Fast Fourier Sine and Cosine Transforms

Tatsuo Torii*

1. Introduction

In this paper, we propose methods of the fast Fourier sine and cosine transforms utilizing properties of input data such as reality and symmetry.

Fourier coefficients of a complex valued function $X(t)$ of period 2π are defined by

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} X(t)\exp(-ikt)\,dt \quad , \quad k = 0, 1, \cdots . \tag{1}$$

The integral of Fourier coefficient is approximated by the trapezoidal rule of N sample points:

$$C_k = \frac{1}{N}\sum_{0\le k<N} X_j\,\overline{W}(jk) \quad , \quad 0 \le k < N, \tag{2}$$

where

$$X_j = X(\tfrac{2\pi}{N}j),$$

$$W(j) = \exp(\tfrac{2\pi i}{N}j) = \overline{W}(-j).$$

For a composite number $N=r_1r_2\cdots r_n$, the number of complex multiplications can be reduced to $N(r_1+r_2+\cdots+r_n)$ by Cooley-Tukey algorithm.[1] We have already shown in a previous paper that the number of operations of sine or cosine transform based on the trapezoidal rule can be reduced to about $\frac{N}{16}(r_1+r_2+\cdots+r_n)$ where N is not necessarily a power of 2. Further, our algorithm will be applied to the approximation and the integration of periodic functions. The method may be considered as the fast generalized Clenshaw-Curtis integration.[2,3]

2. FFT table

We consider an interpolation of $X(t)$ by a trigonometric polynomial on N points

$$\frac{2\pi}{N}(j+\alpha), \quad 0 \le j < N,$$

where α is a nonnegative constant.

Let A_k's be coefficients of interpolating polynomial of $X(t)$. Then, we have, by the definition of A_k

$$A_k = \frac{1}{N}\sum_{0\le j<N} X_{j+\alpha}\,\overline{W}(k(j+\alpha)) \quad , \quad 0 \le k < N, \tag{3}$$

This paper first appeared in Japanese in Joho—Shori (Journal of the Information Processing Society of Japan), Vol. 15, No. 9 (1974), pp. 670~679.

* Department of Information Science, Faculty of Engineering, Nagoya University

where

$$X_{j+\alpha} = X\left(\frac{2\pi}{N}(j+\alpha)\right), \quad W(j+\alpha) = \exp\left(\frac{2\pi i}{N}(j+\alpha)\right).$$

The solution of the above linear simultaneous equations can be obtained as

$$A_k = \frac{1}{N}\sum_{0\le j<N} X_{j+\alpha}\overline{W}(k(j+\alpha)), \quad 0\le k<N, \tag{4}$$

from the discrete orthogonality of trigonometric functions. This formula is usually called discrete transform of $X(t)$.

In the case $\alpha = \frac{1}{2}$, the formula (4) is evidently the midpoint rule for the integral of Fourier coefficients, so we denote A_k by B_k.

In the case $\alpha = 0$, the formula (4) is nothing but the discrete transform based on the trapezoidal rule, i.e., $A_k = C_k$.

Assume that N is decomposed into $N = r_1 r_2 \cdots r_n$. Putting

$$N_0 = \tilde{N}_n = 1,$$

$$N_\ell = r_1 r_2 \cdots r_\ell, \quad \tilde{N}_\ell = N/\tilde{N}_\ell, \quad \ell = 1, 2, \cdots n,$$

and rewriting, for brevity

$$X(j+\alpha) = X_{j+\alpha}, \quad A(k) = A_k,$$

we define a table

$$X^\ell(j+\alpha, k) = \sum_{0\le q<N_\ell} X(j+q\tilde{N}_\ell+\alpha)\,\overline{W}(j+q\tilde{N}_\ell+\alpha)^k, \tag{5}$$

