Minimization of Hazard-Free Switching Networks

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1. *Introduction*

With the development of extremely high speed switching networks, malfunctions due to spurious transient outputs, that is, hazards are becoming increasingly critical. Problems of hazards have been discussed by Huffman(1), McCluskey(2) and others. They have shown existence condition of hazards and detecting procedure of them.

I this paper we try to design the minimal hazard-free switching networks. At first, the definition of hazards and their existence conditions are given along the lines of McCluskey's work(2), and we propose a designing procedure of the minimal hazard-free 2-level AND-OR networks, which is suitable for computer programming. This procedure consists of two steps, i. e., the one is determination of the set of prime implicants satisfying hazard-free conditions and the other is minimal covering. Among them the latter is formalized by the linear programming with 0-1 variables.

Secondly, it is shown that the existence condition of hazards is invariant for the transformation of variables and complementation of variables and functions. Noting this point, for use in synthesis we construct the absolutely minimal hazard-free forms for the representatives of functions of incompletely specified three variables and completely specified four variables. These forms are listed together with invariants in Table 3 and 4. Furthermore we propose the synthesis procedure of the minimal hazard-free networks by the table-look-up scheme using these tables.

2. *Design of Minimal Hazard-Free Combinational Networks*

2. 1. *Hazard-Free Condition****

A network whose 1-sets (0-sets) satisfy the following conditions never contains any static or dynamic hazards:

(1) There are no 1-sets (0-sets) that contain exactly one pair of complementary literals.

(2) For each pair of adjacent input states that both produce a 1-output (0-output), there is at least one 1-set (0-set) that includes both input states of the pair.

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*** The readers could refer the McCluskey's article(2) concerning the definition of hazard and the relating terminology.

2. 2. *1-sets Satisfying Condition (2) of Sec. 3. 1 and the Procedure Obtaining the Prime Implicants Covering Them*

Under the assumption that the input states producing 1-output and don't care input states (unspecified input states) are given using the minimal term representation, we consider the design problem of the minimal hazard-free 2-level AND-OR networks. We can put aside the condition (1) of sec. 2. 1 from our consideration because the minimal term which produces 1-output can be a stable 1-set. Therefore we only consider the condition (2) of sec. 2. 1. That is, our aim in this section becomes to construct 1-sets which satisfy condition (2) and the prime implicants covering them. In the following we will show a systematic method suitable for digital computers.

Any logical function of n variables $f(x_1, \dots x_n)$ can be expressed by sum of products expression as follows:

$$f(x_1, x_2, \dots, x_n)=\sum_{\boldsymbol{e}} f(\boldsymbol{e}) x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}, \tag{1}$$

where $e_i \in \{0, 1, 2\}$, $\boldsymbol{e}=(e_1, e_2, \dots, e_n)$

$$x_i^{e_i}=\begin{cases} x_i' (\text{complement of } x); & e_i=0 \\ x_i & ; e_i=1 \\ 1 & ; e_i=2 \end{cases}$$

and $f(\boldsymbol{e})$ is equal to 1 only when the product $x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$ is contained as a term in the product-sum form.

While a vector $\boldsymbol{e}$ determines a logical product, it also geometrically determines a vertex, edge, plane, cube etc. of n dimensional hypercube.

[Definition 1] The dimension of cube $\boldsymbol{e}$ is the number of its coordinates equal to 2. The cube whose dimension is k will be called k-cube.

[Definition 2] For two cubes $\boldsymbol{a}=(a_1, a_2, \dots, a_n)$ and $\boldsymbol{b}=(b_1, b_2, \dots, b_n)$, $a_i, b_i \in \{0, 1, 2\}$, we define an operation $\boldsymbol{a} \circ \boldsymbol{b}$ as follows:

$$\boldsymbol{a} \circ \boldsymbol{b}=(a_1 \circ b_1, \dots, a_n \circ b_n), \tag{2}$$

where $a_i \circ b_i$ is defined as shown in Table 1.

Table 1. $\circ$ operation.

b_i \ a_i	0	1	2
0	0	y	y
1	y	1	y
2	y	y	2

[Definition 3] The number of y contained in the vector $\boldsymbol{a} \circ \boldsymbol{b}$ defied for two cubes $\boldsymbol{a}$ and $\boldsymbol{b}$ will be indicated by $\#_y(\boldsymbol{a} \circ \boldsymbol{b})$. If $\#_y(\boldsymbol{a} \circ \boldsymbol{b})=1$, a cube $\boldsymbol{a} * \boldsymbol{b}$ will be defined by changing y of $\boldsymbol{a} \circ \boldsymbol{b}$ to 2. If $\#_y(\boldsymbol{a} \circ \boldsymbol{b}) \geqq 2$, cube $\boldsymbol{a} * \boldsymbol{b}$ does not exist.

[Definition 4] Cube $\boldsymbol{a}$ is said to include $\boldsymbol{b}$ when the condition of either $a_i=b_i$ or $a_i=2$ whenever $a_i \neq b_i$ is satisfied.

We note from the above definitions that cube $\boldsymbol{a} * \boldsymbol{b}$ is the one that includes both of $\boldsymbol{a}$

and $\boldsymbol{b}$ and whose dimension is greater than them by 1.

Therefore we can get all the desired 1-sets and prime implicants according to the following procedures A and B.

Procedure A *Procedure of obtaining 1-sets which satisfy condition* (2) *of sec.* 2. 1.

(1) Represent the given input states which produce 1-output by the vectors $\boldsymbol{e}$ used in Eq. (1). The vectors obtained are 0-cubes.

(2) Find all 1-cubes $\boldsymbol{a}*\boldsymbol{b}$ for all pairs of cubes $\boldsymbol{a}$, $\boldsymbol{b}$. They are the 1-sets which cover the adjacent input states.

(3) List up all 1-cubes obtained in (2) and 0-cubes which are not included by any of them.

Procedure B *Procedure of obtaining prime implicants*

(1) Represent the given input states which produce 1-output and don't care input states by vector $\boldsymbol{e}$ used in Eq. (1).

(2) Begin with $k=0$ and find $(k+1)$-cubes by doing $*$ operation for all pairs of k-cubes. Proceed this procedure till the number of $(k+1)$-cubes newly obtained becomes zero.

(3) List up all k-cubes which are not included by any of $(k+1)$-cubes.

2. 3. *Procedure for Designing Minimal Hazard-Free Networks*

To construct a minimal hazard-free network can be reduced to the minimal covering problem finding the minimal set of the prime implicants in the set of the prime implicants obtained by the procedure B, which covers 0-cubes and 1-cubes obtained by the procedure A. Furthermore this minimal covering problem can be formalized by 0-1 variables linear programming.

2. 3. 1. *Formulation by the Linear Programming*

Let the 0-cubes and 1-cubes obtained by the procedure A be $c_1, c_2, \ldots, c_m$ and let the prime implicants cubes obtained by the procedure B be $r_1, r_2, \ldots, r_n$. Next define the constants $a_{ij} (i=1, \ldots, m;\ j=1, \ldots, n)$ as

$$a_{ij}=\begin{cases}1, & \text{if } r_j \text{ includes } c_i,\\ 0, & \text{otherwise,}\end{cases}$$

and also the 2-valued variables $z_j (j=1, \ldots, n)$ as

$$z_j=\begin{cases}1, & \text{if } r_j \text{ is contained in the desired set of the prime implicants,}\\ 0, & \text{otherwise,}\end{cases}$$

then our problem can be formulated as follows. That is,

Procedure C Under the constraints

$$\sum_{j=1}^{n} a_{ij} z_j \geqq 1 \quad (i=1, \ldots, m) \tag{3}$$

find $z_j (=0 \text{ or } 1,\ j=1, \ldots, n)$ minimizing the objective functions,

$$(1) \quad y=1+\sum_{j=1}^{n} z_j \quad \text{(total number of AND and OR gates)} \tag{4}$$

or

$$(2) \quad y=\sum_{j=1}^{n} w_j z_j+\sum_{j=1}^{n} z_j \quad \text{(total number of diodes),} \tag{5}$$

where $w_j = N - d_j$, d_j is the dimension of cube r_j and N is the number of input variables.

2. 3. 2. *Reduction of Number of the Constraints and Variables*

Many algorithms of the integer linear programming have been proposed and, for examples, Gomory's[3] and Balas'[4] methods can be employed to execute the procedure C. However, in any execution it is of course desirable to decrease the number of constraints and variables. For this purpose we utilize the following three properties.

(Property 1) If $a_{ij_1}=1$ and $a_{ij}=0(j \neq j_1)$ then $z_{j_1}=1$. Therefore, the variable z_{j_1} and the constraints such that $a_{kj_1}=1$ can be put aside from our consideration.

(Property 2) If for the i_1-th and i_2-th constraints, $a_{i_2j}=1$ whenever $a_{i_1j}=1$, the i_2-th constraint can be omitted.

(Property 3) If for the coefficients of z_{j_1} and z_{j_2}, $a_{ij_2}=1$ whenever $a_{ij_1}=1$, z_{j_1} is equal to 0.

These properties (1), (2) and (3) respectively correspond to (1) essential term, (2) column dominance, and (3) row dominance found in the prime implicant table which is obtained from the $m \times n$ matrix (a_{ij}) by replacing 1-elements by $\checkmark$ symbols and removing 0-elements.

