Short Note

An Approximation of Real Analytic Functions Based on Cauchy's Integral Representation

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A new method based on Cauchy's integral representation is developed for approximating analytic functions. This method gives a very simple computer algorithm in comparison with the usual approximations of analytic functions such as Taylor series and orthogonal polynomial expansions. Numerical results are also discussed.

1. Introduction

It has already been recognized that rational functions are useful for the approximation of functions. In such an approximation, there are a number of difficult problems from both theoretical and practical points of view, because it is generally nonlinear with respect to the unknown parameters, and this nonlinearity depends on the property of the functions to be approximated [1], [2], [3].

On the other hand, as long as we restrict ourselves to the case of analytic functions usually encountered in physical problems, there is a possibility that the theory of analytic functions can be applied effectively [4]. In this paper we will propose a new method for approximating analytic functions based on the Cauchy's integral representation. The approximation consists of a linear system of rational functions which have simple poles originating in the Cauchy's kernel. The unknown parameters can be directly determined by discretizing the integral.

2. Rational Expansion of Analytic Functions

Consider a real analytic function $f(x)$ defined in the closed interval $K=[-1, 1]$ on the real axis. And let the corresponding complex function $f(z)$ be analytic on and within a contour Γ in the complex plane which encloses the interval K. The starting point of our approach is Cauchy's integral representation

$$f(x)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)}{z-x}dz, \quad x\in K, \tag{1}$$

where the path of integration Γ is taken in the positive sense. It can be expected to approximate the integral (1) in the form

$$f_N(x)=\sum_{j=1}^{N}\frac{C_j}{z_j-x}, \tag{2}$$

where $z_1, z_2, \cdots, z_N, z_{N+1}=z_1$ are successive points on Γ, so chosen that $|z_{j+1}-z_j|$ approaches zero uniformly with respect to x as $N\to\infty$. And C_j are proper complex constants to be determined by the approximation scheme. The formula (2) is regared as a simple expansion based on a system of rational functions, i.e., discrete values of Cauchy's kernel.

The path of integration Γ may be taken arbitrary if only the condition concerning (1) is satisfied, so that there are a number of possible sets of the poles z_j in (2). However, it is practically convenient to specify the form of contour Γ to the particular cases. In the present paper, we shall consider two cases where Γ is a circle, and where it is an ellipse.

3. Approximation with Circular Contour Γ

Let Γ be a circle $|z|=R>1$, R being a positive constant. Then the integral in (1) is rewritten as

$$f(x)=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{zf(z)}{z-x}d\theta, \tag{3}$$

with $z=R\cdot e^{i\theta}$.

Further, the Eq. (3) can be approximate by the trapezoidal rule, so that

$$f_N(x)=\frac{1}{N}\sum_{j=1}^{N}\frac{z_jf(z_j)}{z_j-x}, \tag{4}$$

where

$$z_j=R\cdot e^{i\theta_j}$$

and

$$\theta_j=\frac{2(j-1)\pi}{N}.$$

This is the approximation formula for a circular contour. It can be used for approximating analytic functions, unless they have singularities on and within the circle $|z|=R$. Additionally, the formula (4) can be reduced to a real form if we take z_j symmetric with respect to the real and the imaginary axes.

As is well known from the classical approximation theory [5],

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$$\left|\int_{\theta_j}^{\theta_{j+1}} g_x(\theta)d\theta-\frac{\pi}{N}\{g_x(\theta_j)+g_x(\theta_{j+1})\}\right|\leqq\frac{2}{3}M_j\left(\frac{\pi}{N}\right)^3, \quad (5)$$

where

$$g_x(\theta)\equiv\frac{zf(z)}{2\pi(z-x)}$$

and

$$M_j\equiv\max_{\theta_j\leqq\theta\leqq\theta_{j+1}}|g_x''(\theta)|.$$

From (5), we have

$$|\varepsilon(x)|\equiv|f(x)-f_N(x)| \leqq\frac{2\pi^3}{3N^2}M, \quad (6)$$

with

$$M=\frac{1}{N}\sum_{j=1}^{N}M_j.$$

This is the error estimate for the circular case.