$$0\le j<\tilde{N}_\ell, \quad 0\le k<N_\ell.$$

After a troublesome algebra, we have a recurrence formula

$$X^0(j+\alpha, 0) = X(j+\alpha),$$

$$X^\ell(j+\alpha, k+sN_{\ell-1}) = \overline{W}(j+\alpha)^{sN_{\ell-1}} \sum_{0\le t<r_\ell} X^{\ell-1}(j+t\tilde{N}_\ell+\alpha, k)\,\overline{W}\left(\frac{Nst}{r_\ell}\right),$$

$$0\le j<\tilde{N}_\ell, \quad 0\le k<N_{\ell-1}. \tag{6}$$

from which we obtain all the coefficients:

$$A(k) = X^n(\alpha, k)/N, \quad 0\le k<N.$$

Here, the number of operations is bounded by $N(r_1+r_2+\cdots+r_n)$. Therefore, we will call $\{X^\ell(j+\alpha,k)\}$ FFT table of function $X(t)$, which plays an essential role in the reduction of the number of operations.

Now, we assume that the input data are all real. Then, we have following relations in the FFT table. The element of FFT table $X^\ell(j+\alpha,k)$ is real for any j when k is 0.

Otherwise there hold relations

$$X^\ell(j+\alpha, N_\ell-k) = \overline{X}^\ell(j+\alpha, k)\,\overline{W}(N_\ell(j+\alpha)), \quad 0\le j<\tilde{N}_\ell, \quad 0<k<N_\ell. \tag{7}$$

This can be proved from the definition of FFT table and the periodicity of the trigonometric functions.

Next, we consider symmetry of FFT table. For Hermite symmetrical data

$$X(N-j-\alpha)=\overline{X}(j+\alpha)\ ,\quad 0\leq j<N,$$

we have similar property such that

$$X^{\ell}(\tilde{N}_{\ell}-j-\alpha\ ,k)=\overline{X}^{\ell}(j+\alpha,k)\ ,\quad 0\leq j<\tilde{N}_{\ell}\ ,\quad 0\leq k<N_{\ell}\ . \tag{8}$$

For skew Hermite symmetrical data

$$X(N-j-\alpha)=-\overline{X}(j+\alpha)\ ,\quad 0\leq j<N,$$

we find such a skew symmetry as

$$X^{\ell}(N_{\ell}-j-\alpha,k)=-\overline{X}^{\ell}(j+\alpha,k)\ ,\quad 0\leq j<\tilde{N}_{\ell}\ ,\quad 0\leq k<N_{\ell}\ . \tag{9}$$

In formulas (8) and (9), α must be 0 or $\frac{1}{2}$, because both $j+\alpha$ and $\tilde{N}_{\ell}-j-\alpha$ are to be included in the set $\{j+\alpha;\ 0\leq j<N\}$.

3. Algorithms

For convenience, we describe algorithms based on the midpoint rule, only in the case where N is a power of 2.

(a) Complex data

$$\begin{aligned}
&X^{0}(j+\tfrac{1}{2},0)=X(j+\tfrac{1}{2})\ ,\\
&X^{\ell}(j+\tfrac{1}{2},k)=X^{\ell-1}(j+\tfrac{1}{2},k)+X^{\ell-1}(j+\tilde{N}_{\ell}+\tfrac{1}{2},k)\ ,\\
&X^{\ell}(j+\tfrac{1}{2},k+N_{\ell-1})=\overline{W}(j+\tfrac{1}{2})^{k}\{X^{\ell-1}(j+\tfrac{1}{2},k)-X^{\ell-1}(j+\tilde{N}_{\ell}+\tfrac{1}{2},k)\}\ ,\\
&\qquad 0\leq j<\tilde{N}_{\ell}\ ,\quad 0\leq k<N_{\ell-1}\ ,\quad \ell=1,2,\cdots,n\ ,\\
&B(k)=X^{n}(\tfrac{1}{2},k)/N\ ,\quad 0\leq k<N.
\end{aligned}$$

Here, the number of operations is $\frac{N}{2}\log_2 N$.

