Some Poised Lacunary Interpolation Polynomials

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An interpolation similar to the (0, 2)-polynomial is investigated. This interpolation is defined by a polynomial of degree at most $k+1$ whose value is prescribed at two end-points, x_{1k} and x_{kk} together with its second derivatives at k points, $x_{1k}, x_{2k}, \cdots, x_{kk}$. These points are arbitrary real numbers ordered as

$$-1 \leq x_{1k} < x_{2k} < \cdots < x_{kk} \leq 1.$$

In the present paper, such interpolation for any integer $k \geq 2$ is constructed in an explicit form and the uniform xonvergence of the interpolation is discussed for some classes of functions.

Specifically, two convergence theorems are obtained when both end-points, x_{1k} and x_{kk}, are -1 and 1, respectively. Namely, one theorem gives a sufficient condition under which the interpolation for every $f \in C^2[-1, 1]$ converges uniformly to f in the interval as the number of interpolation points increases infinitely. In the other theorem, another sufficient condition is also given. In particular, this condition is useful in the case where interpolation points consist of zeroes of the polynomial of the form,

$$(x^2-1)p_{k-2}(x),$$

where p_{k-2} is the $(k-2)$-nd orthogonal polynomial over the open interval $(-1, 1)$. Here, a class of sufficiently smooth function is assumed.

A computational algorithm for the interpolation polynomial is also constructed.

1. Introduction

Let k and n be positive integers and $E_{kn}=(e_{ij})$, ($i=1, 2, \cdots, k$ and $j=0, 1, \cdots, n-1$), be a matrix where $e_{ij}=0$ or 1 and

$$\sum_{i=1}^{k} \sum_{j=0}^{n-1} e_{ij}=n.$$

Let $\{x_{ik}\}_{i=1}^{k}$ be arbitrary real numbers ordered as,

$$A: -1 \leqq x_{1k} < x_{2k} < \cdots < x_{kk} \leqq 1.$$

These are points of interpolation and A is an infinite triangular matrix as $k=1, 2, \cdots$.

The general problem of interpolation by means of a polynomial can be described as follows; When a matrix E_{kn} is specified, find a polynomial $p(x)$ of degree at most $n-1$ which satisfies,

$$P^{(j)}(x_{ik})=y_{ik}^{j}, \text{ for } (i, j) \in e_k \equiv \{(i, j): e_{ij}=1\}, \quad (1.1)$$

for given interpolation points $\{x_{ik}\}_{i=1}^{k}$, where y_{ik}^{j} are arbitrary real numbers prescribed. In this problem, for every choice of the real numbers $\{x_{ik}\}_{i=1}^{k}$ and for every choice of $\{y_{ik}^{j}: (i, j) \in e_k\}$, if there exists a unique polynomial $p(x)$ of degree at most $n-1$ satisfying conditions (1.1), then the problem is said to be poised. On the other hand, if the problem has a unique polynomial solution for only certain choice of interpolation points, then it is said to be conditionally poised. We shall call E_{kn} the incidence matrix of the problem.

In 1955, Surányi and Turán [1] investigated an interesting problem, the so-called (0, 2)-interpolation problem, where the incidence matrix $E_{kn}=(e_{ij})$, ($i=1, 2, \cdots, k$ and $j=0, 1, \cdots, 2k-1$), is specified by $e_{ij}=1$ for $i=1, 2, \cdots, k$ and $j=0$, 2 and otherwise $e_{ij}=0$. Their problem is not poised, but is conditionally poised [2]-[4]. Subsequently, Balázs and Turán [5] dealt with the uniform convergence of (0, 2)-interpolation over the very special matrix of interpolation points in which each element of its k-th row consists of zeroes of polynomial,

$$(x^2-1)p'_{k-1}(x), \quad (1.2)$$

where p_{k-1} is the $(k-1)$-st Legendre polynomial with $p_{k-1}(1)=1$.

In the present paper, an interpolation problem having the following incidence matrix is investigated; $E_{kn}=(e_{ij})$, $i=1, 2, \cdots, k$ and $j=0, 1, \cdots, k+1$, where $e_{10}=1$, $e_{20}=e_{30}=\cdots=e_{k-1,0}=0$, $e_{k0}=1$, $e_{i2}=1$ for all i and otherwise $e_{ij}=0$. This problem is similar to the (0, 2)-interpolation problem. However, there is an essential difference between the two problems. That is, the (0, 2)-problem is not poised. On the other hand, our problem is poised although because it is a lacunary interpolation. In fact, our E_{kn} can be decomposed into the horizontal sum of two incidence matrices of Lagrange interpolation problems which are known to be poised. Thus, according to Lemma 4 of Sharma's paper [2], it is evident that this E_{kn} is poised. We shall call solutions to our problems a quasi-(0, 2) interpolation.

