# Topology Preserving Parallel Operations on Hexagonal Lattice Points

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A group of parallel local operations with asymmetric windows working on binary images are considered. A specific condition for those operations, called L-condition, which requires not only the quasi-preservation of topological structure of binary pictures but also the ultimate annihilation of all components except for the background, is formulated in terms of adjacency relation among the components of pictures. The operation defined by the majority function of three-argument is proved to be the one and only one operation which satisfies L-condition on the hexagonal lattice.

## 1. Topology Preserving Operation

It is well known that certain types of local parallel operations preserve some properties of binary pictures which are defined as subsets of the set of lattice points in two dimensional space [2, 5]. Binary pictures are usually represented by *characteristic functions* whose domain is the set of all lattice points in question, and whose range is {0, 1}, '1' and '0' indicating whether a lattice point is included in a picture or not, respectively. Operations on binary pictures essentially calculate new characteristic functions from old ones. By the term 'parallel' we mean that the values of new function on all lattice points are determined simultaneously from the values of old function.

Among the various properties of binary pictures, their topology has been considered important because the basic structures of the pictures are represented by it. These topologies have been defined by introducing adjacency relation among the points [3]. One of the simplest topologies is the *adjacency tree* [4]. The nodes of the adjacency tree correspond to connected 1- or 0-components of the picture. Two nodes are connected by an arc when the corresponding two components are adjacent. The root of the adjacency tree corresponds to the (possibly) infinite component of the background of the picture. It has been proved that the graph thus defined is actually a tree [4].

There is a group of local parallel operations, called *shrinking*. A shrinking operation preserves the topology of pictures in terms of the adjacency tree. Repeated application of a shrinking operation changes a binary picture from its original forms into the simplest form which has the same adjacency tree as the original. Shrinking operation use, in general, some small symmetric windows for the 'local' calculation. In the case of the square lattice space, $3\times 3$ or $5\times 5$ windows, the target point being located at their centers, are often used.

Local parallel operations with asymmetric windows have also been studied, among which another type of topology preserving operation is included [1]. This operation, which we will name L-operation for the name of its inventor, also preserves the adjacency tree of pictures *except for* the annihilation of single-point-components. It is this annihilation that enables L-operation to shrink any non-background component to a single-point (and then to annihilate it) even if the original component is not simply connected. The adjacency tree of the picture is gradually simplified by the deletions of single point nodes until only the background node remains.

The purpose of this paper is to determine a necessary and sufficient condition for an operation with the same window as L-operation to perform a topology preserving shrinking in the above meaning. For simplicity, operations on the hexagonal lattice are concerned. The same problem on the conventional square lattice would be more complex because of the difference of cardinalities of possible parallel operations on both types of lattices, i.e., $2^{2^3}$ to $2^{2^4}$.

## 2. Definitions

Let $I\times I$ be the set of pairs of integers. A *point* is a member of this set, denoted by the form $(i, j)$. The six points

$$(i-1, j-1), (i-1, j), (i, j-1),$$
$$(i, j+1), (i+1, j), (i+1, j+1)$$

are called the *neighbor* of the point $p=(i, j)$, and also are said *adjacent* to $p$. Two subsets $S_1$ and $S_2$ of $I\times I$ are *adjacent* if and only if there exist two points $s_1\in S_1$ and $s_2\in S_2$ such that $s_1$ is adjacent to $s_2$. The concepts of *path*, *connectedness*, and *component* are defined in the conventional way [3]. A 1- or 0-component $C$ is said *singular* if it consists of only one point. We assume that there is one and only one component whose size is infinite. We call it the *background* component.

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As mentioned before, we use the characteristic function scheme for the representation of pictures. Let a point name, say $p$, represent the value (0 or 1) of the characteristic function at the point as well as the point itself. Using this convention, we define our parallel operations by 3-argument binary functions as

$$a_{ij} \leftarrow f(a_{ij}, a_{i-1,j-1}, a_{i-1,j})$$

where $a_{ij}=(i, j)$. The geometrical interpretation of this definition is shown in Fig. 1 in which an oblique coordinate system is used. Let $p^*$ denote the value of $p$ *after* a single application of an operation.