4. Approximation with Elliptic Contour Γ

In order to release ourselves from the above-mentioned restriction that $f(x)$ must not have singularities in $|z|\leqq R$, we introduce a conformal transformation from the z-plane to the w-plane, with a mapping function defined by

$$z=\frac{1}{2}\left(w+\frac{1}{w}\right)$$

where z is a complex variable defined in Sec. 2. This function maps the ellipse which has foci $z=\pm 1$ and semi-axes $(R+R^{-1})/2$ and $(R-R^{-1})/2$ onto the circle $|w|=R>1$. Thereby, (1) results in

$$f(x)=\frac{1}{2\pi}\int_0^{2\pi}\frac{vf(z)}{z-x}d\theta, \quad (7)$$

where

$$v=\frac{1}{2}\left(w-\frac{1}{w}\right)$$

and

$$w=R\cdot e^{i\theta}.$$

Similarly as before,

$$f_N(x)=\frac{1}{N}\sum_{j=1}^{N}\frac{v_jf(z_j)}{z_j-x}, \quad (8)$$

where

$$z_j=\frac{1}{2}\left(w_j+\frac{1}{w_j}\right),$$
$$w_j=R\cdot e^{i\theta_j}$$

and

$$v_j=\frac{1}{2}\left(w_j-\frac{1}{w_j}\right).$$

This formula can be used even if the singularities of the function are located in the neighbourhood of the line segment $[-1, 1]$ on the real axis if we take R close to 1. The formula (8) can also be reduced to a real form if we take z_j symmetric with respect to the real and the imaginary axes.

The error estimate for this approximation is the same as that for the circular case, i.e. (6), excepting that

$$g_x(\theta)\equiv\frac{vf(z)}{2\pi(z-x)}.$$

5. Numerical Examples

In this section we will give some numerical examples.

Values of $\log_{10}\|\varepsilon(x)\|_\infty$ for $f(x)=\sin\pi x$ are listed in Table 1, where $\|\varepsilon(x)\|_\infty$ designates the maximum norm of $\varepsilon(x)=f(x)-f_N(x)$, $x\in K$, with N taken as powers of 2. In this case, we performed the experiment with the parameters R from 1.1 to 3.0 in the increments of 0.1. However, in the table only the data for optimum value of R at $N=16$ are given. Within our experiments, the results in the elliptic case can be generally considered better in accuracy than those in the circular case.

Furthermore, we shall show an example for which only the elliptic algorithm is useful. Table 2 indicates values of $\log_{10}\|\varepsilon(x)\|_\infty$ for $f(x)=\arctan x$, which has singularities at $\pm i$. In this simulation, we selected $R=1.1$, 1.6 and 2.3 which give ellipse semi-axes of nearly 0.1, 0.5 and 0.9, respectively. As is shown in this table, the error norm decreases rapidly with an increase of N, and the approximation at $R=1.6$ yields extremly good accuracy, as can be expected.

We performed the experiments in double precision arithmetic on FACOM M200 at the Kyoto University Computer Center.

6. Conclusion

Since we need a fairly large number of terms when

Table 1 Values of $\log_{10}\|\varepsilon\|_\infty$ in the approximation of $f(x)=\sin\pi x$.

Number of Terms, N	Circular		Elliptic	
	$R=1.5$	$R=2.3$	$R=1.5$	$R=2.3$
4	1.1	2.1	− 0.3	0.5
8	0.3	1.7	− 1.2	− 0.7
16	− 3.2	− 0.3	− 2.5	− 5.1
32	− 7.2	− 9.0	− 5.4	−11.3
64	−13.4	−14.4	−11.0	−22.9

Table 2 Values of $\log_{10}\|\varepsilon\|_\infty$ in the approximation of $f(x)=\arctan x$, with elliptic contour Γ.

Number of Terms, N	$R=1.1$	$R=1.6$	$R=2.3$
4	0.5	− 0.5	−0.5
8	0.1	− 1.4	−1.0
16	−0.4	− 3.0	−1.6
32	−1.1	− 6.3	−2.3
64	−2.5	−12.8	−3.3

evaluating $f_N(x)$ for a given x with high accuracy, there is some problem in the efficiency. In the present method, however, unknown parameters of the linear system of rational functions based on the Cauchy's integral can be directly determined by discretizing the integral formula, i.e., this method is very simple and particularly useful for approximations of such complicated functions that their higher derivatives in Taylor's series are difficult to obtain.

References

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