In Section 2, a quasi-(0, 2) interpolation for given interpolation points $\{x_{ik}\}_{i=1}^{k}$, $k \geqq 2$, is constructed in an explicit form. Section 3 is devoted to discussion of the uniform convergence of quasi-(0, 2) interpolation for the class of twice-continuously differentiable functions defined on the interval $[-1, 1]$. Then a convergence theorem is obtained when both end-points, x_{1k} and

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x_{kk} are -1 and 1, respectively. Strictly speaking, the theorem gives a sufficient condition under which the quasi-(0, 2) interpolation of the function f belonging to the class converges uniformly to f in the interval. In Section 4, another sufficient condition is also given. This condition is useful in the case where interpolation points consist of zeroes of the polynomial in the form,

$$(x^2-1)p_{k-2}(x),$$

where p_{k-2} is the $(k-2)$-nd orthogonal polynomial over the interval $(-1, 1)$. Here, a class of sufficiently smooth functions is assumed. A computational algorithm for the quasi-(0, 2) interpolation is also constructed in Section 5.

Finally, before proceeding to detailed discussions, it seems to be necessary to describe motivations of the investigation of our problems. The most important purpose is to apply the quasi-(0, 2) interpolation as a tool for solving numerically the following two-point boundary value problems for the non-linear ordinary differential equation of the second order,

$$y''(x)=f(x, y(x)), \text{ in } x\in[-1, 1],$$
$$y(-1)=y(1)=0.$$

In fact, for a certain class of this boundary problem, solutions can be well-approximated by means of the quasi-(0, 2) interpolation. Also, the uniform convergence of such approximate solution to its exact solution can be proved for some infinite triangular matrix of interpolation points. But this application is not discussed in this paper.

2. Construction of Quasi-(0, 2) Interpolation

As shown in the Introduction, the quasi-(0, 2) interpolation problem is poised. Thus for any k points of interpolation, there exists a unique solution, namely a unique polynomial of degree at most $k+1$, satisfying conditions (1.1). It is evident that all solutions form a linear space of $(k+2)$-dimension which consists of all polynomials of degree $\leqq k+1$. Conversely, from well-known properties of a basis within a linear space, every solution can be expressed as a linear combination of $k+2$ linearly independent polynomials contained in the space.

Now, it is clear that the set of $k+2$ polynomials, $r_{1k}(x; A)$, $r_{kk}(x; A)$ and $s_{ik}(x; A)$, $i=1, 2, \cdots, k$, of degree at most $k+1$ with respect to the variable x, satisfying the following conditions, is a basis determined by the k-th row $(x_{1k}, x_{2k}, \cdots, x_{kk})$ of the matrix A of interpolation points.

CONDITIONS:

(1) $r_{jk}(x_{ik}; A)=\delta_{ij}$, for $i, j=1$ and k,

(2) $r''_{jk}(x_{ik}; A)=0$, for $i=1, 2, \cdots, k$ and $j=1, k$,

(3) $s_{jk}(x_{ik}; A)=0$, for $i=1, k$ and $j=1, 2, \cdots, k$,

(4) $s''_{jk}(x_{ik}; A)=\delta_{ij}$, for $i, j=1, \cdots, k$,

where δ_{ij} is the Kronecker delta. Using this basis, therefore, every quasi-(0, 2) interpolation can be uniquely written in the form,

$$q_k(x; A)=r_{1k}(x; A)y_{1k}+r_{kk}(x; A)y_{kk}+\sum_{i=1}^{k} s_{ik}(x; A)y_{ik}^2, \tag{2.1}$$

where y_{1k}, y_{kk} and y_{ik}^2 $(i=1, 2, \cdots, k)$ are $k+2$ arbitrary real numbers.

We shall construct the basis of polynomials which satisfy the above Conditions. Let $r_{1k}(x; A)$ and $r_{kk}(x; A)$ be polynomials as follows;

$$r_{1k}(x; A)=\frac{(x-x_{kk})}{(x_{1k}-x_{kk})},$$
$$r_{kk}(x; A)=\frac{-(x-x_{1k})}{(x_{1k}-x_{kk})},$$

then it is easy to show that these are plynomials satisfying conditions, (1) and (2).

