On a Numerical Calculation of the Finite Fourier Integrals with the High Frequency

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Abstract

In the machine calculation of the finite Fourier integrals

$$F(\omega)\equiv\int_a^b f(t)e^{-i\omega t}dt$$

by the ordinary approximation formulas of numerical integration such as Simpson's, a difficulty arises for high frequency ω, because the stepsize must be taken smaller than in low frequency in order to secure the accuracy of the method in such a case. It should be noted that the same situation also occurs in applying faster algorithm like as Goertzel's or Cooley-Tukey's. For these values to be calculated must be exactly the same as those by the ordinary approximation.

Filon's method which approximates $f(t)$ by series of parabolic arcs and uses the integration by parts on the series covers this fault to the extent of the truncation error $O(1/\omega^2)$ as $\omega\to\infty$. But as long as such an approximation is used, it seems that improvement on this error bound can not be expected.

Extending the Filon's idea, the author produced a formula which agrees with $F(\omega)$ within the truncation error of $O(1/\omega^{2K+1})$ as $\omega\to\infty$, where K stands for any desiared positive integer. The formula does not make it necessary to compute any derivatives, but it consists in combining values of the analytic continuation of $f(t)$ which are taken at different points in the complex domain.

1. *Introduction*

The finite Fourier integral is defined by the form

$$F(\omega)\equiv\int_a^b f(t)e^{-i\omega t}dt, \tag{1}$$

where $-\infty<a<b<+\infty$, frequency ω is a real number, and $f(t)$ is a real and analytic function of real parameter t on the interval (a, b).

From here to the end of this section, let us put $a=0$, $b=2\pi$, $\omega=\Omega$, where Ω is an integer. Dividing the interval $(0, 2\pi)$ into N equal sized subintervals, and putting $f_n=f(nh)$, where stepsize $h=2\pi/N$, then the step function of Ω

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$$\tilde{F}_1(\Omega)=\frac{2\pi}{N}\sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi}{N}\Omega n} \tag{2}$$

is obtained as an approximation formula of the form (1). Simpson's rule is also written in the form (2) replacing f_n by suitable quantities.

In the equation (2) it is desirable, if possible, to choose the number N so that the function $f(t)$ may not include the frequency greater than or equal to $N/2$. If such a number is not selectable, the formula (2) may have an unignorably large error by the aliasing phenomenon.

For example, if $f(t)$ is taken such as

$$f(t)=-\frac{1}{2}(t-\pi) \qquad (0<t<2\pi),$$

$$f(0)=f(2\pi)=0,$$

then the followings are obtained

$$F(\Omega)=-i\pi/\Omega, \tag{3}$$

$$\tilde{F}_1(\Omega)=-i\frac{\pi h}{2}\cot\frac{1}{2}\Omega h$$

$$(\Omega \neq mN, \text{ where } m \text{ is an integer}). \tag{4}$$

These values are shown in Table 1 in the case of $N=32$. The left side of each

Table 1. Table of values of $\tilde{F}_1(\Omega)$, $F(\Omega)$, $E_1(\Omega)$ for the function $f(t)=-\frac{1}{2}(t-\pi)(0<t<2\pi)$, $f(0)=f(2\pi)=0$