3. *The Same Class Transformation and Number of Classes of Functions*

3. 1. *The Same Class Transformation and Hazard*

The transformation consisting of (1) complementation of variable, (2) permutation of variables and (3) complementation of function will be called the same class transformation of Boolean function. The relation among the functions which are convertible to each other by finite use of operations of the same class transformation is indicated by R.

This relation R is an equivalence relation on the family of Boolean functions and thus the functions are resolved into equivalence classes. The functions which are involved in a same class will be called the same class function.

An incompletely specified n variables Boolean function is a many to one mapping from $\{0, 1\}^n$ (n order direct product of the space $\{0, 1\}$) to the space $\{0, 1, d\}$. In other words, the function assigns one of 0, 1 and d to each of 2^n vertices in n dimensional hypercube.

Considering the fact that permutation and negation of variables, which involved in the same class transformation, are irrelevant to the Hamming distances between vertices, we can say that if two figures whose vertices are assigned 0, 1 and d (a part of n dimensional hypercube) are congruent, they correspond to the same class functions.

The same class transformation is nothing but a combination of relabeling transformations of variables and transformations between 0 and 1. Permutation of variables changes the labels of elements in 1-sets (0-sets), while complementation of variables yields complementation of the corresponding elements in 1-sets (0-sets) and complementation of function makes the elements of 1-set (0-set) be complemented and become the element of 0-set (1-set). Therefore, neither permutation nor complementation evidently produce a new hazard. Furthermore, from the fact that 1-sets and 0-sets have mutually dual relations it is also clear that any new hazard is not yielded even by complementation of function.

Thus we get the following assertion concerning the effect of the same class transformation on hazard.

[Assertion 1] There is no possibility of any hazard in any network which is derived from a hazard-free network by using the same class transformation.

3. 2. *Enumeration of Incompletely Specified Boolean Functions*

It is natural to ask for the number of classes of incompletely specified Boolean functions in connection with their design algorithm. Many results and techniques(5) concerning the problem for completely specified functions have been already obtained by using the group theory. The same techniques can be extended to the case of incompletely specified functions, yielding more general results as is expected. The latter results include the former as the results of the special case.

Numbers of classes of incompletely and completely specified functions are tabulated in Table 2 for several values of n.

Table 2. Number of classes.

n	completely specified function*			incomplely specified function		
	total	complement, permutation of variable	containing complement of function	total	complement, permutation of variable	containing complement of function
2	16	6	4	81	21	13
3	256	22	14	6,561	267	155
4	65,536	402	238	43,046,721	132,102	about 60,000
5	4,294,967,296	1,228,158	about 600,000	—	—	—

* the results for the completely specified function are taken from [5].

3. 3. *Invariant of Incompletely Specified Function and Representative Function*

The invariants which characterize the class of functions will be computed in this section. The weight of a function f will be denoted by a pair of $w_1(f)$ and $w_d(f)$, the former represents the number of 1-vertices of a function and the latter stands for the sum of numbers of 1-vertices and d-vertices of the function.

Given a function f of n variables, its invariants are computed as follows:

[Algorithm of computing the invariants]

(1) Compute the sequence of the weights

$$\{w_d(f),\ w_d(f\oplus x_1), \dots,\ w_d(f\oplus x_n),\ w_d(f\oplus x_1\oplus x_2), \dots,$$
$$w_d(f\oplus x_{n-1}\oplus x_n), \dots,\ w_d(f\oplus x_1\oplus \dots \oplus x_n),\ w_1(f),$$
$$w_1(f\oplus x_1), \dots,\ w_1(f\oplus x_1\oplus \dots \oplus x_n)\},$$

where $\oplus$ denotes the ring sum (exclusive or).

(2) If $w_1(f)+w_d(f)>2^{n-1}$, convert 0-vertices into 1-vertices and vice versa, where d-vertices are not changed, and compute again the sequence of weights. This procedure corresponds to the complementation of the function.

(3) If $w_d(f \oplus x_i) > 2^{n-1}$, complement all those weights which involve x_i, *i. e.*, w goes to $2^n - w$. When $w_d(f \oplus x_i) = 2^{n-1}$, if complementation of all the weights involving x_i makes the weight sequence smaller than ever in lexicographic order, do this operation. This procedure corresponds to complementation of variables.

(4) Permute $w_d(f \oplus x_i)$ and $w_d(f \oplus x_j)$ so that they are in ascending order. The permutation of the relevant variables must be consistently applied to all the other weights that involve these variables.

If some of the first-order weights $w_d(f \oplus x_i)$ are the same, try all permutations of these weights to obtain the weight sequence which is smallest in lexicographic order.

This procedure corresponds to the permutation of variables.

(5) After the smallest sequence is obtained concerning w_d, apply the procedure (3) and (4) to the sequence of w_1 as far as w_d is unchanged.

The final sequence obtained in this fashion is the smallest sequence of weights and will be called the invariant of a function f. A function f whose sequence of weights is the smallest sequence is said to be of the normal form. We will choose the functions of normal form as the representative functions.

4. *Representative Functions and Synthesis of the Minimal Hazard-Free Networks Using Invariants*

4. 1. *Synthesis Procedure*

Given a table in which the minimal hazard-free networks for the representative functions of n variables and their invariants are tabulated, the minimal hazard-free network of any function f of n variables can be obtained as follows:

(1) Compute the sequence of weights of a given function f.

(2) Compute the invariants of f using the transformation described in sec. 3. 3, and register them sequentially.

(3) Find the function with the same invariants in the table.

(4) Realize f by applying the inverse of the transformations registered in the step (2) to the network given in the table.

4. 2. *The Minimal Hazard-Free Networks for the Representative Functions*

Choosing the functions of normal forms as the representatives we calculated their invariants and constructed the minimal hazard-free network by use of the techniques described in secs. 2, 3 and 4. This computation was done by NEAC 2203 computer. The results are shown in Tables 3 and 4. That is, all the representatives of incompletely specified 3 variables functions and of completely specified 4 variables functions are tabulated in Tables 3 and 4, respectively.

In the case of completely specified 4 variables functions, the results were obtained without using the linear programming. It took about 2 hours to obtain the invariants and the minimal networks of 238 functions of 4 variables because of the low performance rate of the computer used.

Table 3. Invariants for incompletely specified 3 variables functions and their minimal hazard-free forms.