On the other hand, the remaining polynomials $s_{ik}(x; A)$, $(i=1, 2, \cdots, k)$, can be obtained by solving two-point boundary-value problems of second order differential equations. In fact, the polynomial $s_{ik}(x; A)$ satisfying conditions, (3) and (4), is characterized as a particular solution to the following two-point boundary value problem,

$$\begin{aligned} &s''_{ik}(x; A)=L_{ik}(x; A), \text{ in } x\in[x_{1k}, x_{kk}],\\ &s_{ik}(x_{1k}; A)=s_{ik}(x_{kk}; A)=0, \end{aligned} \tag{2.2}$$

where $L_{ik}(x; A)$ is a fundamental polynomial of the Lagrange interpolation and is defined as follows;

$$L_{ik}(x; A)=\frac{\pi_k(x; A)}{(x-x_{ik})\pi'_k(x_{ik}; A)}, \tag{2.3}$$

here, $\pi_k(x; A)$ is the polynomial of degree k,

$$\pi_k(x; A)=\text{constant}\cdot(x-x_{1k})(x-x_{2x})\cdots(x-x_{kk}). \tag{2.4}$$

From well-known theory about ordinary differential equations, for example [6], by using the Green function,

$$G(x, u; x_{1k}, x_{kk})=\begin{cases} \dfrac{(x-x_{\cdot k})(u-x_{kk})}{(x_{kk}-x_{1k})}, & \text{for } x\leqq u\leqq x_{kk},\\[2ex] \dfrac{(x+x_{\cdot k})(u+x_{kk})}{(x_{kk}-x_{1k})}, & \text{for } x_{1k}\leqq u<x, \end{cases}$$

the solution to the boundary value problem (2.2) can be obtained exactly and is expressed in the form,

$$s_{ik}(x; A)=\int_{x_{1k}}^{x_{kk}} G(x, u; x_{1k}, x_{kk})L_{ik}(u, A)du. \tag{2.5}$$

It is also easy to prove that this solution is the polynomial satsfying conditions (3) and (4).

Procedures for interpolating a function by means of a quasi-(0, 2) interpolation become the following. It is assumed that the function $f(x)$ is at least twice-differentiable. Then, in the equation (2.1), we can replace y_{1k} and y_{kk} by values of $f(x)$ at $x=x_{1k}$ and x_{kk}, respectively, and also y_{ik}^2 by values of the second derivative $f''(x)$ at

$x=x_{ik}$ $(i=1, 2, \cdots, k)$. Consequently, the equation (2.1) can be rewritten in the form,

$$q_k(x, f; A)=r_{1k}(x; A)f(x_{1k})+r_{kk}(x; A)f(x_{kk}) + \sum_{i=1}^{k} \int_{x_{1k}}^{x_{kk}} G(x, u; x_{1k}, x_{kk})L_{ik}(u; A)du f''(x_{ik}), \tag{2.6}$$

with $s_{ik}(x; A)$ represented in the equation (2.5). This representation is the quasi-(0, 2) interpolation process to the function $f(x)$ over the matrix A of interpolation points.

The quasi-(0, 2) interpolation process $q_k(x, f; A)$ has the following properties:

PROPERTY 1. If $f(x)$ is a polynomial of degree at most $k+1$, then $q_k(x, f; A)$ is identical with f.

PROPERTY 2. The quasi-(0, 2) interpolation process $q_k(x, f; A)$ has a property of linear operators with respect to f; that is, for two arbitrary twice-differentiable functions, f and g,

$$q_k(x, f+g; A)=q_k(x, f; A)+q_k(x, g; A).$$

Proofs of these properties are clear and these properties are useed in the proof of Theorem 1 in Section 3.

3. Main Theorem for Uniform Convergence

In this section, a sufficient condition for the uniform convergence of quasi-(0, 2) interpolation process is investigated in the class $C^2[-1, 1]$ of twice-continuously differentiable functions defined on the interval $I=[-1, 1]$. First of all, the difference between $f(x)$ belonging to $C^2[-1, 1]$ and the interpolation process $q_k(x, f; A)$ are estimated over the matrix A of interpolation points. A sufficient condition under which the interpolation process $q_k(x, f; A)$ for every $f \in C^2[-1, 1]$ converges uniformly to f in I is described in Theorem 2.