(b) Real and Hermite symmetrical data

$$\begin{aligned}
&X^{2}(j+\tfrac{1}{2},0)=\{X(j+\tfrac{1}{2})+X(\tfrac{N}{2}-j-\tfrac{1}{2})\}+\{X(\tfrac{N}{4}+j+\tfrac{1}{2})+X(\tfrac{N}{4}-j-\tfrac{1}{2})\}\ ,\\
&X^{2}(j+\tfrac{1}{2},2)W(j+\tfrac{1}{2})^{2}=\{X(j+\tfrac{1}{2})+X(\tfrac{N}{2}-j-\tfrac{1}{2})\}-\{X(\tfrac{N}{4}+j+\tfrac{1}{2})+X(\tfrac{N}{4}-j-\tfrac{1}{2})\}\ ,\\
&X^{2}(j+\tfrac{1}{2},1)=\overline{W}(j+\tfrac{1}{2})\{X(j+\tfrac{1}{2})-X(\tfrac{N}{2}-j-\tfrac{1}{2})+\overline{W}(\tfrac{N}{4})(X(\tfrac{N}{4}+j+\tfrac{1}{2})-X(\tfrac{N}{4}-j-\tfrac{1}{2}))\}\ ,\\
&X^{\ell}(j+\tfrac{1}{2},N_{\ell-1})W(j+\tfrac{1}{2})^{N_{\ell-1}}=X^{\ell-1}(j+\tfrac{1}{2},0)-X^{\ell-1}(\tilde{N}_{\ell}-j-\tfrac{1}{2},0)\ ,\\
&X^{\ell}(j+\tfrac{1}{2},k)=X^{\ell-1}(j+\tfrac{1}{2},k)+\overline{X}^{\ell-1}(\tilde{N}_{\ell}-j-\tfrac{1}{2},k)\ ,\\
&X^{\ell}(j+\tfrac{1}{2},N_{\ell-1}-k)=\overline{W}(j+\tfrac{1}{2})^{N_{\ell-1}}\{\overline{X}^{\ell-1}(j+\tfrac{1}{2},k)-X^{\ell-1}(\tilde{N}_{\ell}-j-\tfrac{1}{2},k)\}\ ,\\
&X^{\ell}(j+\tfrac{1}{2},N_{\ell-2})=\overline{W}(j+\tfrac{1}{2})^{N_{\ell-2}}\{X^{\ell-1}(j+\tfrac{1}{2},N_{\ell-2})W(j+\tfrac{1}{2})^{N_{\ell-2}}\\
&\qquad\qquad -\overline{W}(\tfrac{N}{4})X^{\ell-1}(\tilde{N}_{\ell}-j-\tfrac{1}{2},N_{\ell-2})W(\tilde{N}_{\ell}-j-\tfrac{1}{2})^{N_{\ell-2}}\}\ ,\\
&\qquad 0\leq 2j<\tilde{N}_{\ell}\ ,\quad 0<2k<N_{\ell-1}\ ,\quad \ell=3,4,\cdots,n-1\ ,\\
&B(0)=2X^{n-1}(\tfrac{1}{2},0)/N\ ,\quad B(\tfrac{N}{4})=\sqrt{2}\,X^{n-1}(\tfrac{1}{2},\tfrac{N}{4})W(\tfrac{N}{8})/N\ ,\quad B(\tfrac{N}{2})=0\ ,\\
&B(k)=2\,\mathrm{Re}\,X^{n-1}(\tfrac{1}{2},k)/N\ ,\\
&B(\tfrac{N}{2}-k)=-2\,\mathrm{Im}\,X^{n-1}(\tfrac{1}{2},k)/N\ ,\quad 0<k<\tfrac{N}{4}\ .
\end{aligned}$$

This algorithm may be called the fast Fourier cosine transform of the even function $X(t)$, whose number of operations is equal to $\frac{N}{8}(\log_2 N - 2)$.

The similar algorithm of the fast Fourier sine transform, shown in our Japanese version, is omitted in this paper.

4. Approximation and integration

We construct a sequence of interpolating polynomials of $X(t)$ with increasing order of power of 2. This algorithm can be formulated into an iteration of the dicrete Fourier transform based on the midpoint rule under some initial conditions. The resulting polynomial is a discrete Fourier series of $X(t)$ based on the trapezoidal rule whose degree of approximation essentially depends on the differential property of the given function $X(t)$. Assume that $X(t)$ is an analytic and periodic function on $(-\infty,\infty)$.

