A Note on Interpolation of Multivariable Functions

Kozo Ichida* and Takeshi Kiyono*

Abstract

The problem of numerical interpolation of multivariable functions, when their values are assumed to be given on discrete lattice points, has been solved by the Monte Carlo method, because the deterministic method takes too much time to calculate interpolated values. This paper describes a more stable algorithm than that of [3] for nonlinear interpolation of multivariable functions.

1. Introduction

There are few books that treat numerical interpolation for more than one variable. It is possible to express interpolation formula for multivariables as a Cartesian product form of one variable, but the number of terms increases exponentially with the number of variables, and even modern fast computers take too much time to calculate interpolated values. Consequently this problem has been solved by the Monte Carlo method [1,2,3]. In nonlinear interpolation there exist negative coefficients, and the main difficulty in Monte Carlo sampling is how to treat these coefficients that are regarded as negative probabilities [4]. We may solve this problem by sorting out positive and negative coefficients and sampling in proportion to their absolute values [3]. But this method loses accuracy as the number of variables increases. To overcome this difficulty we provide a sampling method which combines positive and negative coefficients. This scheme may be applied to other problems for treating such negative probabilities.

2. Nonlinear Interpolation

Let $f(x_1,\ldots,x_k)$ be a multivariable function defined in a domain D

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* Department of Information Science, Faculty of Engineering, Kyoto University

of the k-dimensional Euclidean space. The problem is to estimate, by interpolation, the values of the unknown function f at an arbitrary point in D. The data are assumed to be given at the following $\prod_{r=1}^{k} n_r$ lattice points:

$$x_r = x_r^{(1)}, \ \ldots \ , x_r^{(n_r)} \qquad (r = 1,2,\ldots,k).$$

We use the following polynomial which holds exactly at the data points [3].

$$f(x_1,\ldots,x_k) = \sum_{i_1=1}^{n_1}\cdots\sum_{i_k=1}^{n_k} L(x_1,..,x_k;i_1,..,i_k)f(x_1^{(i_1)},..,x_k^{(i_k)}), \quad (1)$$

where

$$L(x_1,..,x_k;i_1,..,i_k) = \prod_{r=1}^{k} L(x_r,i_r) \quad (2)$$

is the coefficient of k-dimensional Lagrange interpolation formula.

Simple sampling from the numerous terms of eq.(1) does not converge easily on account of large variance [1,2]. So we rewrite eq.(1) as

$$f(x_1,\ldots,x_k) = \sum_{L>0} |L(x_1,..,x_k;i_1,..,i_k)|\, f(x_1^{(i_1)},..,x_k^{(i_k)})$$
$$- \sum_{L<0} |L(x_1,..,x_k;i_1,..,i_k)|\, f(x_1^{(i_1)},..,x_k^{(i_k)}). \quad (3)$$

The value of f may be estimated by the following equation:

$$f(x_1,\ldots,x_k) = \frac{\mathcal{L}'}{N'} \sum_{L>0} f(x_1^{(i_1)},..,x_k^{(i_k)})$$
$$- \frac{\mathcal{L}''}{N''} \sum_{L<0} f(x_1^{(i_1)},..,x_k^{(i_k)}), \quad (4)$$

where $\mathcal{L}'$ and $\mathcal{L}''$ are the sums of the positive and negative coefficients, N' and N'' are the numbers of terms in the summations of eq.(4).

3. Combined Sampling

The reason that we sort out terms as eq.(3) is that a simple sampling from these terms of eq.(3) does not converge easily on account of large variance due to negative coefficients. This sampling method is indeed feasible. But $\mathcal{L}'$ and $\mathcal{L}''$ become very large as k increases. Accordingly loss of significant figures increases by subtracting the same order of large numbers.