In the sequel, the interpolation points $\{x_{ik}\}_{i=1}^{k}$ are restricted as follows;

$$-1=x_{1k}<x_{2k}<\cdots<x_{kk}=1,$$

for every $k \geqq 2$, that is, both end-points are fixed as $x_{1k}=-1$ and $x_{kk}=1$. Let $\omega(\delta)$ denote the modulus of continuity for functions, where δ is a sufficientlly small positive real number. If the function is continuous, then $\omega(\delta) \to 0$ as $\delta \to 0$. It is convenient to introduce a function defined by,

$$M_k(A)=\max_{x \in I}\left\{\sum_{i=1}^{k} L_{ik}(x; A)^2\right\}^{1/2}, \tag{3.1}$$

where $L_{ik}(x; A)$ are polynomials defined by the equation (2.3).

Now, we describe the first theorem.

THEOREM 1. Let k be any integer $\geqq 2$. If $f \in C^2[-1, 1]$, then the quasi-(0, 2) interpolation process $q_k(x, f; A)$ satisfies the following inequality,

$$|f(x)-q_k(x, f; A)| \leqq 3\omega\left(\frac{1}{k}\right)+3M_k(A)\omega\left(\frac{1}{\sqrt{k}}\right), \tag{3.2}$$

for every matrix A of interpolation points and for any $x \in I$, where ω denotes the modulus of continuity for the second derivative of f.

It should be noted that the inequality (3.2) gives a priori evaluations of errors in the interpolation process $q_k(x, f; A)$ of $f(x)$. Two lemmas are used in order to prove Theorem 1. The first lemma is immediately derived from the following Jackson's theorem.

Jackson's Theorem [7]: Let k be any integer $\geqq 1$ and $u(x)$ an arbitrary continuous function defined on the interval $[-1, 1]$. Then, there exists a polynomial $p_k(x)$ with degree at most k which satisfies the inequality,

$$|u(x)-p_k(x)| \leqq 6\omega\left(\frac{1}{k}\right),$$

for every $x \in [-1, 1]$, where ω denotes the modulus of continuity of the function $u(x)$.

In order to construct the polynomial satisfying the above inequality, the constructive method by de la Vallée-Poussin can be used [7], [8]: Let m be any integer $\geqq 1$ and introduce a function $Q_{2m-1}(t; u)$ derived from the given function $u(x)$ as follows,

$$Q_{2m-1}(t; u)=\frac{3}{2\pi}\int_{-\infty}^{\infty} u\left(\cos\left(t+\frac{2y}{m}\right)\right)\left(\frac{\sin y}{y}\right)^4 dy.$$

Then, this Q_{2m-1} is a polynomial of degree at most $2m-1$ with respect to x when the variable t is replaced by $\cos^{-1} x$. In addition, $Q_{2m-1}(\cos^{-1} x; u)$ satisfies the inequality of Jackson's Theorem if m is chosen in such a way $m=[(k+1)/2]$, where [.] denotes the Gaussian symbol. The following lemma is a direct consequence of Jackson's Theorem.

LEMMA 1. Let k be any integer ≥ 2. If a function $f(x)$ belongs to $C^2[-1, 1]$, then $Q_{2m-1}(\cos^{-1} x; f'')$ satisfies the inequality,

$$|f''(x)-Q_{2m-1}(\cos^{-1} x; f'')| \leqq 6\omega\left(\frac{1}{k}\right),$$

for energy $x \in I$, where $m=[(k+1)/2]$ and ω is the modulus of continuity for the second derivative of f.

The second lemma is as follows;

LEMMA 2. Let f be any function belonging to $C^2[-1, 1]$ and k any integer $\geqq 2$. If $p(x)$ is a polynomial of degree at most $k+1$ whose second derivative is $Q_{2m-1}(\cos^{-1} x; f'')$ where $m=[(k+1)/2]$, then the following evaluation holds for every infinite triangular matrix A of interpolation points. Namely, for any $x \in I$,

$$|q_k''(x, p-f; A)| \leqq 6M_k(A)\omega\left(\frac{1}{\sqrt{k}}\right),$$

where ω is the modulus of continuity for the second derivative of $f(x)$ and $q_k''(x, p-f; A)$ is as follows,

$$q_k''(x, p-f; A)=\sum_{i=1}^{k} L_{ik}(x; A)\{p''(x_{ik})-f''(x_{ik})\}. \tag{3.3}$$

PROOF of Lemma 2: Using Schwarz' inequality, the equation (3.3) is estimated as follows,

$$|q_k''(x, p-f; A)| \leqq M_k(A)\left\{\sum_{i=1}^{k} |p''(x_{ik})-f''(x_{ik})|^2\right\}^{1/2},$$
$$\leqq M_k(A)\sqrt{k} \max_{x\in I} |p''(x)-f''(x)|. \quad (3.4)$$

By applying Lemma 1 to the last term of the above equation, the proof of this lemma is complete.