No.	Invariant		1-vertex	d-vertex	Function
1	04444444	04444444	-	-	0
2	13335553	04444444	-	7	0
3	22246444	04444444	-	67	0
4	22444426	04444444	-	56	0
5	24442264	04444444	-	34	0
6	31335535	04444444	-	567	0
7	33333373	04444444	-	347	0
8	33333337	04444444	-	356	0
9	40444444	04444444	-	4567	0
10	44440444	04444444	-	2345	0
11	44444440	04444444	-	1247	0
12	42224446	04444444	-	3567	0
13	42244264	04444444	-	3467	0
14	42442246	04444444	-	3456	0
15	53335557	04444444	-	03567	0
16	53331553	04444444	-	23457	0
17	51333355	04444444	-	34567	0
18	64442224	04444444	-	123456	0
19	62442442	04444444	-	124567	0
20	62242444	04444444	-	234567	0
21	73333335	04444444	-	1234567	0
22	84444444	04444444	-	01234567	0
23	13335553	13335553	7	-	XYZ
24	22246444	13335553	7	6	XY
25	22444426	13355335	6	5	XYZ※
26	24442264	13553353	4	3	XY※Z※
27	31335535	13355335	6	57	XY
28	31335535	13335553	7	56	XY,XZ
29	33333373	13335553	7	34	YZ
30	33333373	13553353	4	37	XY※Z※
31	33333373	15333355	3	47	YZ
32	33333337	13355335	6	35	XYZ※
33	40444444	13335553	7	456	X
34	44440444	13533535	5	234	XY※
35	44444440	13335553	7	124	XYZ
36	42224446	13355335	6	357	XY
37	42224446	13335553	7	356	XY,YZ,XZ
38	42244264	13335553	7	346	XY,YZ
39	42244264	13553353	4	367	XZ※
40	42442244	13355335	6	345	XZ※
41	42442244	15333355	3	456	X※YZ
42	42442244	13553353	4	356	XY※
43	53335557	13335553	7	0356	XY,YZ,XZ
44	53335557	13355335	6	0357	XY
45	53335557	15555555	0	3567	X※Y※Z※
46	53331553	13335553	7	2345	YZ,XZ
47	53331553	13533535	5	2347	XY※,XZ
48	53331553	13553353	4	2357	XY※
49	51333355	13335553	7	3456	X
50	51333355	13355335	6	3457	X
51	51333355	13553353	4	3567	X
52	51333355	15333355	3	4567	YZ
53	64442224	13355335	6	12345	YZ※,XZ※
54	62444422	13335553	7	12456	X,Y
55	62444422	13355335	6	12457	X
56	62444422	15353533	2	14567	YZ※
57	62242444	13335553	7	23456	X,Y
58	62244444	13553353	5	23467	X
59	73333333	13335553	7	123456	X,Y,Z
60	73333333	13355335	6	123457	X,Y
61	73333333	13553355	4	123567	X
62	22246444	22246444	67	-	XY
63	22444426	22444426	56	-	XYZ※+XY※Z
64	24442264	24442264	34	-	XY
65	31335535	22246444	67	5	XY
66	31335535	22444426	56	7	XY+XZ
67	33333373	24442264	34	7	YZ+XY※Z※
68	33333373	24224464	37	4	YZ
69	33333373	22444462	47	3	YZ+XY※Z※
70	33333337	22444426	56	3	XY※Z+XYZ※
71	40444444	22246444	67	45	X
72	40444444	22444426	56	47	X
73	44440444	26242444	23	45	X※Y
74	44440444	24442264	34	25	X※Y+XY※
75	44440444	24422446	35	24	X※Y+XY※
76	44444440	22444422	47	12	XYZ+XY※Z※
77	42224446	22246444	67	35	XY
78	42224446	22444426	56	37	XY+XZ
79	42244264	22422444	37	46	YZ
80	42244264	24442264	34	67	YZ+XZ※
81	42244264	22246444	67	34	XY
82	42244264	22444462	47	36	YZ+XZ※,XY+XZ※
83	42442246	24244246	36	45	X※YZ+XZ※
84	42442246	24442264	34	56	X※YZ+XZ※,X※YZ+XY※
85	42442246	22464244	46	35	XZ※
86	42442246	22444426	56	34	XY※+XZ※
87	51333355	22444426	56	347	X
88	51333355	24244246	36	457	X+YZ
89	51333355	24224464	37	456	YZ
90	51333355	22246444	67	345	X
91	51333355	22444462	47	356	X
92	51333355	22464244	46	357	X
93	51333355	24442264	34	567	X+YZ
94	53331553	22424644	57	234	XZ
95	53331553	22444462	47	235	XZ+XY※,XY※+YZ
96	53331553	24422446	35	247	X※Y+XY※,X※Y+XZ,YZ+XZ,XY※+YZ
97	53331553	24442264	34	257	X※Y+XY※,XZ+XY※
98	53331553	22642444	45	237	XY※
99	53331553	24462442	24	357	X※Y+XY※
100	53335557	24466446	06	357	X※Y※Z※+XY
101	53335557	22444426	56	037	XY+XZ
102	53335557	24446664	07	356	X※Y※Z※+YZ,X※Y※Z※+XY,X※Y※Z※+XZ
103	53335557	22246444	67	035	XY
104	64442224	22444426	56	1234	XZ※+XY※,YZ※+XY※
105	64442224	22464244	46	1235	XZ※
106	64442224	24442264	34	1256	X※Y+XY※,X※Y+XZ※,X※Z+XZ※,X※Z+XY※
107	62444422	26444422	12	4567	YZ※+Y※Z
108	62444422	24244642	27	1456	YZ※+X
109	62444422	24264424	26	1457	YZ※
110	62444422	24442624	25	1467	YZ※+X
111	62444422	22246444	67	1245	X
112	62444422	22444426	56	1247	X
113	62444422	22444462	47	1256	X
114	62242444	22642444	45	2367	X
115	62242444	24422442	35	2467	X+Y
116	62242444	24442264	34	2567	X+Y
117	62242444	22424644	57	2346	X
118	62242444	22444426	56	2347	X
119	62242444	22246444	67	2345	X
120	31335535	31335535	567	-	XY+XZ
121	33333373	33333373	347	-	XY※Z※+YZ
122	33333337	33333337	356	-	X※YZ+XYZ※+XY※Z
123	40444444	31335535	567	4	X
124	44440444	33551533	245	3	X※Y+XY※
125	44444440	33353551	247	1	X※YZ※+XY※Z※+XYZ
126	42224446	31335535	567	3	XY+XZ
127	42224446	33333337	356	7	YZ+XY+XZ
128	42244264	33333373	347	6	YZ+XZ※
129	42244264	31355353	467	3	XY+XZ※
130	42442246	33353155	346	5	X※YZ+XZ※
131	42442246	31553335	456	3	XY※+XZ※
132	42442246	33333337	356	4	X※YZ+XZ※+XY※
133	51333355	33333373	347	56	X+YZ
134	51333355	31355353	367	45	X+YZ
135	51333355	31553335	456	37	X
136	51333355	33353155	346	57	X+YZ
137	51333355	33333337	356	47	X+YZ
138	51333355	33135355	367	45	X+YZ
139	51333355	31335535	567	34	X
140	53331553	33551533	245	37	X※Y+XY※
141	53331553	33531355	345	27	X※Y+XY※,XY※+YZ
142	53331553	33353551	247	35	X※Y+XY※+YZ,X※Y+XY※+XZ
143	53331553	33333373	347	25	YZ+XY※
144	53331553	31533553	457	23	XY※+XZ
145	53331553	33313555	357	24	YZ+XZ
146	53335557	31335535	567	03	XY+XZ
147	53335557	33357555	067	35	XY+X※Y※Z※
148	53335557	33333337	356	07	YZ+XY+XZ
149	53335557	33555537	056	37	X※Y※Z※+XY+XZ
150	40444444	40444444	4567	-	X
151	44440444	44440444	2345	-	XY※+X※Y
152	44444440	44444440	1247	-	X※Y※Z+X※YZ※+XY※Z※+XYZ
153	42224446	42224446	3567	-	XY+YZ+XZ
154	42244264	42244264	3467	-	XY+YZ+XZ※
155	42442246	42442246	3456	-	X※YZ+XZ※+XY※

(Note) 1. 1-vertexes and 0-vertexes are represented as decimal numbers corresponding to binary numbers which represent vertexes.

2. W※ is the complement of W, XYZ is the logcal product of X, Y and Z, X+Y is the logical sum of X and Y.

Table 4. Invariants for 4 variables functions and their minimal hazard-free forms.