Given a 1-component $C$, we define the following 3 sets derived from $C$.

$$C^0=\{p|p \in C, p^*=1\},$$
$$C^-=\{p|p \in C, p^*=0\} \text{ and}$$
$$C^+=\{p|p \notin C, p^*=1, \alpha(p)_\cap C \neq \phi\}$$

where

$$\alpha(a_{ij})=\{a_{i-1,j-1}, a_{i-1,j}\}.$$

The new set corresponding to $C$ is now defined as

$$C^*=C^0 \cup C^+.$$

It is worth noting that $C^*$ may not be a 1-component though $C$ is. For a 0-component $D$, the derived sets $D^0$, $D^-$, $D^+$, and $D^*$ are similarly defined.

We finally define the condition which characterizes the L-operation as follows. We will call it L-condition.

*L-condition*

L1: The adjacency tree of the picture *after* a single application of the operation is identical to what it was *before* the operation, except for the changes due to L2.

L2: Simply connected components may be annihilated by a single application of the operation.

L3: The 'lifetime' of all components except for the background are finite, i.e., we can erase all non-background components by applying the operation a sufficient number of times.

Subcondition L1 alone characterizes the conventional shrinking operations. L2 and L3 are the characteristic features of L-operation. L3 is important for L-condition, because the identity operation satisfies both L1 and L2.

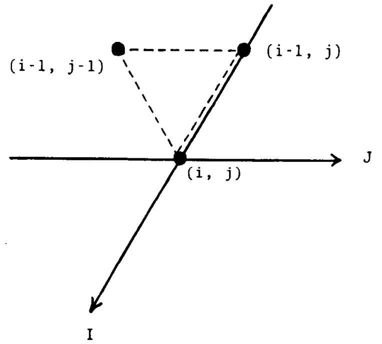


Fig. 1 Argument configuration of operations.

## 3. Necessary Condition

We will use several 'test patterns' in order to obtain the set of subcondition of the necessary condition for an operation $f$ to satisfy L-condition. In what follows, we will denote $f$ by the set of 8 values

$$f_0, f_1, \cdots, f_7$$

where

$$f(a_{ij}, a_{i-1,j-1}, a_{i-1,j}) \text{ is denoted by}$$
$$f_{a_{ij}+2a_{i-1,j-1}+4a_{i-1,j}}.$$

*Condition N1*

Subconditions L1 and L2 imply that no new components can be created. Then it is seen that

$$f_0=0$$

and

$$f_7=1.$$

*Condition N2.*

Considering L3 together with the 'isolated point' pattern, it is seen that

$$f_1=0$$

and

$$f_6=1.$$

If $f_1$ is 1, for example, the original pattern with only one single-point component will never be converted to uniform background even if various patterns are generated *below* the point.

*Condition N3.*

Let us consider a pattern in which only the two points $a_{i-1,j-1}$ and $a_{i-1,j+1}$ are '1', while the rest are '0's (Fig. 2(a)). After the operation, the configulation of values is changed as Fig. 2(b). Note that we use the conditions N1 and N2, i.e.,

$$f_0=f_1=0.$$

From the facts that, in Fig. 2(b),

$$a_{ij}=f_2, \quad a_{i,j+1}=f_4, \text{ and}$$
$$a_{ij} \text{ is adjacent to } a_{i,j+1},$$

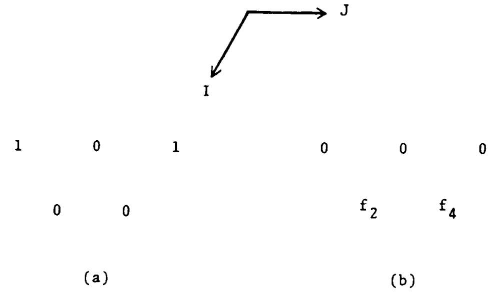


Fig. 2 Two point pattern. Before (a) and after the operation (b).

it is seen that only one of $f_2$ and $f_4$ can be 1. It is because subconditions L1 and L2 say that no two components should be merged into one. Similarly, only one of $f_3$ and $f_5$ can be 0.