PROOF of Theorem 1: Define a function $R_k(x,f; A)$ on the interval $[-1, 1]$ such that $R_k(x,f; A)=f(x)-q_k(x,f; A)$, then this function is twice-continuously differentiable with respect to variable x and has zeroes at $x=x_{1k}$, $(=-1)$, and x_{kk}, $(=1)$. Thus $R_k(x,f; A)$ can be regarded as a particular solution to the following boundary value problem;

$$R_k''(x,f; A)=f''(x)-q_k''(x,f; A), \text{ in } x\in[-1, 1],$$
$$R_k(-1,f; A)=R_k(1,f; A)=0.$$

Then, as stated in Section 2, the solution can be written in the form,

$$R_k(x,f; A)=\int_{-1}^{1} G(x, u; -1, 1)\{f''(u)-q_k''(u,f; A)\}du.$$

Now, $R_k(x,f; A)$ can be evaluated from the above equation as follows,

$$|R_k(x,f; A)|$$
$$\leqq \max_{x\in I}\int_{-1}^{1} |G(x, u; A)|du \max_{u\in I}|f''(u)-q_k''(u,f; A)|,$$
$$\leqq \frac{1}{2}\max_{u\in I} |f''(u)-q_k''(u,f; A)|, \quad (3.5)$$

since

$$\max_{x\in I}\int_{-1}^{1} |G(x, u; -1, 1)|du=\frac{1}{2}. \quad (3.6)$$

On the other hand, it is clear that

$$|f''(x)-q_k''(x,f; A)|$$
$$\leqq |f''(x)-p''(x)|+|q_k''(x,f; A)-p''(x)|,$$

for all polynomials $p(x)$, where $p''(x)$ is the second derivative of p. In addition, when $p(x)$ is a polynomial of degree at most $k+1$, it is also true from Properties 1 and 2 that

$$|f''(x)-q_k''(x,f; A)| \leqq |f''(x)-p''(x)|+|q_k''(x,f-p; A)|. \quad (3.7)$$

In the inequality (3.7), we have freedom in the choice of $p(x)$, since $p(x)$ is an arbitrary polynomial of degree at most $k+1$. In other words, we may take a polynomial $p(x)$ such that its second derivative is identical with $Q_{2m-1}(\cos^{-1}x; f'')$ defined for $f''(x)$, where the degree of Q_{2m-1} is at most $k-1$ as $m=[(k+1)/2]$. Then the first term of right hand side in the equation (3.7) can be evaluated by Lemma 1 as follows,

$$|f''(x)-p''(x)|=|f''(x)-Q_{2m-1}(\cos^{-1} x; f'')| \leqq 6\omega\left(\frac{1}{\sqrt{k}}\right), \quad (3.8)$$

for every $x\in I$. The second term is also evaluated by Lemma 2 as follows,

$$\max_{x\in I} |q_k''(x,f-p; A)| \leqq 6M_k(A)\omega\left(\frac{1}{\sqrt{k}}\right), \quad (3.9)$$

Combining equations (3.7), (3.8) and (3.9), the proof of Theorem 1 is complete.

Theorem 1 leads to an interesting result as follows;

THEOREM 2. Let f be a function belonging to $C^2[-1, 1]$. If the triangular matrix A of interpolation points satisfies

$$\lim_{k\to\infty} M_k(A)<\infty, \quad (3.10)$$

then the quasi-(0, 2) interpolated polynomials $q_k(x,f; A)$ converge uniformly to $f(x)$ in the interval $[-1, 1]$ as $k\to\infty$.

PROOF: The proof of this theorem is simple. The difference between $f(x)$ and $q_k(x,f; A)$ can be evaluated by the inequality (3.2) in Theorem 1. On the other hand, since $f''(x)$ is continuous and $M_k(A)$ is bounded for every $k\geq 2$, the right hand side in the inequality converges to zero as $k\to\infty$.

