A Method of Multi-dimensional Linear Interpolation

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1. *Introduction*

Let $\boldsymbol{G}$ be the set of lattice points in the n-dimensional Euclidean space $\boldsymbol{R}^n$, i. e. those points $\boldsymbol{x}=(x_1, \ldots, x_n)$ whose coordinates x_i are integers. Furthermore, we shall denote by $\overline{\boldsymbol{G}}_0$ the union of those unit hypercubes each of which has its 2^n vertices in $\boldsymbol{G}_0$ (which is a subset of $\boldsymbol{G}$), and by $(\boldsymbol{S}_0, \boldsymbol{S}_1, \ldots)$ a partition of $\overline{\boldsymbol{G}}_0$ into n-dimensional simplexes $\boldsymbol{S}_i$'s each of which is contained in one of these unit hypercubes in $\overline{\boldsymbol{G}}_0$ with its $n+1$ vertices belonging to $\boldsymbol{G}_0$. The problem is to extend a given real-valued function g defined on $\boldsymbol{G}_0$ to a function f defined on $\overline{\boldsymbol{G}}_0$, i. e. to obtain an f defined on $\overline{\boldsymbol{G}}_0$ such that

$$f(\boldsymbol{y})=g(\boldsymbol{y}) \qquad \text{for every} \qquad \boldsymbol{y}\in\boldsymbol{G}_0. \tag{1.1}$$

This process of extension may be called the interpolation of g by f. We shall confine ourselves to such an interpolation that $f(\boldsymbol{x})$ is continuous and that $f(\boldsymbol{x})$ is piecewise linear, i. e. linear in each of the $\boldsymbol{S}_i$'s.

The above specifications determine an f in the one-dimensional case but are not sufficient

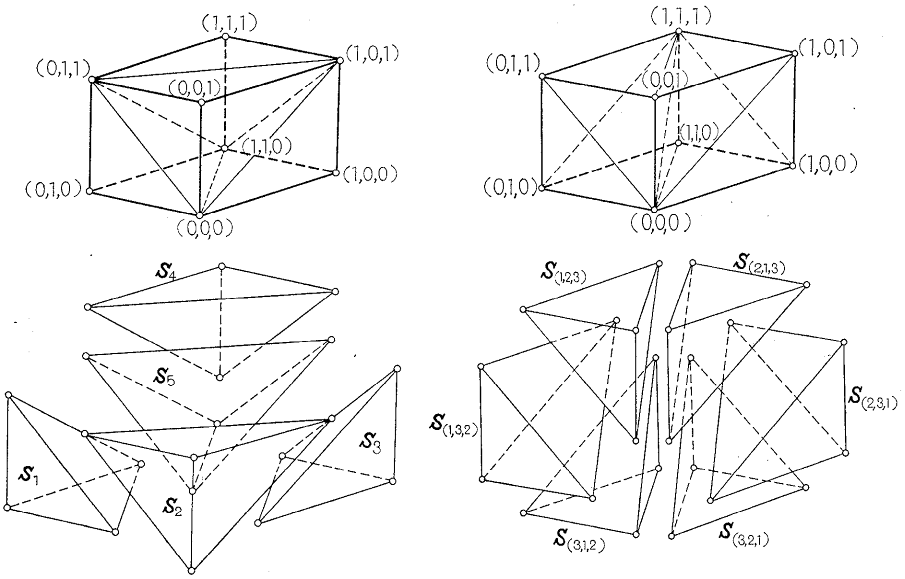

Fig. 1.

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to fix an f in the multi-dimensional case, as is exemplified in Fig. 1 by the two different partitions of a (three-dimensional) cube into (respectively, five and six) simplexes.

Among many possible interpolation methods (in the above sense), a computationally simple and efficient one will be presented in § 2 and its accuracy of approximation will be discussed in § 3.