No.	Invariant																1-vertex	Decimal representation of function	Function
1	0	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	NULL		0
2	1	7	7	7	7	9	9	9	9	9	9	7	7	7	7	9	15	40	WXYZ
3	2	6	6	6	8	10	10	8	10	8	8	6	8	8	8	8	14 15	41	WXY
4	2	6	6	8	8	10	8	8	8	8	6	8	8	10	10	6	13 14	37 39	WXY※Z+WXYZ※
5	2	6	8	8	8	8	8	8	6	6	10	10	10	6	8	8	11 12	31 36	WX※YZ+WXY※Z※
6	2	8	8	8	8	6	6	6	10	10	10	8	8	8	8	6	7 8	13 27	W※XYZ+WX※Y※Z※
7	3	5	5	7	7	11	9	9	9	9	7	7	7	9	9	7	13 14 15	41 43	WXY+WXZ
8	3	5	7	7	7	9	9	9	7	7	11	9	9	5	7	9	11 12 15	49 36	WYZ+WXY※Z※
9	3	5	7	7	7	9	9	9	7	7	7	9	9	9	11	5	11 13 14	31 37 39	WX※YZ+WXY※Z+WXYZ※
10	3	7	7	7	7	7	7	7	11	11	11	7	7	7	7	7	7 8 15	67 27	XYZ+WX※Y※Z※
11	3	7	7	7	7	7	7	11	11	7	7	7	7	7	7	11	6 9 15	12 28 40	W※XYZ※+WX※Y※Z+WXYZ
12	3	7	7	7	7	7	7	7	7	7	11	11	11	7	7	7	7 11 12	13 31 36	W※XYZ+WX※YZ+WXY※Z※
13	4	4	4	8	8	12	8	8	8	8	8	8	8	8	8	8	12 13 14 15	44	WX
14	4	4	8	8	8	8	8	8	4	8	8	12	8	8	8	8	10 11 12 13	38 32	WXY※+WX※Y
15	4	4	8	8	8	8	8	8	8	8	8	8	8	8	4	12	9 10 12 15	28 30 36 40	WX※Y※Z+WX※YZ※+WXY※Z※+WXYZ
16	4	8	8	8	8	4	4	8	12	8	8	8	8	8	8	8	6 7 8 9	29 14	WX※Y※+W※XY
17	4	8	8	8	8	4	8	8	8	8	4	8	8	8	8	12	5 6 9 10	10 12 28 30	W※XY※Z+W※XYZ※+WX※Y※Z+WX※YZ※
18	4	8	8	8	8	4	8	8	8	8	8	8	8	4	12	8	5 6 8 11	10 12 27 31	W※XY※Z+W※XYZ※+WX※Y※Z※+WX※YZ
19	4	4	6	6	6	10	10	10	8	8	8	8	8	8	10	6	11 13 14 15	41 43 49	WXY+WXZ+WYZ
20	4	4	6	6	8	10	10	8	8	6	10	8	10	6	8	8	11 12 14 15	41 42 49	WXY+WXZ※+WYZ
21	4	4	6	8	8	10	8	8	6	6	8	10	10	8	10	6	11 12 13 14	42 38 31	WXZ※+WXY※+WX※YZ
22	4	6	6	6	6	8	8	8	8	8	12	10	10	6	6	8	7 11 12 15	49 67 36	WYZ+XYZ+WXY※Z※
23	4	6	6	6	6	8	8	8	8	8	8	10	10	10	10	4	7 11 13 14	13 31 37 39	W※XYZ+WX※YZ+WXY※Z+WXYZ※
24	4	6	6	6	8	8	8	6	12	10	10	6	8	8	8	6	7 8 14 15	41 67 27	WXY+XYZ+WX※Y※Z※
25	4	6	6	6	8	8	8	6	8	6	10	10	12	8	8	6	7 11 12 14	42 31 13	WXZ※+WX※YZ+W※XYZ
26	4	6	6	8	8	12	6	6	6	6	8	8	8	10	10	8	3 12 13 14	42 38 4	WXZ※+WXY※+W※X※YZ
27	4	6	6	8	8	8	6	6	10	10	12	8	8	6	6	8	7 8 12 15	45 67	WY※Z※+XYZ
28	4	6	6	8	8	8	6	6	6	10	8	12	8	10	6	8	7 10 12 13	38 30 13	WXY※+WX※YZ※+W※XYZ
29	4	6	6	8	8	8	6	6	10	10	8	8	8	10	10	4	7 8 13 14	13 27 37 39	W※XYZ+WX※Y※Z※+WXY※Z+WXYZ※
30	4	6	8	8	8	6	6	6	8	8	12	10	10	6	8	6	7 8 11 12	45 31 13	WY※Z※+WX※YZ+W※XYZ
31	4	6	8	8	8	6	6	6	8	8	8	10	10	10	4	10	7 9 10 12	13 28 30 36	W※XYZ+WX※Y※Z+WX※YZ※+WXY※Z※
32	5	3	5	7	7	11	9	9	7	7	9	9	9	7	9	7	11 12 13 14 15	44 49	WX+WYZ
33	5	3	7	7	7	9	9	9	5	9	9	11	7	7	7	9	10 11 12 13 15	43 38 49 32	WXZ+WXY※+WYZ+WX※Y
34	5	3	7	7	7	9	9	9	9	9	9	7	7	7	11	5	8 11 13 14 15	41 43 49 27	WXY+WXZ+WYZ+WX※Y※Z※
35	5	5	5	7	7	13	7	7	7	7	9	7	7	9	9	9	3 12 13 14 15	44 4	WX+W※X※YZ
36	5	5	7	7	7	7	7	7	9	9	13	9	9	5	7	7	7 8 11 12 15	49 45 67	WYZ+WY※Z※+XYZ
37	5	5	7	7	7	7	7	7	5	9	9	13	9	9	7	7	7 10 11 12 13	38 32 13	WXY※+WX※Y+W※XYZ
38	5	5	7	7	7	7	7	7	9	9	9	9	9	9	3	11	7 9 10 12 15	67 36 30 28	XYZ+WXY※Z※+WX※YZ※+WX※Y※Z
39	5	5	7	7	7	7	7	7	9	9	9	9	9	9	11	3	7 8 11 13 14	13 27 31 37 39	W※XYZ+WX※Y※Z※+WX※YZ+WXY※Z+WXYZ※
40	5	7	7	7	7	5	5	9	13	9	9	7	7	7	7	9	6 7 8 9 15	29 67 14	WX※Y※+XYZ+W※XY
41	5	7	7	7	7	5	9	9	9	9	13	7	7	7	7	5	4 7 8 11 15	49 67 27 9	WYZ+XYZ+WX※Y※Z※+W※XY※Z※
42	5	7	7	7	7	9	9	9	9	9	13	11	11	7	7	9	0 7 11 12 15	49 67 36 0	WYZ+XYZ+WXY※Z※+W※X※Y※Z※
43	5	7	7	7	7	5	9	9	9	9	9	7	7	3	11	9	5 6 8 11 15	49 27 12 10	WYZ+WX※Y※Z※+W※XYZ※+W※XY※Z
44	5	7	7	7	7	5	9	9	9	9	5	7	7	7	7	13	5 6 9 10 15	10 12 28 30 40	W※XY※Z+W※XYZ※+WX※Y※Z+WX※YZ※+WXYZ
45	5	5	5	5	5	7	9	9	9	9	9	9	11	9	9	5	7 11 13 14 15	41 43 49 67	WXY+WXZ+WYZ+XYZ
46	5	5	5	5	7	9	9	9	7	7	11	9	11	7	7	7	7 11 12 14 15	41 42 49 67	WXY+WXZ※+WYZ+XYZ
47	5	5	5	7	7	9	7	11	11	7	5	7	7	9	9	9	6 9 13 14 15	41 43 46 66	WXY+WXZ+WYZ※+XYZ※
48	5	5	5	7	7	9	7	7	11	11	9	7	7	9	9	5	7 8 13 14 15	41 43 67 27	WXY+WXZ+XYZ+WX※Y※Z※
49	5	5	5	7	7	9	7	7	7	11	9	11	7	9	5	9	7 10 12 13 15	43 38 67 30	WXZ+WXY※+XYZ+WX※YZ※
50	5	5	5	7	7	9	7	7	7	7	9	11	11	9	9	5	7 11 12 13 14	42 38 31 13	WXZ※+WXY※+WX※YZ+W※XYZ
51	5	5	7	7	7	11	11	11	9	9	9	9	9	9	11	7	0 11 13 14 15	41 43 49 0	WXY+WXZ+WYZ+W※X※Y※Z※
52	5	5	7	7	7	7	11	11	5	9	9	9	5	9	7	7	4 10 11 13 15	43 49 32 9	WXZ+WYZ+WX※Y+W※XY※Z※
53	5	5	7	7	7	7	7	11	9	5	9	9	9	9	11	7	6 9 11 13 14	46 34 66	WY※Z+WX※Z+XYZ※
54	5	5	7	7	7	7	7	11	9	5	9	9	9	5	7	11	6 9 11 12 15	49 34 36 12	WYZ+WX※Z+WXY※Z※+W※XYZ※
55	5	5	7	7	7	7	7	11	9	9	9	9	5	5	11	7	6 8 11 13 15	43 49 27 12	WXZ+WYZ+WX※Y※Z※+W※XYZ※
56	5	7	7	7	7	5	5	9	9	5	9	11	11	7	7	9	6 7 9 11 12	34 14 36	WX※Z+W※X※Y+WXY※Z※
57	5	7	7	7	7	5	5	9	9	9	9	11	11	11	11	5	6 7 8 11 13	14 37 31 27	W※XY+WXY※Z+WX※YZ+WX※Y※Z※
58	5	7	7	7	7	9	9	9	9	9	9	11	7	7	11	5	0 7 11 13 14	0 13 31 37 39	W※X※Y※Z※+W※XYZ+WX※YZ+WXY※Z+WXYZ※
59	6	2	6	6	8	10	10	8	6	8	8	10	8	8	8	8	10 11 12 13 14 15	44 50	WX+WY
60	6	2	6	8	8	10	8	8	8	8	6	8	8	10	6	10	9 10 12 13 14 15	44 46 48	WX+WY※Z+WYZ※