This theorem asserts that the equation (3.10) is a sufficient condition for the uniform convergence of quasi-(0, 2) interpolation process for every twice-continuously differentiable function defined on the interval $[-1, 1]$. We can find an example of the infinite triangular matrix satisfying the sufficient condition. Namely, let L' be the matrix whose k-th row consists of zeroes of the polynomials,

$$(x^2-1)p_{k-1}'(x), \quad (3.11)$$

where $p_{k-1}'(x)$ is the first derivative of the $(k-1)$-st Legendre polynomial. Then, we can show that the matrix L' satisfies the inequality (3.1). In fact,

$$\lim_{k\to\infty}\max_{x\in I}\left\{\sum_{i=1}^{k} L_{ik}(x; L')^2\right\}^{1/2} \leqq 1. \quad (3.12)$$

This inequality was proven by L. Fejér in 1932 [9]. Then, it follows by Theorem 2 that the quasi-(0, 2) interpolated polynomials $q_k(x,f; L')$ for any $f(x)$ belonging to $C^2[-1, 1]$ converge uniformly to f in the interval $[-1, 1]$.

4. Uniform Convergence over A Class of Restricted Functions

In a special case where an infinite triangular matrix has k zeroes of the following polynomial as its k-th row;

$$(x^2-1)p_{k-2}(x), \quad (4.1)$$

where $p_{k-1}(x)$ is an arbitrary $(k-2)$-nd orthogonal polynomial over the interval $(-1, 1)$, it is not easy to prove that the matrix of interpolation points satisfies the uniform convergence condition of Theorem 2 [10]. But if the function to be processed by means of the quasi-(0, 2) interpolation is restricted, then a simpler sufficient condition for the uniform convergence can be derived.

In this section, we shall consider the uniform convergence property of the interpolation process for any function belonging to the class $C^{(\infty)}[-1,1]$: For any $f \in C^{(\infty)}[-1,1]$, we assume

(1) f is an infinite-times differentiable function defined on the interval $[-1,1]$,

(2) there is a bounded constant $C_1 \geqq 0$ such that, for every integer $n \geqq 0$,

$$\sup_{x\in I}\frac{|f^{(n)}(x)|}{n!} \leqq C_1. \tag{4.2}$$

It should be noted that this class contains analytic functions, but the converse is not necessarily true.

In order to describe a theorem, a useful notion is introduced for the matrix A of interpolation points. Let $\Theta_k(x, A)$ be a monic polynomial of degree k whose zeroes consist of all elements of the k-th row of the matrix A. Then the set of these polynomials for $k=2$, $3, \cdots$ forms an infinite sequence, that is, $\{\Theta_i(x; A)\}_{i=2}^{\infty}$. For this sequence, if there exists a real number $\rho>0$ such that for every $i \geqq 2$,

$$\sup_{x\in I}|(i+1)(i+2)\Theta_k(x; A)| \leqq C_2 i^{-\rho}, \tag{4.3}$$

with a bounded constant $C_2>0$, then the matrix A is said to be convergent with order ρ.

The following theorem is an interesting result;

THEOREM 3. Let f be a function belonging to $C^{(\infty)}[-1,1]$ and $k \geqq 2$. If the matrix A of interpolation points is convergent with order $\rho>0$. Then the quasi-(0, 2) interpolated polynomials $q_k(x, f; A)$ converge uniformly to $f(x)$ in the interval $[-1,1]$. In addition, its convergence rate is greater than or equal to $k^{-\rho}$.

PROOF: Let $R_k(x, f; A)=f(x)-q_k(x, f; A)$. This $R_k(x, f; A)$ is an infinite-times differentiable function defined on the interval $[-1,1]$ and its second derivative is zero at the interpolation points, $x_{1k}, x_{2k}, \cdots, x_{kk}$. Thus, under these conditions, the following equation can be derived by Rolle's theorem;

$$R_k''(x, f; A)=\frac{1}{k!}\Theta_k(x; A) f^{(k+2)}(\xi_k), \tag{4.4}$$

where ξ_k is a certain point in the open interval $(-1,1)$. On the other hand, by the definition of $q_k(x, f; A)$, $R_k(x, f; A)$ is zero at $x=x_{1k}$ and x_{kk} where $x_{1k}=-1$ and $x_{kk}=1$. That is,

$$R_k(-1, f; A)=R_k(1, f; A)=0. \tag{4.5}$$

Therefore, equations (4.4) and (4.5) form a two-point boundary value problem for the ordinary differential equation of second order. Then as stated in Section 2, the solution to this problem is written in the form,

$$R_k(x, f; A)=\frac{1}{k!}\int_{-1}^{1} G(x, u; -1, 1)\Theta_k(u; A) f^{(k+2)}(\xi_k)du. \tag{4.6}$$