2. *A Method of Multi-dimensional Piecewise Linear Interpolation*

It suffices to consider the case where $\overline{\boldsymbol{G}}_0$ is a unit hypercube:

$$\boldsymbol{C}=\{\boldsymbol{x}=(x_1, \ldots, x_n) \mid 0 \leqq x_i \leqq 1 (i=1, \ldots, n)\}, \tag{2.1}$$

$\boldsymbol{G}_0$ being its 2^n vertices. We partition the $\boldsymbol{C}$ into $n!$ simplexes $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_{n!}$, where $\boldsymbol{S}_j$'s are put in a one-to-one correspondence with the permutations $(i_1, \ldots, i_n)$'s of $(1, \ldots, n)$ and $\boldsymbol{S}_{(i_1, \ldots, i_n)}$ is defined by

$$\boldsymbol{S}_{(i_1, \ldots, i_n)}=\{\boldsymbol{x}=(x_1, \ldots, x_n) \mid 0 \leqq x_{i_1} \leqq x_{i_2} \leqq \cdots \leqq x_{i_n} \leqq 1\}. \tag{2.2}$$

If we denote by $\boldsymbol{e}_i$ the unit vector whose i-th component is 1 and all the other components are 0 and put

$$\left.\begin{aligned}
&\boldsymbol{d}_0^{(i_1, \ldots, i_n)} \equiv \boldsymbol{d}_0=(1, 1, \ldots, 1), \\
&\boldsymbol{d}_r^{(i_1, \ldots, i_n)}=\sum_{s=r+1}^{n} \boldsymbol{e}_{i_s} \qquad (r=1, \ldots, n-1), \\
&\boldsymbol{d}_n^{(i_1, \ldots, i_n)}=(0, 0, \ldots, 0), \\
&i_{n+1} \equiv n+1, \quad i_0 \equiv 0, \quad x_{n+1} \equiv 1, \quad x_0 \equiv 0,
\end{aligned}\right\} \tag{2.3}$$

then a point

$$\boldsymbol{x}=(x_1, \ldots, x_n)=\sum_{i=1}^{n} x_i \boldsymbol{e}_i \qquad \epsilon \boldsymbol{S}_{(i_1, \ldots, i_n)} \tag{2.4}$$

is expressed as a weighted mean of vertices $\boldsymbol{d}_r^{(i_1, \ldots, i_n)}$'s:

$$\boldsymbol{x}=\sum_{r=0}^{n}(x_{i+1}-x_{i_r}) \boldsymbol{d}_r^{(i_1, \ldots, i_n)}, \tag{2.5}$$

where it is noted that

$$\left.\begin{aligned}
&x_{i_{r+1}}-x_{i_r} \geqq 0 \quad (r=0, \ldots, n), \\
&\sum_{n=0}^{n}(x_{i_{r+1}}-x_{i_n})=x_{n+1}-x_0=1.
\end{aligned}\right\} \tag{2.6}$$

From (2.5) we directly have the interpolation formula:

$$\boxed{\begin{aligned} f(\boldsymbol{x})=\sum_{r=0}^{n}(x_{i_{r+1}}-x_{i_r}) g(\boldsymbol{d}_r^{(i_1, \ldots, i_n)}) \\ \text{when} \quad \boldsymbol{x} \epsilon \boldsymbol{S}_{(i_1, \ldots, i_n)}. \end{aligned}} \tag{2.7}$$

3. *Accuracy of Approximation*

Let us suppose that the given function g (on $\boldsymbol{G}_0$) is the restriction to $\boldsymbol{G}_0$ of a function (to be denoted by the same symbol g) defined on some domain including $\overline{\boldsymbol{G}}_0$. (We shall assume that the g has the continuous second derivatives.) Then, the discrepancy:

$$\varepsilon(\boldsymbol{x}) \equiv f(\boldsymbol{x})-g(\boldsymbol{x}) \tag{3.1}$$

between the original function g and the interpolating function f may be estimated as follows. Without loss in generality we may assume that

$$0\equiv x_0<x_1\leqq x_2\leqq\dots\leqq x_n\leqq x_{n+1}=1 \tag{3.2}$$

since the case where $0=x_1=\dots=x_m$ can be reduced to the problem of lower dimension by deleting $x_1, \dots, x_m$, as is evident from (2.2) and (2.5).