61	6	2	8	8	8	8	8	8	6	6	6	10	10	10	8	8	9	10	11	12	13	14		42 38 48 32 46 34	WXZ※+WXY※+WYZ※+WX※Y+WY※Z+WX※Z
62	6	6	6	6	8	6	6	8	14	8	8	6	8	8	8	8	6	7	8	9	14	15		68 29	XY+WX※Y※
63	6	6	6	6	8	6	6	8	6	8	8	14	8	8	8	8	6	7	10	11	12	13		38 32 14	WXY※+WX※Y+W※XY
64	6	6	6	8	8	14	8	8	8	8	6	8	8	6	6	6	1	2	12	13	14	15		44 3 1	WX+W※X※YZ※+W※X※Y※Z
65	6	6	6	8	8	6	8	8	8	8	2	8	8	10	10	10	5	6	9	10	13	14		48 46 66 64	WYZ※+WY※Z+XYZ※+XY※Z
66	6	6	6	8	8	6	8	8	8	8	6	8	8	6	14	6	5	6	8	11	13	14		66 64 31 27	XYZ※+XY※Z+WX※YZ+WX※Y※Z※
67	6	6	6	8	8	6	8	8	8	8	6	8	8	6	6	14	5	6	9	10	12	15		10 12 28 30 36 40	W※XY※Z+W※XYZ※+WX※Y※Z+WX※YZ※+WXY※Z※+WXYZ
68	6	6	8	8	8	8	8	8	2	6	10	10	6	10	8	8	3	4	10	11	12	13		38 32 63 58	WXY※+WX※Y+XY※Z※+X※YZ
69	6	6	8	8	8	8	8	8	6	6	6	6	6	14	8	8	3	4	9	10	13	14		48 46 9 4	WYZ※+WY※Z+W※XY※Z※+W※X※YZ
70	6	8	8	8	8	2	6	6	10	10	6	8	8	8	8	10	5	6	7	8	9	10		33 29 14 16	WX※Z※+WX※Y※+W※XY+W※XZ
71	6	8	8	8	8	6	6	6	6	6	6	8	8	8	8	14	3	5	6	9	10	12		4 10 12 28 30 36	W※X※YZ+W※XY※Z+W※XYZ※+WX※Y※Z+WX※YZ※+WXY※Z※
72	6	4	4	6	6	10	8	8	8	8	10	10	10	8	8	6	7	11	12	13	14	15		44 67 49	WX+XYZ+WYZ
73	6	4	4	6	8	10	8	6	8	10	8	10	8	10	6	8	7	10	12	13	14	15		44 67 48	WX+XYZ+WYZ※
74	6	4	4	8	8	10	6	6	10	10	10	8	8	8	8	6	7	8	12	13	14	15		44 67 45	WX+XYZ+WY※Z※
75	6	4	6	6	6	8	8	12	10	6	6	8	8	8	10	8	6	9	11	13	14	15		52 66 41	WZ+XYZ※+WXY
76	6	4	6	6	6	8	8	8	6	10	10	12	8	8	6	8	7	10	11	12	13	15		43 38 49 32 67	WXZ+WXY※+WYZ+WX※Y+XYZ
77	6	4	6	6	6	8	8	8	10	10	10	8	8	8	10	4	7	8	11	13	14	15		41 43 49 67 27	WXY+WXZ+WYZ+XYZ+WX※Y※Z※
78	6	4	6	6	8	8	12	6	6	8	8	8	10	6	8	10	5	10	11	12	14	15		50 42 10	WY+WXZ※+W※XY※Z
79	6	4	6	6	8	8	8	6	10	8	12	8	10	6	8	6	7	8	11	12	14	15		41 42 49 45 67	WXY+WXZ※+WYZ+WY※Z※+XYZ
80	6	4	6	6	8	8	8	6	6	8	8	12	10	10	8	6	7	10	11	12	13	14		42 38 48 32 13	WXZ※+WXY※+WYZ※+WX※Y+W※XYZ
81	6	4	6	6	8	8	8	6	10	8	8	8	10	10	4	10	7	9	10	12	14	15		41 42 48 67 28	WXY+WXZ※+WYZ※+XYZ+WX※Y※Z
82	6	4	6	8	8	12	6	6	8	8	10	6	6	8	10	8	3	8	12	13	14	15		44 45 4	WX+WY※Z※+W※X※YZ
83	6	4	6	8	8	8	6	6	8	12	10	10	6	8	6	8	7	8	10	12	13	15		43 38 45 33 67	WXZ+WXY※+WY※Z※+WX※Z※+XYZ
84	6	4	6	8	8	8	6	6	8	8	6	10	10	12	6	8	7	9	10	12	13	14		42 38 48 46 13	WXZ※+WXY※+WYZ※+WY※Z+W※XYZ
85	6	4	6	8	8	8	6	6	8	8	10	10	10	8	10	4	7	8	11	12	13	14		42 38 45 31 13	WXZ※+WXY※+WY※Z※+WX※YZ+W※XYZ
86	6	4	8	8	8	6	6	6	6	10	10	12	8	8	8	6	7	8	10	11	12	13		38 32 45 33 13	WXY※+WX※Y+WY※Z※+WX※Z※+W※XYZ
87	6	4	8	8	8	6	6	6	10	10	10	8	8	8	4	10	7	8	9	10	12	15		45 33 29 67	WY※Z※+WX※Z※+WX※Y※+XYZ
88	6	6	6	6	8	6	10	4	10	12	8	6	8	8	8	8	5	7	8	10	14	15		41 48 33 67 16	WXY+WYZ※+WX※Z※+XYZ+W※XZ
89	6	6	6	6	8	6	6	8	10	4	8	10	12	8	8	8	6	7	9	11	12	14		42 34 66 14	WXZ※+WX※Z+XYZ※+W※XY
90	6	6	6	6	8	10	10	4	10	8	8	10	8	8	4	8	1	7	10	12	14	15		41 42 48 67 1	WXY+WXZ※+WYZ※+XYZ+W※X※Y※Z
91	6	6	6	6	8	6	10	8	10	8	4	6	8	8	8	12	5	6	9	10	14	15		41 48 66 28 10	WXY+WYZ※+XYZ※+WX※Y※Z+W※XY※Z
92	6	6	6	6	8	6	10	8	10	8	8	6	8	4	12	8	5	6	8	11	14	15		41 49 66 27 10	WXY+WYZ+XYZ※+WX※Y※Z※+W※XY※Z
93	6	6	6	6	8	6	6	8	10	8	8	10	8	8	4	12	6	7	9	10	12	15		67 14 36 30 28	XYZ+W※XY+WXY※Z※+WX※YZ※+WX※Y※Z
94	6	6	6	8	8	10	4	4	8	8	10	8	8	10	10	6	3	7	8	12	13	14		42 38 45 22	WXZ※+WXY※+WY※Z※+W※YZ
95	6	6	6	8	8	6	4	8	12	8	6	8	8	10	10	6	6	7	8	9	13	14		46 29 66 14	WY※Z+WX※Y※+XYZ※+W※XY
96	6	6	6	8	8	10	4	8	8	4	6	8	8	10	10	10	3	6	9	12	13	14		42 38 46 66 4	WXZ※+WXY※+WY※Z+XYZ※+W※X※YZ
97	6	6	6	8	8	6	4	8	8	8	6	12	8	10	6	10	6	7	9	10	12	13		38 46 14 30	WXY※+WY※Z+W※XY+WX※YZ※
98	6	6	6	8	8	10	4	8	8	8	6	8	12	10	6	6	2	7	9	12	13	14		42 38 46 13 3	WXZ※+WXY※+WY※Z+W※XYZ+W※X※YZ※
99	6	6	6	8	8	10	8	8	8	8	6	4	12	6	6	10	2	5	9	12	14	15		41 42 28 10 3	WXY+WXZ※+WX※Y※Z+W※XY※Z+W※X※YZ※
100	6	6	8	8	8	4	4	8	10	6	10	10	10	6	8	8	6	7	8	9	11	12		34 45 29 14	WX※Z+WY※Z※+WX※Y※+W※XY
101	6	6	8	8	8	4	8	8	6	6	10	10	10	10	4	8	4	7	9	10	11	12		32 34 63 13	WX※Y+WX※Z+XY※Z※+W※XYZ
102	6	6	8	8	8	4	8	8	6	6	6	10	10	6	8	12	5	6	9	10	11	12		32 34 36 12 10	WX※Y+WX※Z+WXY※Z※+W※XYZ※+W※XY※Z
103	6	6	8	8	8	8	8	8	6	6	10	6	6	10	4	12	3	4	9	10	12	15		63 40 30 28 4	XY※Z※+WXYZ+WX※YZ※+WX※Y※Z+W※X※YZ
104	6	8	8	8	8	6	6	6	6	6	10	8	8	4	12	10	3	5	6	8	11	12		45 58 12 10	WY※Z※+X※YZ+W※XYZ※+W※XY※Z
105	6	6	6	6	6	10	10	10	10	10	10	10	10	10	10	6	0	7	11	13	14	15		41 43 49 67 0	WXY+WXZ+WYZ+XYZ+W※X※Y※Z※
106	6	6	6	6	6	6	6	10	10	10	10	10	6	6	10	6	6	7	8	11	13	15		43 49 67 14 27	WXZ+WYZ+XYZ+W※XY+WX※Y※Z※
107	6	6	6	6	6	6	6	10	10	6	10	10	10	6	6	10	6	7	9	11	12	15		49 34 67 14 36	WYZ+WX※Z+XYZ+W※XY+WXY※Z※
108	6	6	6	6	6	6	6	10	10	6	6	10	10	10	10	6	6	7	9	11	13	14		46 34 66 14	WY※Z+WX※Z+XYZ※+W※XY
109	7	1	7	7	7	9	9	9	7	7	7	9	9	9	7	9	9	10	11	12	13	14	15	44 50 52	WX+WY+WZ
110	7	7	7	7	7	7	7	7	7	7	15	7	7	7	7	7	3	4	7	8	11	12	15	76 63 45	YZ+XY※Z※+WY※Z※
111	7	7	7	7	7	7	7	7	7	7	7	7	7	7	15	7	3	5	6	8	11	13	14	66 64 58 27	XYZ※+XY※Z+X※YZ+WX※Y※Z※
112	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	15	3	5	6	9	10	12	15	4 10 12 28 30 36 40	W※X※YZ+W※XY※Z+W※XYZ※+WX※Y※Z+WX※YZ※+WXY※Z※+WXYZ
113	7	3	5	5	7	9	9	7	7	9	9	11	9	9	7	7	7	10	11	12	13	14	15	44 50 67	WX+WY+XYZ
114	7	3	5	7	7	9	7	11	9	5	7	9	9	7	9	9	6	9	11	12	13	14	15	44 52 66	WX+WZ+XYZ※
115	7	3	5	7	7	9	7	7	9	9	11	9	9	7	9	5	7	8	11	12	13	14	15	44 67 45 49	WX+XYZ+WY※Z※+WYZ
116	7	3	5	7	7	9	7	7	9	9	7	9	9	11	5	9	7	9	10	12	13	14	15	44 67 46 48	WX+XYZ+WY※Z+WYZ※
117	7	3	7	7	7	7	11	11	7	7	7	7	7	11	7	7	4	9	10	11	13	14	15	50 52 9	WY+WZ+W※XY※Z※
118	7	3	7	7	7	7	7	11	11	7	7	7	7	7	11	7	6	8	9	11	13	14	15	52 66 29 41	WZ+XYZ※+WX※Y※+WXY
119	7	3	7	7	7	7	7	7	7	11	11	11	7	7	7	7	7	8	10	11	12	13	15	43 38 49 32 45 33 67	WXZ+WXY※+WYZ+WX※Y+WY※Z※+WX※Z※+XYZ
120	7	3	7	7	7	7	7	11	7	7	7	11	7	7	7	11	6	9	10	11	12	13	15	52 32 38 12	WZ+WX※Y+WXY※+W※XYZ※