Now, R_k can be easily evaluated from the equation (4.6) as follows,

$$|R_k(x, f; A)| \leqq \frac{\max_{x\in I}|f^{(k+2)}(x)|}{(k+2)!}\max_{x\in I}|(k+1)(k+2)\Theta_k(x; A)| \times \max_{x\in I}\int_{-1}^{1}|G(x, u; -1, 1)|du, \tag{4.7}$$

for every $x \in I$. By the assumption, there exists a constant C_2 such that for every $k \geqq 2$,

$$\max_{x\in I}|(k+1)(k+2)\Theta_k(x; A)| \leqq C_2 k^{-\rho}. \tag{4.8}$$

Therefore, from equations (3.6), (4.7) and (4.8), a bound of $R_k(x, f; A)$ is given as follows;

$$|R_k(x, f; A)| \leqq \frac{1}{2}C_1 C_2 k^{-\rho}, \tag{4.9}$$

for every $x \in I$, since f belongs to $C^{(\infty)}[-1,1]$. Then the right hand side of the equation (4.9) converges to zero with the rate $k^{-\rho}$ for any $x \in I$. The proof of this theorem is complete.

We can show some examples of matrices of interpolation points that are convergent with order $\rho>0$. As the first example, let T be a matrix of interpolation points whose element of k-th row consists -1, 1 and zeroes of the $(k-2)$-nd orthogonal Chebyshev polynomial over the interval $(-1,1)$. Then the corresponding monic polynomial $\Theta_k(x; T)$ is evaluated as follows; for any $k \geqq 2$,

$$\sup_{x\in I}|(k+1)(k+2)\Theta_k(x; T)| \leqq \frac{96}{(k+1)}.$$

Thus the order of this matrix T is 1. On the other hand, in the case of the matrix L' discussed in the previous section, the evaluation of the corresponding monic polynomial $\Theta_k(x; L')$ is as follows;

$$\sup_{x\in I}|(k+1)(k+2)\Theta_k(x; L')| \leqq \frac{192}{(k+3)},$$

for any $k \geqq 2$. The order of the matrix L' is also 1. As the last example, we shall investigate the matrix L of interpolation points whose elements of its k-th row consist of -1, 1 and zeroes of the $(k-2)$-nd Legendre polynomial. In this case, the corresponding monic polynomial $\Theta_k(x; L)$ is evaluated as

$$\sup_{x\in I}|(k+1)(k+2)\Theta_k(x; L)| \leqq \frac{126\sqrt{2}}{\sqrt{k-3}},$$

for every $k>3$. Therefore, it shows that the order of this matrix L is 1/2.

Note that the above inequalities can be derived by simple computations.

5. Formula for Numerical Computation

This section is devoted to discussing an algorithm for computing values of the quasi-(0, 2) interpolation at any point $x^* \in [-1,1]$. An essential part of the computations is the evaluations of polynomials $\{s_{ik}(x; A)\}_{i=1}^{k}$ at the point x^* and remaining parts are very simple.

Thus, the algorithm of computations is discussed only for polynomials $s_{ik}(x; A)$.

By employing the equation (2.5), the value of $s_{ik}(x^*; A)$ is obtained by numerical computation. However it is expensive from points of view of the computational accuracy and its processing time to carry out directly the integration contained in the equation.

In order to derive numerically computable formula of the polynomial $s_{ik}(x; A)$, suppose that $s_{ik}(x; A)$ is expanded as follows,

$$s_{ik}(x; A)=(x-x_{1k})(x-x_{kk})\sum_{j=0}^{k-1} C_j(i, k)x^j, \quad (5.1)$$

where $C_j(i, k)$ are coefficients defined by i and interpolation points, $x_{1k}, x_{2k}, \cdots, x_{kk}$. These coefficients are determined as the polynomial (5.1) satisfies conditions (3) and (4) in section 2. However, $C_j(i, k)$ are, actually, defined only by the condition (4) since the polynomial (5.1) is already satisfying the condition (3) since it has common factors $(x-x_{1k})$ and $(x-x_{kk})$. In other words, coefficients $C_j(i, k)$ are determined as the second order derivative $s''_{ik}(x; A)$ of the polynomial (5.1) satisfying the following equation;

$$s''_{ik}(x)=L_{ik}(x; A), \quad (5.2)$$

for all real numbers x, where $L_{ik}(x; A)$ is defined by the equation (2.3) and can also be expanded in the form with power series of x as follows,

$$L_{ik}(x; A)=\sum_{j=0}^{k-1} \alpha_j(i, k; A)x^j, \quad (5.3)$$

where,

$$\alpha_j(i, k; A)=\frac{\Sigma[x_{1k}, x_{2k}, \cdots, x_{kk}]_i^{k-j-1}}{\prod_{p=1,\neq i}^{k} (x_{ik}-x_{pk})}, \quad (5.4)$$