Here we consider a sampling method which combines positive and negative coefficients to avoid this difficulty. In one dimension the interpolation formula is expressed as

$$f(x) = \sum_{t=1}^{m} |L(x,j_t)|\, f(j_t) - \sum_{t=m+1}^{n} |L(x,j_t)|\, f(j_t), \tag{5}$$

where $f(j_t) = f(x^{(j_t)})$, and $L(x,j_t)$ is assumed to be positive for $1 \leqslant t \leqslant m$, negative for $m+1 \leqslant t \leqslant n$. The coefficients satisfy the following identity.

$$\sum_{t=1}^{m} |L(x,j_t)| - \sum_{t=m+1}^{n} |L(x,j_t)| = 1. \tag{6}$$

We rewrite eq.(5) into the following form.

$$\begin{aligned} f(x) = {} & [\,|L(x,j_1)| - 2\,|L(x,j_{m+1})|\,]\,f(j_1) \\ & + |L(x,j_{m+1})|\,[2f(j_1) - f(j_{m+1})] \\ & + [\,|L(x,j_2)| - 2\,|L(x,j_{m+2})|\,]\,f(j_2) \\ & + |L(x,j_{m+2})|\,[2f(j_2) - f(j_{m+2})] + \ldots\ldots \\ & + |L(x,j_n)|\,[2f(j_{m-1}) - f(j_n)] + |L(x,j_m)|\,f(j_m). \end{aligned} \tag{7}$$

In eq.(7), $[\,|L(x,j_1)| - 2\,|L(x,j_{m+1})|\,]$, $|L(x,j_{m+1})|$, ... , $|L(x,j_m)|$ are nonnegative and their sum is unity by eq.(6). Regarding these coefficients as probabilities, we can estimate the value of f(x) as the mean value of $f(j_1)$, $[2f(j_1) - f(j_{m+1})]$, ... , $f(j_m)$. In order to express eq.(5) as eq.(7), the following equation must be satisfied:

$$\sum_{t=1}^{m} |L(x,j_t)| \geqslant 2 \sum_{t=m+1}^{n} |L(x,j_t)|\,. \tag{8}$$

Using eq.(6), eq.(8) becomes

$$\sum_{t=1}^{n} |L(x,j_t)| \leqslant 3. \tag{9}$$

We perform the above procedure for each variable. There is no subtraction of large numbers like eq.(4) and variance does not inflate inordinately. But when we calculate the quantity $2f(\ldots,x_{r_i}^{(i)},\ldots) - f(\ldots,x_{r_i}^{(i')},\ldots)$ ($\equiv 2f_{r_i} - f_{r'_i}$) for s variables x_{r_i} (i=1,2,...,s) out of k variables, it is necessary to calculate the following quantity.

$$\begin{aligned} w = {} & 2^s f_{r_1..r_s} - 2^{s-1}(f_{r_1..r_{s-1}r'_s} + f_{r_1..r'_{s-1}r_s} + \ldots + f_{r'_1r_2..r_s}) \\ & + 2^{s-2}(f_{r_1..r_{s-2}r'_{s-1}r'_s} + \ldots..) - \ldots + (-1)^{s-1}2(f_{r_1r'_2..r'_s} + \ldots \end{aligned}$$

$$+ f_{r'_1 \ldots r'_{s-1} r_s}) + (-1)^s f_{r'_1 \ldots r'_s}. \tag{10}$$

$2f_{r_i} - f_{r'_i}$ is the point that divides f_{r_i} and $f_{r'_i}$ externally with ratio 1:2 (also w in eq.(10) is understood to be such a point). It may be considered that we calculate the numerical interpolation using such external points. This circumstance resembles the extrapolation of multivariable function [2]. Accordingly, the expression of standard error that Hammersley gives in linear interpolation is applicable in this case [2], namely if grad f is uniformly bounded, then

$$\sigma < [(\frac{1}{2} + \log k)\frac{M}{2N}]^{\frac{1}{2}} \tag{11}$$

is satisfied and σ increases slowly with the number of dimensions.

Some numerical examples have been calculated with this algorithm and the results support the utility of this method.

References

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