Ω	① $\tilde{F}_1(\Omega)$((2))*		② $F(\Omega)$((3))		③ $E_1(\Omega)$(=①-②)		④ $\tilde{F}_1(\Omega)$((4))	
0	0.28573E-11	0.	0.	0.	0.28573E-11	0.	0.	0.
1	-0.14286E-10	-0.31315E 01	0.	-0.31416E 01	-0.14286E-10	0.10100E-01	0.	-0.31315E 01
2	-0.14286E-11	-0.15506E 01	0.	-0.15708E 01	-0.14286E-11	0.20238E-01	0.	-0.15506E 01
3	0.30001E-10	-0.10167E 01	0.	-0.10472E 01	0.30001E-10	0.30456E-01	0.	-0.10167E 01
4	0.10395E-22	-0.74460E 00	0.	-0.78540E 00	0.10395E-22	0.40794E-01	0.	-0.74460E 00
5	-0.18572E-10	-0.57702E 00	0.	-0.62832E 00	-0.18572E-10	0.51296E-01	0.	-0.57702E 00
6	0.71432E-11	-0.46159E 00	0.	-0.52360E 00	0.71432E-11	0.62008E-01	0.	-0.46159E 00
7	0.29287E-10	-0.37582E 00	0.	-0.44880E 00	0.29287E-10	0.72982E-01	0.	-0.37582E 00
8	-0.42859E-11	-0.30843E 00	0.	-0.39270E 00	-0.42859E-11	0.84274E-01	0.	-0.30843E 00
9	-0.85718E-11	-0.25312E 00	0.	-0.34907E 00	-0.85718E-11	0.95948E-01	0.	-0.25312E 00
10	-0.71432E-11	-0.20608E 00	0.	-0.31416E 00	-0.71432E-11	0.10808E 00	0.	-0.20608E 00
11	0.16631E-21	-0.16486E 00	0.	-0.28560E 00	0.16631E-21	0.12074E 00	0.	-0.16486E 00
12	0.10395E-22	-0.12775E 00	0.	-0.26180E 00	0.10395E-22	0.13405E 00	0.	-0.12775E 00
13	-0.42859E-11	-0.93560E-01	0.	-0.24166E 00	-0.42859E-11	0.14810E 00	0.	-0.93560E-01
14	0.21429E-10	-0.61350E-01	0.	-0.22440E 00	0.21429E-10	0.16305E 00	0.	-0.61350E-01
15	0.40002E-10	-0.30377E-01	0.	-0.20944E 00	0.40002E-10	0.17906E 00	0.	-0.30377E-01
16	-0.57145E-11	0.	0.	-0.19635E 00	-0.57145E-11	0.19635E 00	0.	0.
17	0.85718E-11	0.30377E-01	0.	-0.18480E 00	0.85718E-11	0.21518E 00	0.	0.30377E-01
18	0.10000E-10	0.61350E-01	0.	-0.17453E 00	0.10000E-10	0.23588E 00	0.	0.61350E-01
19	-0.42859E-11	0.93560E-01	0.	-0.16535E 00	-0.42859E-11	0.25891E 00	0.	0.93560E-01
20	0.10395E-22	0.12775E 00	0.	-0.15708E 00	0.10395E-22	0.28483E 00	0.	0.12775E 00
21	-0.71432E-11	0.16486E 00	0.	-0.14960E 00	-0.71432E-11	0.31446E 00	0.	0.16486E 00
22	0.14286E-11	0.20608E 00	0.	-0.14280E 00	0.14286E-11	0.34888E 00	0.	0.20608E 00
23	0.10715E-10	0.25312E 00	0.	-0.13659E 00	0.10715E-10	0.38971E 00	0.	0.25312E 00
24	-0.42859E-11	0.30843E 00	0.	-0.13090E 00	-0.42859E-11	0.43932E 00	0.	0.30843E 00
25	0.25715E-10	0.37582E 00	0.	-0.12566E 00	0.25715E-10	0.50148E 00	0.	0.37582E 00
26	0.42859E-11	0.46159E 00	0.	-0.12083E 00	0.42859E-11	0.58242E 00	0.	0.46159E 00
27	-0.25715E-10	0.57702E 00	0.	-0.11636E 00	-0.25715E-10	0.69338E 00	0.	0.57702E 00
28	0.10395E-22	0.74460E 00	0.	-0.11220E 00	0.10395E-22	0.85680E 00	0.	0.74460E 00
29	0.18572E-10	0.10167E 01	0.	-0.10833E 00	0.18572E-10	0.11251E 01	0.	0.10167E 01
30	-0.24287E-10	0.15506E 01	0.	-0.10472E 00	-0.24287E-10	0.16553E 01	0.	0.15506E 01
31	-0.80003E-10	0.31315E 01	0.	-0.10134E 00	-0.80003E-10	0.32328E 01	0.	0.31315E 01

* using (2) by Cooley-Tukey's method (N=32)

column denotes the real part of complex numbers, and the light side denotes the imaginary parts.

Tables used in this report were made by the electronic computer TOSBAC–

3400-30E (48 bits/word, mantissa part=36 bits).