121	7	3	7	7	7	7	7	7	7	7	7	11	11	11	7	7	7	9	10	11	12	13	14		42 38 48 32 46 34 13	WXZ※+WXY※+WYZ※+WX※Y+WY※Z+WX※Z+W※XYZ
122	7	5	5	5	7	7	7	9	7	9	9	13	7	7	7	9	6	7	10	11	12	13	15		43 38 49 32 67 14	WXZ+WXY※+WYZ+WX※Y+XYZ+W※XY
123	7	5	5	7	7	7	5	9	13	9	7	7	7	9	9	7	6	7	8	9	13	14	15		68 29 46 43	XY+WX※Y※+WY※Z+WXZ
124	7	5	5	7	7	7	9	9	9	9	3	7	7	9	9	11	5	6	9	10	13	14	15		41 43 48 46 66 64	WXY+WXZ+WYZ※+WY※Z+XYZ※+XY※Z
125	7	5	5	7	7	7	9	9	9	9	7	7	7	5	13	7	5	6	8	11	13	14	15		41 43 49 66 64 27	WXY+WXZ+WYZ+XYZ※+XY※Z+WX※Y※Z※
126	7	5	5	7	7	7	9	9	9	9	11	7	7	9	9	3	4	7	8	11	13	14	15		41 43 49 67 27 9	WXY+WXZ+WYZ+XYZ+WX※Y※Z※+W※XY※Z※
127	7	5	7	7	7	5	9	13	7	7	7	9	5	9	7	9	4	6	9	10	11	13	15		52 15 32	WZ+W※XZ※+WX※Y
128	7	5	7	7	7	9	9	13	11	7	7	9	9	9	11	9	0	6	9	11	13	14	15		52 66 41 0	WZ+XYZ※+WXY+W※X※Y※Z※
129	7	5	7	7	7	5	9	9	7	7	3	9	9	9	11	9	5	6	9	10	11	13	14		48 32 46 34 66 64	WYZ※+WX※Y+WY※Z+WX※Z+XYZ※+XY※Z
130	7	5	7	7	7	9	9	9	3	7	11	9	5	9	7	9	3	4	10	11	12	13	15		43 38 49 32 63 58	WXZ+WXY※+WYZ+WX※Y+XY※Z※+X※YZ
131	7	5	7	7	7	9	9	13	7	7	7	5	9	9	7	5	2	4	9	11	13	14	15		52 41 9 3	WZ+WXY+W※XY※Z※+W※X※YZ※
132	7	5	7	7	7	5	5	9	7	7	7	13	9	9	7	9	6	7	9	10	11	12	13		38 32 46 34 14	WXY※+WX※Y+WY※Z+WX※Z+W※XY
133	7	5	7	7	7	5	9	9	7	7	11	9	9	9	3	9	4	7	9	10	11	12	15		49 32 34 67 63	WYZ+WX※Y+WX※Z+XYZ+XY※Z※
134	7	5	7	7	7	9	9	9	7	11	11	13	9	9	7	9	0	7	10	11	12	13	15		43 38 49 32 67 0	WXZ+WXY※+WYZ+WX※Y+XYZ+W※X※Y※Z※
135	7	5	7	7	7	5	9	9	7	7	7	9	9	13	7	5	4	7	9	10	11	13	14		48 32 46 34 13 9	WYZ※+WX※Y+WY※Z+WX※Z+W※XYZ+W※XY※Z※
136	7	5	7	7	7	9	9	9	7	7	7	5	5	13	7	9	3	4	9	10	13	14	15		41 43 48 46 9 4	WXY+WXZ+WYZ※+WY※Z+W※XY※Z※+W※X※YZ
137	7	5	7	7	7	5	9	9	7	7	7	9	9	5	7	13	5	6	9	10	11	12	15		49 32 34 36 12 10	WYZ+WX※Y+WX※Z+WXY※Z※+W※XYZ※+W※XY※Z
138	7	7	7	7	7	3	7	11	11	7	11	7	7	7	7	7	4	6	7	8	9	11	15		49 34 29 67 14 15	WYZ+WX※Z+WX※Y※+XYZ+W※XY+W※XZ※
139	7	7	7	7	7	3	7	7	11	11	7	7	7	7	7	11	5	6	7	8	9	10	15		33 29 67 14 16	WX※Z※+WX※Y※+XYZ+W※XY+W※XZ
140	7	7	7	7	7	3	7	7	7	11	7	11	7	7	11	7	5	6	7	8	10	11	13		32 33 14 64 16	WX※Y+WX※Z※+W※XY+XY※Z+W※XZ
141	7	7	7	7	7	3	7	7	7	7	7	11	11	7	7	11	5	6	7	9	10	11	12		32 34 14 16 36	WX※Y+WX※Z+W※XY+W※XZ+WXY※Z※
142	7	7	7	7	7	7	7	7	7	11	11	7	3	11	7	7	3	4	7	8	10	13	15		43 33 67 22 9	WXZ+WX※Z※+XYZ+W※YZ+W※XY※Z※
143	7	7	7	7	7	7	7	7	7	7	11	7	7	3	11	11	3	5	6	8	11	12	15		49 45 58 12 10	WYZ+WY※Z※+X※YZ+W※XYZ※+W※XY※Z
144	7	7	7	7	7	3	7	7	7	11	7	11	7	7	11	7	2	4	7	8	11	13	15		43 49 67 27 9 3	WXZ+WYZ+XYZ+WX※Y※Z※+W※XY※Z※+W※X※YZ※
145	7	7	7	7	7	7	7	7	7	7	11	7	7	11	11	3	3	4	7	8	11	13	14		58 22 39 37 27 9	X※YZ+W※YZ+WXYZ※+WXY※Z+WX※Y※Z※+W※XY※Z※
146	7	5	5	5	5	7	7	11	11	7	7	9	9	9	9	7	6	7	9	11	13	14	15		52 68	WZ+XY
147	7	5	5	5	7	7	7	9	11	5	9	9	11	7	7	9	6	7	9	10	13	14	15		68 46 48 43	XY+WY※Z+WYZ※+WXZ
148	7	5	5	5	7	7	7	9	11	9	9	9	7	7	11	5	6	7	8	11	13	14	15		68 49 43 27	XY+WYZ+WXZ+WX※Y※Z※
149	7	5	5	7	7	11	5	5	9	9	11	7	7	9	9	7	3	7	8	12	13	14	15		44 22 67 45	WX+W※YZ+XYZ+WY※Z※
150	7	5	5	7	7	11	5	9	9	5	7	7	7	9	9	11	3	6	9	12	13	14	15		44 66 46 4	WX+XYZ※+WY※Z+W※X※YZ
151	7	5	5	7	7	11	5	9	9	9	7	7	11	9	5	7	2	7	9	12	13	14	15		44 67 46 3	WX+XYZ+WY※Z+W※X※YZ※
152	7	5	5	7	7	11	9	9	9	9	11	11	11	9	9	7	0	7	11	12	13	14	15		44 67 49 0	WX+XYZ+WYZ+W※X※Y※Z※
153	7	5	5	7	7	7	5	9	9	5	7	11	11	9	9	7	6	7	9	11	12	13	14		42 38 46 34 66 14	WXZ※+WXY※+WY※Z+WX※Z+XYZ※+W※XY
154	7	5	5	7	7	7	5	9	9	9	11	11	7	5	9	7	6	7	8	11	12	13	15		43 38 49 45 67 14	WXZ+WXY※+WYZ+WY※Z※+XYZ+W※XY
155	7	5	5	7	7	7	5	9	9	9	7	11	7	9	5	11	6	7	9	10	12	13	15		43 38 46 67 14 30	WXZ+WXY※+WY※Z+XYZ+W※XY+WX※YZ※
156	7	5	7	7	7	5	5	9	11	7	11	9	9	5	7	9	6	7	8	9	11	12	15		49 34 45 29 67 14	WYZ+WX※Z+WY※Z※+WX※Y※+XYZ+W※XY
157	7	5	7	7	7	5	5	9	11	7	7	9	9	9	11	5	6	7	8	9	11	13	14		46 34 29 66 14	WY※Z+WX※Z+WX※Y※+XYZ※+W※XY
158	7	5	7	7	7	5	9	9	7	11	11	9	5	9	7	5	4	7	8	10	11	13	15		43 49 32 33 67 9	WXZ+WYZ+WX※Y+WX※Z※+XYZ+W※XY※Z※
159	7	5	7	7	7	9	9	9	11	11	11	9	9	9	11	5	0	7	8	11	13	14	15		41 43 49 67 54	WXY+WXZ+WYZ+XYZ+X※Y※Z※
160	7	5	7	7	7	5	9	9	7	11	7	9	5	5	11	9	5	6	8	10	11	13	15		43 49 32 33 64 12	WXZ+WYZ+WX※Y+WX※Z※+XY※Z+W※XYZ※
161	7	5	7	7	7	9	9	9	7	7	11	5	5	9	11	5	3	4	8	11	13	14	15		41 43 49 58 27 9	WXY+WXZ+WYZ+X※YZ+WX※Y※Z※+W※XY※Z※
162	7	7	7	7	7	7	7	11	11	11	11	11	7	7	11	7	0	6	7	8	11	13	15		43 49 67 14 54	WXZ+WYZ+XYZ+W※XY+X※Y※Z※
163	7	7	7	7	7	7	7	11	11	7	11	11	11	7	7	11	0	6	7	9	11	12	15		49 34 67 14 36 0	WYZ+WX※Z+XYZ+W※XY+WXY※Z※+W※X※Y※Z※
164	7	7	7	7	7	7	7	11	11	7	7	11	11	11	11	7	0	6	7	9	11	13	14		46 34 66 14 0	WY※Z+WX※Z+XYZ※+W※XY+W※X※Y※Z※
165	8	0	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	9	10	11	12	13	14	15	53	W
166	8	8	8	8	8	0	8	8	8	8	8	8	8	8	8	8	4	5	6	7	8	9	10	11	35 17	WX※+W※X
167	8	8	8	8	8	8	8	8	8	8	8	0	8	8	8	8	2	3	4	5	8	9	14	15	41 29 11 5	WXY+WX※Y※+W※XY※+W※X※Y
168	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	0	1	2	4	7	8	11	13	14	1 3 9 13 27 31 37 39	W※X※Y※Z+W※X※YZ※+W※XY※Z※+W※XYZ +WX※Y※Z※+WX※YZ+WXY※Z+WXYZ※
169	8	2	6	6	6	8	8	8	8	8	8	10	10	10	6	8	7	9	10	11	12	13	14	15	44 50 52 67	WX+WY+WZ+XYZ
170	8	2	6	6	8	8	8	6	8	10	10	10	8	8	8	6	7	8	10	11	12	13	14	15	44 50 51 67	WX+WY+WZ※+XYZ
171	8	2	6	8	8	8	6	6	10	10	8	8	8	10	6	8	7	8	9	10	12	13	14	15	44 51 47 67	WX+WZ※+WY※+XYZ
172	8	2	8	8	8	6	6	6	8	8	8	10	10	10	8	6	7	8	9	10	11	12	13	14	51 47 35 13	WZ※+WY※+WX※+W※XYZ
173	8	6	6	6	6	8	8	8	8	8	8	6	6	6	14	8	3	5	6	8	11	13	14	15	41 43 49 66 64 58 27	WXY+WXZ+WYZ+XYZ※+XY※Z+X※YZ+WX※Y※Z※
174	8	6	6	6	8	8	8	2	8	10	10	6	8	8	8	10	3	5	7	8	10	12	14	15	51 22 16 67 41	WZ※+W※YZ+W※XZ+XYZ+WXY
175	8	6	6	6	8	8	8	6	8	6	6	14	8	8	8	6	1	6	7	10	11	12	13	14	42 38 48 32 66 14 1	WXZ※+WXY※+WYZ※+WX※Y+XYZ※+W※XY+W※X※Y※Z
176	8	6	6	6	8	8	8	6	8	6	6	6	8	8	8	14	3	5	6	9	10	12	14	15	41 42 48 66 28 10 4	WXY+WXZ※+WYZ※+XYZ※+WX※Y※Z+W※XY※Z+W※X※YZ
177	8	6	6	8	8	8	2	6	10	6	8	8	8	10	10	8	3	6	7	8	9	12	13	14	47 22 14 66 42	WY※+W※YZ+W※XY+XYZ※+WXZ※
178	8	6	6	8	8	8	6	6	6	6	8	8	8	6	14	8	3	5	6	8	11	12	13	14	42 38 45 66 64 58	WXZ※+WXY※+WY※Z※+XYZ※+XY※Z+X※YZ
179	8	6	8	8	8	2	6	6	8	8	8	10	10	6	8	10	5	6	7	8	9	10	11	12	35 16 14 45	WX※+W※XZ+W※XY+WY※Z※
180	8	6	8	8	8	6	6	6	8	8	8	6	6	14	8	6	3	4	7	8	9	10	13	14	48 46 33 29 22 9	WYZ※+WY※Z+WX※Z※+WX※Y※+W※YZ+W※XY※Z※

