a_{2n+r}+a_{6n-r}+\cdots\cdots.$$

On the other hand, substituting $2n$ into n in (6 b), we find that

$$c_r'=a_r+(a_{4n-r}+a_{4n+r})+\cdots\cdots,$$

$$c_{2n-r}'=a_{2n-r}+a_{2n+r}+a_{6n-r}+\cdots\cdots.$$

These trapezoidal coefficients $\{c_r'\}$ determine a truncated Chebyshev series of degree $2n$.

Therefore, we have simple relations between $\{b_r\}$, $\{c_r\}$ and $\{c_r'\}$ such that

$$c_r'=(c_r+b_r)/2 \tag{7 a}$$

$$c_{2n-r}'=(c_r-b_r)/2,\quad r=0,1,\cdots\cdots,n-1 \tag{7 b}$$

$$c_n'=c_n/2. \tag{7 c}$$

These relations play an important role in this paper.

2. *Evaluation of the coefficients by midpoint rule*

Taking $x_k'=\cos(k/2n)\pi$, series (4 b) can be rewritten in the form

$$(n/2)b_r=\sum_{k=1}^{n} T_r(x'_{2k-1})f(x'_{2k-1}).$$

Neglecting the detailed description [2], we now consider to evaluate the midpoint coefficients $\{b_r\}$. Let n be an even number. We define

$$F_{2k}\equiv f(x'_{2k-1})+f(-x'_{2k-1}),\quad F_{2k-1}\equiv f(x'_{2k-1})-f(-x'_{2k-1}),$$

$$F^*_{4k}\equiv F_{2k}+F_{n-2k+2},\quad F^*_{4k-2}\equiv F_{2k}-F_{n-2k+2},$$

$$F^*_{4k-1}\equiv x'_{2k-1}F_{2k-1}+x'_{n-2k+1}F_{n-2k+1},\quad F^*_{4k-3}\equiv x'_{2k-1}F_{2k-1}-x'_{n-2k+1}F_{n-2k+1}$$

and

$$\bar{b}_r\equiv(b_{r-1}+b_{r+1})/2,\quad r:\text{ even} \tag{8}$$

Then, the following relations are obtained.

$$(n/2)b_r=T_r(x_1')F_4^*+T_r(x_3')F_3^*+\cdots\cdots+T_r(x_{n/2-1})F_n^*,$$
$$r=0,4,8\cdots\cdots n-4 \tag{9}$$

$$(n/2)b_r = T_r(x_1')F_2^* + T_r(x_3')F_6^* + \cdots\cdots + T_r(x'_{n/2-1})F_{n-2}^*,$$
$$r=2, 6, 10, \cdots\cdots n-2 \tag{10}$$

$$(n/2)\bar{b}_r = T_r(x_1')F_3^* + T_r(x_3')F_4^* + \cdots\cdots + T_r(x_{n/2-1}')F_{n-1}^*,$$
$$r=0, 4, 8, \cdots\cdots, n-4 \tag{11}$$

$$(n/2)\bar{b}_r = T_r(x_1')F_1^* + T_r(x_3')F_5^* + \cdots\cdots + T_r(x'_{n/2-1})F_{n-3}^*,$$
$$r=2, 6, 10, \cdots\cdots n-2. \tag{12}$$

These four series with $n/4$ terms can be evaluated by a recurrence formula. If b_r and $\bar{b}_r$ are obtained for $r=0, 2, 4, \cdots\cdots, n-2$, the coefficients $b_1, b_3, \cdots\cdots, b_{n-1}$ are easily determined by the following formula

$$b_{r+1} = 2\bar{b}_r - b_{r-1}, \quad r=2, 4, \cdots\cdots, n-2$$
$$b_1 = \bar{b}_0. \tag{13}$$

As we stated above, all the coefficients b_r, $r=0, 1, \cdots\cdots, n-1$ are obtained with $n^2/4$ multiplications approximately, when n is multiple of 4. By slight modification, however, all the midpoint coefficients may be obtained for even number n with the same number of operations.

3. *Successive approximation for functions*

We now assume that nth degree truncated Chebyshev series has been determined for given function $f(x)$ by the use of trapezoidal rule. Using this nth degree polynomial, we show a method to construct $2n$th degree polynomial (both polynomials are truncated Chebyshev series with trapezoidal coefficients). Since interpolating points of these polynomials are distributed symmetrically on the range $-1 \leq x \leq 1$ in respect of origin, we concern only on the points on [0, 1].

Let us rewrite the notations as follows. Let $x_k = \cos(k/n)\pi$ be the interpolating point of nth degree polynomial with the trapezoidal coefficients $\{c_k\}$. Similarly, let $x_k' = \cos(k/2n)\pi$ be the interpolating point of $2n$th degree polynomial with the trapezoidal coefficients $\{c_k'\}$. Then, we have

$$x_{2k}' = x_k, \quad k=0, 1, \cdots\cdots, n/2,$$
$$x_1' = \sqrt{(1+x_2')/2}, \tag{14}$$
$$x_{2k-1}' = (x_{2k}' + x_{2k-2}')/(2x_1'), \quad k=2, 3, \cdots\cdots, n/2.$$

Given $\{x_k'\}$ and $f(x_{2k-1}')$, $n-1$th interpolating polynomial is constructed by midpoint rule as stated in paragraph 2. Thus, midpoint coefficients $\{b_r\}$ are obtained. Combining $\{b_r\}$ and $\{c_r\}$ by formula (7), coefficients $\{c_r'\}$, $r=0, 1, \cdots\cdots, 2n$ are easily evaluated. Repeating these operations by substituting $2n$ to n, $\{c_r'\}$ to $\{c_r\}$ and $\{x_k'\}$ to $\{x_k\}$, we construct a sequence of interpolating polynomials whose degrees are the power of 2. Setting $n=2$, starting values necessary for successive approximation are given as follows:

$$x_0 = 1, \quad x_1 = 1, \tag{15 a}$$

and

$$c_0 = \frac{1}{2}f(1) + f(0) + \frac{1}{2}f(-1),$$

$$c_1=\frac{1}{2}f(1) \qquad -\frac{1}{2}f(-1), \tag{15 b}$$

$$c_2=\frac{1}{2}f(1)-f(0)+\frac{1}{2}f(-1).$$

If the given function $f(x)$ is sufficiently smooth on $[-1, 1]$, we can determine the stopping rule

$$|c_{n-1}|+|c_n|<\varepsilon \tag{16}$$

for preassigned accuracy ε.

To construct a polynomial sequence mentioned above taking the degree $n=2^1, 2^2, \cdots\cdots, 2^m$ successively, we require the following operations. Denoting $N=2^m$, we need $N^2/12$ multiplications approximately. Numbers of calculations of square root and of evaluations of the function are $\log_2 N-1$ and $N+1$, respectively.

4. *Conclution*

We can refine the Clenshow-Curtis method [3], and reduce the number of operations as stated above.

We conjecture that the number of multiplication to calculate the midpoint coefficients for the degree 2^k-1 may be more reduced without increasing the complexity of the algorithm, stated in paragraph 2.

References

[1] Cooper, G. J., The evaluation of the coefficients in a Chebyshev expansion. *The Computer Journal, 10* (1967), 94–100.

[2] Torii, T. and S. Makinouchi, An effiicient algorithm for Chebyshev expansion. *Technol. Repts. Osaka Univ., 18* (1968). (in press)

[3] Clenshaw, C. W. and A. R. Curtis, A method for numerical integration on an automatic computer. *Numerische Mathematik, 2* (1960), 197–205.