In general, it is necessary to take small stepsize h and large N to seek $F(\Omega)$ for high frequency by the equation (2). But there is another method that divides the integral interval into several parts at the points where function $f(t)$ varies rapidly and integrates on each divided subintervals and sums these integrated values. The detail will be stated in the following sections.

2. *Extension of Filon's method*

We will follow the Filon's idea.

Let ω be greater than 1 in the equation (1) and interval (a, b) be consisted of subintervals (t_{n-1}, t_n) $(n=1, 2,\cdots, N)$, and suppose that $f(t)=g_n(t)$ on (t_{n-1}, t_n), where $g_n(t)$ is real valued and $K+2$ times continuously differentiable function on a certain open interval which include the closed interval $[t_{n-1}, t_n]$. Then, using the integration by parts, the next formula is obtained:

$$F(\omega)=\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} f(t)e^{-i\omega t}dt=\sum_{n=1}^{N} F_n{}^*(\omega), \tag{5}$$

where

$$\begin{aligned}F_n{}^*(\omega)&=\int_{t_{n-1}}^{t_n} g_n(t)e^{-i\omega t}dt\\&=\left\{\frac{g_n(t_{n-1})}{i\omega}+\frac{g_n{}'(t_{n-1})}{(i\omega)^2}+\cdots+\frac{g_n{}^{(K)}(t_{n-1})}{(i\omega)^{K+1}}\right\}e^{-i\omega t_{n-1}}\\&\quad-\left\{\frac{g_n(t_n)}{i\omega}+\frac{g_n{}'(t_n)}{(i\omega)^2}+\cdots+\frac{g_n{}^{(K)}(t_n)}{(i\omega)^{K+1}}\right\}e^{-i\omega t_n}+E_n{}^*(\omega).\end{aligned}$$

The truncation error $E_n{}^*(\omega)$ is defined as follows:

$$\begin{aligned}E_n{}^*(\omega)&\equiv\int_{t_{n-1}}^{t_n} g_n{}^{(K+1)}(t)\frac{1}{(i\omega)^{K+1}}e^{-i\omega t}dt\\&=O(1/\omega^{K+2}) \qquad (\omega\to\infty).\end{aligned} \tag{6}$$

If we put $g_0(t)=g_{N+1}(t)=0$ and

$$I_n{}^{(l)}\equiv g_{n+1}{}^{(l)}(t_n)-g_n{}^{(l)}(t_n) \qquad (n=0, 1, 2,\cdots, N), \tag{7}$$

an approximation formula of $F(\omega)$ is given by

$$\tilde{F}_2(\omega)\equiv\sum_{n=0}^{N}\left\{\sum_{l=0}^{K}\frac{I_n{}^{(l)}}{(i\omega)^{l+1}}\right\}e^{-i\omega t_n} \tag{8}$$

which is an economical form of calculation on $e^{-i\omega t_n}$. The truncation error is

$$E_2(\omega)\equiv\tilde{F}_2(\omega)-F(\omega)=O(1/\omega^{K+2}). \tag{9}$$

In the Filon's method, $I_n{}^{(l)}$ are put as

$$\left.\begin{aligned}I_n{}^{(1)}&=-\frac{1}{2h}(f_{2n+2}-4f_{2n+1}+6f_{2n}-4f_{2n-1}+f_{2n-2})\\I_n{}^{(2)}&=\frac{1}{h^2}(f_{2n+2}-2f_{2n+1}+2f_{2n-1}-f_{2n-2})\\&(f_n=f(a+nh)).\end{aligned}\right\} \tag{10}$$

It may contain an error even in the 1st derivative of the given function $f(t)$, namely in $I_n^{(1)}$ due to the approximation of $f(t)$ with parabolic arcs. In this case the order of the truncation error is $O(1/\omega^2)$ as $\omega\to\infty$. If other curves of higher order polynomial is adopted in the approximation, the error would be also $O(1/\omega^2)$ as $\omega\to\infty$.

In order to eliminate this fault these approximating curves must not be used but the original $f(t)$ should be used. But then, it is difficult to calculate numericaly derivatives $f'(t)$, $f''(t),\cdots$, etc. so as not to contain such an error.

Is there not any good technique? The answer to this question is in the following theorem.