181	8	6	8	8	8	6	6	6	8	8	8	6	6	6	8	14	3	5	6	8	9	10	12	15	45	33	29	40	12	10	4		WY※Z※+WX※Z※+WX※Y※+WXYZ+W※XYZ※+W※XY※Z+W※X※YZ
182	8	4	4	4	8	8	8	8	8	8	8	12	8	8	8	8	6	7	10	11	12	13	14	15	44	50	68						WX+WY+XY
183	8	4	4	8	8	8	4	8	12	8	8	8	8	8	8	8	6	7	8	9	12	13	14	15	44	47	68						WX+WY※+XY
184	8	4	4	8	8	8	8	8	8	8	4	8	8	8	8	12	5	6	9	10	12	13	14	15	44	64	66	46	48				WX+XY※Z+XYZ※+WY※Z+WYZ※
185	8	4	4	8	8	8	8	8	8	8	8	8	8	4	12	8	5	6	8	11	12	13	14	15	44	64	66	45	49				WX+XY※Z+XYZ※+WY※Z※+WYZ
186	8	4	8	8	8	4	4	8	8	8	8	12	8	8	8	8	6	7	8	9	10	11	12	13	35	47	14						WX※+WY※+W※XY
187	8	4	8	8	8	4	8	8	8	8	4	8	8	8	12	8	5	6	8	9	10	11	13	14	35	64	66	46	48				WX※+XY※Z+XYZ※+WY※Z+WYZ※
188	8	4	8	8	8	8	8	8	4	4	8	8	8	12	8	8	3	4	9	10	11	12	13	14	42	38	48	32	46	34	63	58	WXZ※+WXY※+WYZ※+WX※Y+WY※Z+WX※Z+XY※Z※+X※YZ
189	8	4	8	8	8	4	8	8	8	8	8	8	8	4	8	12	5	6	8	9	10	11	12	15	35	45	49	12	10				WX※+WY※Z※+WYZ+W※XYZ※+W※XY※Z
190	8	4	8	8	8	8	8	8	8	8	8	4	4	12	8	8	3	4	8	9	10	13	14	15	41	43	48	46	33	29	9	4	WXY+WXZ+WYZ※+WY※Z+WX※Z※+WX※Y※+W※XY※Z※+W※X※YZ
191	8	8	8	8	8	4	4	8	8	4	12	8	8	8	8	8	3	4	6	7	8	9	11	12	34	45	29	14	63	15	58	22	WX※Z+WY※Z※+WX※Y※+W※XY+XY※Z※+W※XZ※+X※YZ+W※YZ
192	8	8	8	8	8	4	4	4	8	8	8	8	8	8	8	12	3	5	6	7	8	9	10	12	45	33	29	14	16	22			WY※Z※+WX※Z※+WX※Y※+W※XY+W※XZ+W※YZ
193	8	8	8	8	8	4	4	8	8	8	8	8	4	12	8	8	3	4	6	7	8	9	10	13	46	33	29	14	15	22			WY※Z+WX※Z※+WX※Y※+W※XY+W※XZ※+W※YZ
194	8	8	8	8	8	4	8	8	8	8	8	4	4	8	8	12	3	4	5	6	8	9	10	15	33	29	15	11	40	4			WX※Z※+WX※Y※+W※XZ※+W※XY※+WXYZ+W※X※YZ
195	8	4	4	6	6	8	6	10	10	6	8	10	10	8	8	8	6	7	9	11	12	13	14	15	44	52	68						WX+WZ+XY
196	8	4	4	6	8	8	6	8	10	8	6	10	8	10	6	10	6	7	9	10	12	13	14	15	44	68	46	48					WX+XY+WY※Z+WYZ※
197	8	6	6	4	6	8	10	4	6	8	10	8	10	8	8	10	3	5	7	10	11	12	14	15	50	76	16	42					WY+YZ+W※XZ+WXZ※
198	8	4	6	6	6	6	6	10	12	8	8	8	8	8	10	6	6	7	8	9	11	13	14	15	52	68	29						WZ+XY+WX※Y※
199	8	6	4	6	6	6	8	8	10	10	8	8	8	6	12	6	5	6	7	8	11	13	14	15	68	70	49	27					XY+XZ+WYZ+WX※Y※Z※
200	8	4	6	6	6	6	6	10	8	8	8	12	8	8	6	10	6	7	9	10	11	12	13	15	52	14	67	32	38				WZ+W※XY+XYZ+WX※Y+WXY※
201	8	4	6	6	8	6	10	4	8	10	10	8	10	6	8	8	5	7	8	10	11	12	14	15	50	51	16	67					WY+WZ※+W※XZ+XYZ
202	8	4	6	6	8	10	10	8	4	6	10	8	6	10	8	8	3	4	10	11	12	13	14	15	44	50	58	63					WX+WY+X※YZ+XY※Z※
203	8	4	6	6	8	6	6	8	12	6	10	8	10	6	8	8	6	7	8	9	11	12	14	15	68	29	45	34	49	42			XY+WX※Y※+WY※Z※+WX※Z+WYZ+WXZ※
204	8	4	6	6	8	10	10	8	8	10	10	12	10	10	8	8	0	7	10	11	12	13	14	15	44	50	67	0					WX+WY+XYZ+W※X※Y※Z※
205	8	4	6	6	8	6	6	8	8	6	6	12	10	10	8	8	6	7	9	10	11	12	13	14	42	38	48	32	46	34	66	14	WXZ※+WXY※+WYZ※+WX※Y+WY※Z+WX※Z+XYZ※+W※XY
206	8	4	6	6	8	6	10	8	8	6	10	8	10	10	4	8	4	7	9	10	11	12	14	15	50	63	67	34	42				WY+XY※Z※+XYZ+WX※Z+WXZ※
207	8	4	6	6	8	6	10	8	8	6	6	8	10	6	8	12	5	6	9	10	11	12	14	15	50	66	34	42	10				WY+XYZ※+WX※Z+WXZ※+W※XY※Z
208	8	4	6	8	8	10	4	8	10	6	8	6	6	8	10	10	3	6	8	9	12	13	14	15	44	47	66	4					WX+WY※+XYZ※+W※X※YZ
209	8	4	6	8	8	6	4	8	10	6	8	10	10	8	10	6	6	7	8	9	11	12	13	14	47	14	66	34	42				WY※+W※XY+XYZ※+WX※Z+WXZ※
210	8	4	6	8	8	10	8	8	6	10	4	6	10	8	6	10	2	5	9	10	12	13	14	15	44	57	64	46	48				WX+X※YZ※+XY※Z+WY※Z+WYZ※
211	8	4	6	8	8	6	8	8	6	6	4	10	10	8	10	10	5	6	9	10	11	12	13	14	42	38	48	32	46	34	66	64	WXZ※+WXY※+WYZ※+WX※Y+WY※Z+WX※Z+XYZ※+XY※Z
212	8	4	6	8	8	10	8	8	6	6	8	6	6	12	6	10	3	4	9	10	12	13	14	15	44	63	46	48	4				WX+XY※Z※+WY※Z+WYZ※+W※X※YZ
213	8	4	6	8	8	6	8	8	6	6	8	10	10	12	6	6	4	7	9	10	11	12	13	14	42	38	48	32	46	34	63	13	WXZ※+WXY※+WYZ※+WX※Y+WY※Z+WX※Z+XY※Z※+W※XYZ
214	8	6	6	6	6	4	8	8	8	8	4	10	10	10	10	8	5	6	7	8	10	11	13	15	70	33	14	32	49	27			XZ+WX※Z※+W※XY+WX※Y+WYZ
215	8	6	6	6	6	8	8	12	12	8	8	10	10	10	10	8	0	6	7	9	11	13	14	15	52	68	0						WZ+XY+W※X※Y※Z※
216	8	6	6	6	6	4	8	8	8	8	4	10	10	10	10	8	5	6	7	9	10	11	13	14	48	32	46	34	66	14	64	16	WYZ※+WX※Y+WY※Z+WX※Z+XYZ※+W※XY+XY※Z+W※XZ
217	8	6	6	6	6	8	8	8	8	8	12	6	6	10	10	4	3	4	7	8	11	13	14	15	76	43	41	27	9				YZ+WXZ+WXY+WX※Y※Z※+W※XY※Z※