To begin with, we rewrite (2.5) in the recurrence form:

$$\left.\begin{aligned}&\boldsymbol{y}_0=\boldsymbol{y}_1=\boldsymbol{d}_0,\ \ \boldsymbol{y}_{k+1}=\frac{x_k}{x_{k+1}}\boldsymbol{y}_k+\left(1-\frac{x_k}{x_{k+1}}\right)\boldsymbol{d}_k{}^{(1,\dots,n)},\\ &\boldsymbol{x}=\boldsymbol{y}_{n+1},\end{aligned}\right\} \tag{3.3}$$

where

$$\boldsymbol{y}_{k+1}=\sum_{r=0}^{k}(x_{r+1}-x_r)\boldsymbol{d}_r{}^{(1,\dots,n)}/x_{k+1}\qquad (k=0,\ \dots,\ n). \tag{3.4}$$

Here we note that

$$0\leqq x_k/x_{k+1}\leqq 1. \tag{3.5}$$

Putting

$$\varepsilon_k\equiv\varepsilon(\boldsymbol{y}_k)\qquad (\varepsilon_0=\varepsilon_1=0,\ \ \varepsilon_{n+1}=\varepsilon(\boldsymbol{x})) \tag{3.6}$$

and

$$\left.\begin{aligned}&l_k{}^2=|\boldsymbol{d}_k{}^{(1,\dots,n)}-\boldsymbol{y}_k|^2\equiv\sum_{i=1}^{k}\left(\frac{x_i}{x_k}\right)^2,\\ &p_k(l_k\xi)=g(\xi\boldsymbol{y}_k+(1-\xi)\boldsymbol{d}_k{}^{(1,\dots,n)}),\end{aligned}\right\} \tag{3.7}$$

we have

$$\varepsilon_{k+1}=\frac{x_k}{x_{k+1}}\varepsilon_k+\frac{l_k{}^2}{2}\frac{x_k}{x_{k+1}}\left(1-\frac{x_k}{x_{k+1}}\right)p_k''(l_k\xi_k) \tag{3.8}$$

by the help of Rolle's theorem, where $k\geqq 1$ and ξ_k is in (0, 1).

(3.8) is further rewritten as

$$\begin{aligned}|\varepsilon_{k+1}|&\leqq\frac{x_k}{x_{k+1}}|\varepsilon_k|+\frac{1}{2}M\frac{x_k}{x_{k+1}}\left(1-\frac{x_k}{x_{k+1}}\right)\sum_{i=1}^{k}\left(\frac{x_i}{x_k}\right)^2\\ &=\frac{x_k}{x_{k+1}}\left[|\varepsilon_k|+\frac{M}{2}\left(1-\frac{x_k}{x_{k+1}}\right)\sum_{i=1}^{k}\left(\frac{x_i}{x_k}\right)^2\right],\end{aligned} \tag{3.9}$$

where

$$\left.\begin{aligned}&M=\max_{\substack{\boldsymbol{u}\\ \boldsymbol{x}\varepsilon\bar{G}_0}}\frac{1}{|\boldsymbol{u}|^2}\left|\sum_{i,j}u_iu_j\frac{\partial^2 g(x)}{\partial x_i\partial x_j}\right|,\\ &|\boldsymbol{u}|^2=\sum_{i=1}^{n}u_i{}^2.\end{aligned}\right\} \tag{3.10}$$

From (3.9) we have

$$|\varepsilon_{k+1}|\leqq\frac{M}{2}\sum_{j=1}^{k}\frac{x_j}{x_{k+1}}\left(1-\frac{x_j}{x_{k+1}}\right) \tag{3.11}$$

and, in particular,

$$\begin{aligned}|f(\boldsymbol{x})-g(\boldsymbol{x})|&=|\varepsilon(\boldsymbol{x})|\\ &=|\varepsilon_{n+1}|\leqq\frac{M}{2}\sum_{j=1}^{n}x_j(1-x_j)\leqq\frac{n}{8}M.\end{aligned} \tag{3.12}$$

It should be noted that we cannot in general make the error estimate more strict. In fact, the function

$$g(\boldsymbol{x})=\frac{1}{2}\sum_{i=1}^{n}x_i^2 \tag{3.13}$$

and the point

$$x_1=\cdots=x_n=\frac{1}{2} \tag{3.14}$$

yield

$$\varepsilon(\boldsymbol{x})=\frac{n}{8}M \qquad (M=1). \tag{3.15}$$