3. *Approximation formula* $\tilde{F}_3(\omega)$

Theorem. Let f, g_n be complex functions such that

$$f(z)=g_n(z)\ (\mathrm{Re}\ z\in(t_{n-1},\ t_n))\ (n=1,\ 2,\cdots,\ N) \tag{11}$$

where $g_n(z)$ is regular function on a certain complex region which contains the closed interval $[t_{n-1},\ t_n]$, and let $g_0(z)=g_{N+1}(z)=0$. Then there exists an approximation formula of $F(\omega)$ such that

$$\tilde{F}_3(\omega)\equiv\frac{1}{i\omega}\sum_{n=0}^{N}\left[\sum_{k=1}^{K}c_k\left\{g_{n+1}\left(t_n+\frac{p_k}{i\omega}\right)-g_n\left(t_n+\frac{p_k}{i\omega}\right)\right\}\right]e^{-i\omega t_n}, \tag{12}$$

where the pairs of constants $\{p_k,\ c_k\}$ are defined by the relation

$$\sum_{k=1}^{K}c_k p_k{}^l=l! \qquad (l=0,\ 1,\ 2,\cdots,\ 2K-1), \tag{13}$$

and the truncation error is

$$E_3(\omega)\equiv\tilde{F}_3(\omega)-F(\omega)=O(1/\omega^{2K+1})(\omega\to\infty). \tag{14}$$

Proof. As mentioned before, we have an approximation formula for $F(\omega)$ such that

$$\tilde{F}_*(\omega)\equiv\sum_{n=0}^{N}\left\{\sum_{l=0}^{2K-1}\frac{g_{n+1}{}^{(l)}(t_n)-g_n{}^{(l)}(t_n)}{(i\omega)^{l+1}}\right\}e^{-i\omega t_n} \tag{15}$$

with the truncation error $O(1/\omega^{2K+1})$.

Here we define a polynomial function $\Phi=\Phi(K,\ \varphi,\ t,\ \eta)$ such that the term of η^l is $l\,!$ times as large as that of the Taylor expansion of $\varphi(z)$ $(z=t+\eta)$ at the center t, i. e.

$$\Phi(K,\ \varphi,\ t,\ \eta)\equiv\varphi(t)+\varphi'(t)\eta+\varphi''(t)\eta^2+\cdots+\varphi^{(l)}(t)\eta^l$$
$$+\cdots+\varphi^{(2K-1)}(t)\eta^{2K-1}. \tag{16}$$

Using the polynomial Φ in the formula (15), we have

$$\tilde{F}_*(\omega)=\frac{1}{i\omega}\sum_{n=0}^{N}\Phi\left(K,\ g_{n+1}-g_n,\ t_n,\ \frac{1}{i\omega}\right)e^{-i\omega t_n}, \tag{17}$$

Let us define another function

$$\Psi(K,\ \varphi,\ t,\ \eta)\equiv\sum_{k=1}^{K}c_k\varphi(t+p_k\eta). \tag{18}$$

Then, as described later, we can decide constants $\{p_k,\ c_k\}$ $(k=1,\ 2,\cdots,\ K)$ so that Ψ may coincide with Φ till the term of η^{2K-1}. Hence using Ψ if we put

Table 2. Values of constants $\{p_k, c_k\}$.

K	k	p_k	c_k
2			
	1	0.58578643762690495120E 00	0.85355339059327376220E 00
	2	0.34142135623730950488E 01	0.14644660940672623780E 00
3			
	1	0.41577455678347908331E 00	0.71109300992917301545E 00
	2	0.22942803602790417198E 01	0.27851773356924084880E 00
	3	0.62899450829374791969E 01	0.10389256501586135749E-01
4			
	1	0.32254768961939231180E 00	0.60315410434163360164E 00
	2	0.17457611011583465757E 01	0.35741869243779968664E 00
	3	0.45366202969211279833E 01	0.38887908515005384273E-01
	4	0.93950709123011331292E 01	0.53929470556132745011E-03
5			
	1	0.26356031971814091020E 00	0.52175561058280865248E 00
	2	0.14134030591065167922E 01	0.39866681108317592745E 00
	3	0.35964257710407220812E 01	0.75942449681707595386E-01
	4	0.70858100058588375569E 01	0.36117586799220484544E-02
	5	0.12640800844275782659E 02	0.23369972385776227890E-04
6			
	1	0.22284660417926068948E 00	0.45896467394996359359E 00
	2	0.11889321016726230308E 01	0.41700083077212099410E 00
	3	0.29927363260593140779E 01	0.11337338207404497573E 00
	4	0.57751435691045105021E 01	0.10399197453149074896E-01
	5	0.98374674183825899182E 01	0.26101720281493205940E-03
	6	0.15982873980601701783E 02	0.89854790642962123703E-06