218	8	6	6	6	6	4	8	8	8	8	8	10	10	6	6	12	5	6	7	9	10	11	12	15	49	32	34	67	14	16	36		WYZ+WX※Y+WX※Z+XYZ+W※XY+W※XZ+WXY※Z※
219	8	6	6	6	8	4	8	6	12	10	6	6	8	8	8	10	5	6	7	8	9	10	14	15	68	16	29	33	48				XY+W※XZ+WX※Y※+WX※Z※+WYZ※
220	8	6	6	6	8	8	8	6	12	6	6	10	8	8	4	10	1	6	7	9	10	12	14	15	68	55	48	42					XY+X※Y※Z+WYZ※+WXZ※
221	8	6	6	6	8	4	8	6	8	6	6	10	12	8	8	10	5	6	7	9	10	11	12	14	42	48	32	34	66	14	16		WXZ※+WYZ※+WX※Y+WX※Z+XYZ※+W※XY+W※XZ
222	8	6	6	6	8	8	8	6	12	10	6	10	4	8	8	6	1	6	7	8	10	13	14	15	68	33	48	43	1				XY+WX※Z※+WYZ※+WXZ+W※X※Y※Z
223	8	6	6	6	8	4	8	6	8	10	6	10	8	8	12	6	5	6	7	8	10	11	13	14	48	32	33	66	14	64	16		WYZ※+WX※Y+WX※Z※+XYZ※+W※XY+XY※Z+W※XZ
224	8	6	6	6	8	8	8	6	8	6	10	6	8	4	12	10	3	5	6	8	11	12	14	15	41	42	49	45	66	58	10		WXY+WXZ※+WYZ+WY※Z※+XYZ※+X※YZ+W※XY※Z
225	8	6	4	8	10	10	6	8	12	10	10	10	8	8	10	8	0	6	7	8	12	13	14	15	44	68	54	45					WX+XY+X※Y※Z※+WY※Z※
226	8	6	6	8	8	4	6	6	10	10	4	8	8	10	10	8	5	6	7	8	9	10	13	14	48	46	33	29	66	14	64	16	WYZ※+WY※Z+WX※Z※+WX※Y※+XYZ※+W※XY+XY※Z+W※XZ
227	8	6	6	10	6	12	12	8	8	8	8	10	10	6	10	8	0	1	5	11	12	13	14	15	44	2	19	64	49				WX+W※X※Y※+W※Y※Z+XY※Z+WYZ
228	8	6	6	8	8	4	6	6	6	10	8	12	8	6	10	8	5	6	7	8	10	11	12	13	38	32	45	33	14	64	16		WXY※+WX※Y+WY※Z※+WX※Z※+W※XY+XY※Z+W※XZ
229	8	6	6	8	8	8	6	6	6	10	4	8	12	10	6	8	2	5	7	9	10	12	13	14	42	38	48	46	64	16	57		WXZ※+WXY※+WYZ※+WY※Z+XY※Z+W※XZ+X※YZ※
230	8	6	6	8	8	12	6	6	10	10	8	8	8	6	6	4	1	2	7	8	12	13	14	15	44	67	45	3	1				WX+XYZ+WY※Z※+W※X※YZ※+W※X※Y※Z
231	8	6	6	8	8	4	6	6	10	10	8	8	8	6	6	12	5	6	7	8	9	10	12	15	45	33	29	67	14	16			WY※Z※+WX※Z※+WX※Y※+XYZ+W※XY+W※XZ
232	8	6	6	8	8	8	6	6	6	6	12	8	8	10	10	4	3	4	7	8	11	12	13	14	42	38	45	63	58	22			WXZ※+WXY※+WY※Z※+XY※Z※+X※YZ+W※YZ
233	8	6	6	8	8	8	6	6	6	6	4	8	8	10	10	12	3	5	6	9	10	12	13	14	42	38	48	46	66	64	4		WXZ※+WXY※+WYZ※+WY※Z+XYZ※+XY※Z+W※X※YZ
234	8	6	6	8	8	8	6	6	10	10	8	4	12	6	6	8	2	5	7	8	9	12	14	15	67	41	16	42	45	29	3		XYZ+WXY+W※XZ+WXZ※+WY※Z※+WX※Y※+W※X※YZ※
235	8	6	6	8	8	8	6	6	6	10	8	8	4	6	10	12	3	5	6	8	10	12	13	15	43	38	45	33	64	12	4		WXZ+WXY※+WY※Z※+WX※Z※+XY※Z+W※XYZ※+W※X※YZ
236	8	6	8	8	8	6	6	6	4	8	12	10	6	10	8	6	3	4	7	8	10	11	12	13	38	32	45	33	63	58	22		WXY※+WX※Y+WY※Z※+WX※Z※+XY※Z※+X※YZ+W※YZ
237	8	8	6	8	8	6	4	8	10	6	8	6	6	12	10	6	3	4	6	7	8	9	13	14	46	29	66	14	15	22			WY※Z+WX※Y※+XYZ※+W※XY+W※XZ※+W※YZ
238	8	6	8	8	8	6	6	6	4	8	8	10	6	6	12	10	3	5	6	8	10	11	12	13	38	32	45	33	64	58	12		WXY※+WX※Y+WY※Z※+WX※Z※+XY※Z+X※YZ+W※XYZ※

(Note) 1. 1–vertexes amd functions are same as in Table 3.

2. Decimal representation of function is a set of numbers corresponding to ternary representation of each logical product.

5. *Conclusion*

We have led a design procedure of the minimal hazard-free networks of 2-level AND-OR gates using the linear programming techniques. The networks thus obtained are minimal in the sense that total number of AND, OR gates or diodes to be needed are minimum. Next we have shown that the networks obtained by applying the same class transformation to a hazard-free networks do not have any hazard. The existing techniques for calculating the number of classes of functions and determining the representative functions have been extended to the case of incompletely specified functions.

By use of the techniques proposed by us, the minimal hazard-free forms of the representatives of incompletely specified functions of 3 variables and that of completely specified functions of 4 variables together with all their invariants have been obtained and are tabulated in Tables 3 and 4. Finally we have shown the way to use these tables.

We have left a problem to improve the method applicable to the functions of five or more variables. We have also left the problem to find the easy and effective algorithm to solve the 0–1 variables linear programming. But we will have opportunities to attack these problems some other time.

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