In the equation (5.4), $[x_{1k}, x_{2k}, \cdots, x_{kk}]_i^{k-j-1}$ means all products of $k-j-1$ different elements chosen within the set $\{x_{1k}, \cdots, x_{i-1,k}, x_{i+1,k}, \cdots, x_{kk}\}$ of interpolation points except for x_{ik} and Σ denots the total sum over all numbers of its combination, $k!/\{(k-j-1)!(j+1)!\}$. Therefore, if the left hand side in the equation (5.2) is replaced by the second order derivative of the polynomial (5.1) and the right hand side is replaced by the equation (5.3), then the following equation is obtained.

$$\begin{aligned}&\{(k-1)^2+3(k-1)+2\}C_{k-1}(i, k)x^{k-1}\\ &+\{(k-2)^2+3(k-2)+2\}C_{k-2}(i, k)x^{k-2}\\ &+\sum_{j=0}^{k-3} \{(j^2+3j+2)C_j(i, k)-(j+1)(j+2)C_{j+2}(i, k)\}x^j\\ &\qquad=\sum_{j=0}^{k-1} \alpha_j(i, k; A)x^j, \quad (5.5)\end{aligned}$$

By comparing coefficients of powers of the same order in both sides of the above equation, we can obtain the following recurrence equation for coefficients $C_j(i, k)$,

$$C_j(i, k)=C_{j+2}(i, k)+\frac{1}{j^2+3j+2}\alpha_j(i, k; A), \quad (5.6)$$

for $j=k-3, k-4, \cdots, 1, 0$, where for $j=k-1$ and $k-2$

$$C_{k-1}(i, k)=\frac{1}{(k-1)^2+3(k-1)+2}\alpha_{k-1}(i, k; A), \quad (5.7)$$

and

$$C_{k-2}(i, k)=\frac{1}{(k-2)^2+3(k-2)+2}\alpha_{k-2}(i, k; A). \quad (5.8)$$

The equation (5.5) is valid for all real numbers x if and only if coefficients $C_j(i, k)$ satisfy equations (5.6), (5.7) and (5.8).

Finally, it is easy to solve numerically the recurrence equation (5.6) for initial conditions $C_{k-1}(i, k)$ and $C_{k-2}(i, k)$. Then values of all polynomials $\{s_{ik}(x; A)\}_{i=1}^k$ at any point $x^*\in[-1, 1]$ can be immediately computed by using the form of the equation (5.1). Therefore, the quasi-(0, 2) interpolation can be computed at any point in the interval $[-1, 1]$, since values of two remaining polynomials $r_{1k}(x; A)$ and $r_{kk}(x; A)$ of basis at the point are easily computed.

The present algorithms constructed under the interpolation points corresponding to matrices L, L' or T seem to be stable, judging from several numerical experiments, even though more detailed error analyses are left to be done.

6. Conclusions

An explicit quasi-(0, 2) interpolation has been constructed and two sufficient conditions for its convergence have been obtained: Theorem 2 is applicable to any twice-continuously differentiable function defined over the interval $[-1, 1]$ and it asserts that the uniform convergence of quasi-(0, 2) interpolation process is assured over matrices A of interpolation points, whenever the inequality (3.10) is satisfied for A.

Theorem 3 can be applied to the class of functions satisfying conditions (1) and (2) in Section 4. Namely, this class is smaller than the class of twice-continuously differentiable functions, but broader than that of analytic functions. In such a class, whenever the matrix A of interpolation points has the convergence rate of order $\rho>0$ as is discussed in Section 4, the uniform convergence of the interpolation process is assured over A.

A computational algorithm for the quasi-(0, 2) interpolation was constructed in Section 5. That is, by employing basis polynomials represented in the equation (5.1), the interpolation is easily computed at any point belonging to the interval $[-1, 1]$.

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