Table 3. $\tilde{F}_3(\omega)$, $F(\omega)$, $E_3(\omega)$ using constants of $K=5$ in Table 2 for $f(t)=e^{-t}(0\leq t\leq\pi)$, $e^{t}(\pi<t<2\pi)$.

ω	$\tilde{F}_3(\omega)$		$F(\omega)$		$E_3(\omega)$	
1	279,22399665415	279.09482043981	279.83778107166	278.79456700012	-0.61378441751	0.30025343969
2	102,66184673365	204.55701629444	102.66154983639	204,55767066404	0.00029689725	-0.00065436959
3	55,96756677515	167.27673742547	55.96755628148	167,27674021944	0.00001049367	-0.00000279396
4	30,19457398261	120.32804164569	30.19457352650	120.32804156839	0.00000045611	0.00000007729
5	21,52598326071	107.22867964766	21.52598320622	107,22867963556	0.00000005448	0.00000001210
6	13,87318248068	82.92878540262	13.87318246427	82,92878540512	0.00000001641	-0.00000000279
7	11,19351134484	78.06247876398	11.19351129583	78,06247877422	0.00000004901	-0.00000001024
8	7,89704234810	62.94082174310	7.89704233751	62,94082174589	0.00000001059	-0.00000000279
9	6,82531181251	61.19880739552	6.82531177555	61,19880740297	0.00000003696	-0.00000000745
10	5,08225498499	50.63308679917	5.08225497719	50,63308680197	0.00000000779	-0.00000000279
11	4,58750467316	50.27443012082	4.58750466507	50,27443012315	0.00000000809	-0.00000000232
12	3,54005349974	42.32227668724	3.54005349194	42,32227669097	0.00000000779	-0.00000000372
13	3,29220924476	42.63916907738	3.29220923790	42,63916907878	0.00000000686	-0.00000000139
14	2,60562317536	36.34273336455	2.60562313487	36,34273336874	0.00000004048	-0.00000000419
15	2,47644062194	37.00812836689	2.47644061615	37,00812836783	0.00000000579	-0.00000000093
16	1,99730646016	31.83776975329	1.99730645451	31,83776975539	0.00000000564	-0.00000000209
17	1,92991577544	32.68625958543	1.92991576911	32,68625958636	0.00000000633	-0.00000000093
18	1,57940852381	28.32336978334	1.57940849247	28,32336978637	0.00000003134	-0.00000000302
19	1,54606515313	29.26572803105	1.54606514875	29,26572803268	0.00000000438	-0.00000000162
20	1,28006923443	25.50594397378	1.28006923015	25,50594397517	0.00000000427	-0.00000000139
21	1,26623436198	26.49179143861	1.26623435781	26,49179143924	0.00000000417	-0.00000000093
22	1,05836654643	23.19726162194	1.05836654240	23,19726162287	0.00000000403	-0.00000000093
23	1,05599167991	24.19726430904	1.05599167631	24,19726431020	0.00000000359	-0.00000000116
24	0,88961486562	21.27116159419	0.88961486180	21,27116159535	0.00000000381	-0.00000000116
25	0,89405046814	22.26793666463	0.89405046446	22,26793666579	0.00000000368	-0.00000000116
26	0,75820942334	19.63995360885	0.75820942017	19,63995361002	0.00000000316	-0.00000000116
27	0,76667889821	20.62315975409	0.76667889459	20,62315975478	0.00000000362	-0.00000000069
28	0,65389529888	18.24081139569	0.65389526071	18.24081139802	0.00000003816	-0.00000000232
29	0,66469789499	19.20437635108	0.66469789202	19,20437635178	0.00000000296	-0.00000000069
30	0,56970900520	17.02755305124	0.56970900258	17,02755305217	0.00000000261	-0.00000000093
31	0,58178340183	17.96804901864	0.58178339858	17,96804901934	0.00000000324	-0.00000000069
32	0,50078808488	15.96547673526	0.50078808200	15,96547673596	0.00000000288	-0.00000000069