Noting that discrete Fourier coefficient has relation $C_{-k} = C_{N-k}$ for any k, we approximate $X(t)$ by the polynomial

$$p(z) = \sum_{k=-N/2}^{N/2}{}'' C_k z^k, \quad z = e^{it}, \tag{10}$$

where N is an even number and the notation $\sum''$ means that the first and last summands are to be halved.

We obtain the error bound

$$|X(t) - p(e^{it})| \le |a_{N/2}| + |a_{-N/2}| + 2\sum_{k>N/2}\{|a_k| + |a_{-k}|\},$$

from the discrete orthogonality of the trigonometric functions. By this inequality, we have a practical error bound

$$2\{|C_{N/2-1}| + |C_{N/2+1}| + \tfrac{1}{2}|C_{N/2}|\}.$$

Now, we show an algorithm of the fast and generalized Clenshaw-Curtis integration. When N is 2^n, we rewrite B_k and C_k by B_k^n and C_k^n, respectively. Then we have

$$C_0^1 = \frac{1}{2}\{X(0) + X(\pi)\},$$
$$C_1^1 = \frac{1}{2}\{X(0) - X(\pi)\},$$
$$C_k^{n+1} = \frac{1}{2}\{C_k^n + B_k^n\},$$
$$C_{N+k}^{n+1} = \frac{1}{2}\{C_k^n - B_k^n\},$$
$$0 \le k < 2^n, \quad n = 1, 2, \cdots.$$

This is a recurrence formula for the discrete Fourier coefficients C_k^n's based on the trapezoidal rule. In this iterative method, the previous computed values of function for discrete Fourier coefficients B_k^n's based on the midpoint rule are not wasted. We increase N with the power of 2, until stopping criterion

$$2\{|C^n_{N/2-1}|+|C^n_{N/2+1}|+\frac{1}{2}|C_{N/2}|\}<\varepsilon,$$

is satisfied for a tolerance ε.

The number of operations to obtain all the coefficients $\{C^n_k\}$ for complex input data is counted by

$$\sum_{\ell=2}^{n-1}\ell\,2^{\ell-1}=\frac{N}{2}(\log_2 N-2)\ ,\quad N=2^n,$$

which can be halved for real data.

In the case of fast Fourier sine or cosine transform, the number of operations is reduced to $\frac{N}{8}(\log_2 N-4)$.

As mentioned above, we get an interpolating polynomial

$$x(t)\simeq{\sum_{0\le k\le N/2}}''\{\bar{c}_k e^{-ikt}+c_k e^{ikt}\}$$

for real valued function $x(t)$ with preassigned accuracy.

By the integration of the series term by term, we have, evidently

$$\int_0^t x(t)\,dt\simeq c_0 t+2\sum_{0<k<N/2}\frac{1}{k}\,\mathrm{Im}\{c_k e^{ikt}\}+\frac{2}{N}c_{N/2}\sin\frac{N}{2}t. \tag{11}$$

If $x(t)$ is an even function, all the coefficients c_k's are real. So we obtain a simple formula:

$$\int_0^t x(t)\,dt\simeq c_0 t+2\sum_{0<k\le N/2}\frac{c_k}{k}\sin kt. \tag{12}$$

Integration of periodic functions will be extended to wider class of functions by a variable transformation. Some examples are shown in our Japanese version.

References

1) J.W.Cooley and J.W.Tukey: An Algorithm for the Machine Calculation of Complex Fourier Series, Math. Comp.,vol.19, pp.297-301(1965).

2) T.Torii: On the Fast Fourier Transform Using the Symmetry of Input Data, Journal of Information Processing of Japan, vol.15,pp.516-523(1974).

3) C.W.Clenshaw and A.R.Curtis: A Method for Numerical Integration on an Automatic computer, Numer. Math., vol.2, pp.197-205(1960).