$$\tilde{F}_3(\omega) \equiv \frac{1}{i\omega}\sum_{n=0}^{N} \Psi\left(K,\ g_{n+1}-g_n,\ t_n,\ \frac{1}{i\omega}\right)e^{-i\omega t_n}, \tag{19}$$

the formula $\tilde{F}_3(\omega)$ coincide with $\tilde{F}_*(\omega)$ with the error $O(1/\omega^{2K+1})$, so that the truncation error $E_3(\omega) \equiv \tilde{F}_3(\omega) - F(\omega) = O(1/\omega^{2K+1})$.

4. *Calculation of constants* $\{p_k, c_k\}$

By comparison the coefficient of terms of Φ with that of Taylor expansion of Ψ

$$\Psi = (\sum_{k=1}^{K} c_k)\varphi(t) + (\sum_{k=1}^{K} c_k p_k)\varphi'(t)\eta + \left(\frac{1}{2!}\sum_{k=1}^{K} c_k p_k^2\right)\varphi''(t)\eta^2$$
$$+\cdots+\left(\frac{1}{(2K-1)!}\sum_{k=1}^{K} c_k p_k^{2K-1}\right)\varphi^{(2K-1)}(t)\eta^{2K-1}+\cdots, \tag{20}$$

we have the non-linear equation of p_k and c_k

$$\sum_{k=1}^{K} c_k p_k^l = l! \qquad (l=0,\ 1,\ 2,\cdots,\ 2K-1). \tag{21}$$

Eliminating c_k from the equation (21), we have a linear equation of $S_1, S_2,\cdots, S_k$:

$$\left.\begin{aligned}
&S_K - S_{K-1} + 2!\ S_{K-2} - 3!\ S_{K-3}\\
&\qquad +\cdots+(-1)^{K-1}(K-1)!\ S_1 = (-1)^{K-1}K!,\\
&S_K - 2!\ S_{K-1} + 3!\ S_{K-2} - 4!\ S_{K-3}\\
&\qquad +\cdots+(-1)^{K-1}K!\ S_1 = (-1)^{K-1}(K+1)!,\\
&2!\ S_K - 3!\ S_{K-1} + 4!\ S_{K-2} - 5!\ S_{K-3}\\
&\qquad +\cdots+(-1)^{K-1}(K+1)!\ S_1 = (-1)^{K-1}(K+2)!,\\
&\cdots\cdots,\\
&(K-1)!\ S_K - K!\ S_{K-1} + (K+1)!\ S_{K-2} - (K+2)!\ S_{K-3}\\
&\qquad +\cdots+(-1)^{K-1}(2K-2)!\ S_1 = (-1)^{K-1}(2K-1)!,
\end{aligned}\right\} \tag{22}$$

where $S_1, S_2,\cdots, S_K$ are fundamental symmetric functions of $p_1, p_2,\cdots, p_K$, So, $p_1, p_2,\cdots, p_K$ are given as the solutions of an algebraic equation

$$p^K - S_1 p^{K-1} + S_2 p^{K-2} - \cdots + (-1)^K S_K = 0. \tag{23}$$

On the other hand, $c_1, c_2,\cdots, c_k$ are obtained from a linear equation as follows:

$$\sum_{k=1}^{K} c_k p_k^l = l! \qquad (l=0,\ 1,\ 2,\cdots,\ K-1). \tag{24}$$

Thus we have obtained the values of constants $\{p_k, c_k\}$. These values are listed in Table 2.

An application of the formula $\tilde{F}_3(\omega)$ is shown in Table 3, by using constants of $n=5$ in Table 2 for the function $f(t)=e^{-t}(0 \leq t < \pi)$, $=e^t(\pi \leq t < 2\pi)$.

References

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