*Condition N4.*

Consider again the 'isolated point' pattern. It is easily seen that, if either one of $f_2$ and $f_4$ is 1, the isolated point will travel downward either left ($f_4=1$) or right ($f_2=1$) forever. Then, from L3, it is seen that

$$f_2=f_4=0.$$

Similarly

$$f_3=f_5=1.$$

We summarize N1 to N4 as

$$f_0=f_1=f_2=f_4=0,$$
$$f_7=f_6=f_5=f_3=1.$$

In other words, $f$ must be the simple majority function of three arguments.

In the original paper[1], L-operation is defined on the square lattice. If we are to consider L-operation on the hexagonal lattice, taking account of the unique connectivity, say 6-connectivity, we actually obtain the 'majority' operation derived above. That is, only the operation defined by the majority function *can* satisfy L-condition.

## 4. Sufficient Condition

In this section we prove that the operation derived above satisfies L-condition. Let $L_0$ be the operation defined by the majority function. We first prove that $L_0$ satisfies the comparatively simple subconditions L2 and L3 of L-condition.

*Proposition 1*

A non-singular component $S$ is not annihilated by a single application of $L_0$.

*Proof*

Let $s$ be a point in $S$. Then there is another point $t$ in $S$, which is adjacent to $s$. Suppose that $s^*=t^*=0$. In this case it is seen that

$$s \notin \alpha(t) \text{ and } t \notin \alpha(s),$$

i.e., $s$ and $t$ are horizontally adjacent. Then the point $p$ such that $\alpha(p)=\{s, t\}$ is seen to be included in $S^*$ because

$$f_6=f_7=1$$

in $L_0$. If either $s^*$ or $t^*$ or both is 1, on the other hand, $S^*$ is not empty. In any case, it is seen that $S^*$ is non empty.

Multiply connected components are obviously non-singular. Then, from Proposition 1, $L_0$ is proved to satisfy L2.

*Proposition 2*

Given a picture, possibly consisting of many components, there exists an integer $n$ such that at most $n$ applications of $L_0$ to the picture erase all components except for the background.

*Proof*

We will make a proof for the case of 0-background. The other case (1-background) can similarly be proved by reversing the 'polarity'. If there is no component in the picture, $n=0$. Otherwise, consider the following three values.

$$a=\max\,(j|(i,j) \notin \text{background}),$$
$$b=\min\,(i|(i,j) \notin \text{background}),$$
$$c=\max\,(i-j|(i,j) \notin \text{background}).$$

The three lines

$$j=a,\ i=b, \text{ and } i-j=c$$

surround the triangular area in which 1-components exist (Fig. 3). When $L_0$ is applied once,

$a$ is not increased because $f_0=f_2=0$,

$c$ is not increased because $f_0=f_4=0$,

$b$ is increased at least 1 because $f_0=f_1=0$.

Then, it is seen that the surrounding triangle will collapse to a single point by applying $L_0$ at most $c+a-b$ times. Then $n=c+a-b+1$ is seen.

Proposition 2 shows that $L_0$ satisfies L3.

Regarding L1, all of the following subconditions must be examined.

1) If $S$ is a non-singular component, $S^*$ is a connected component.

2) If $S_1$ and $S_2$ are different components, $S_1^*$ and $S_2^*$ are disjoint and not adjacent (including the cases that one or both of $S_1$ and $S_2$ is singular).

3) If $C$ and $D$ are non-singular 1- and 0-component, respectively, and if $C$ is adjacent to $D$, $C^*$ is adjacent to $D^*$.

We omit the details of the proof that $L_0$ satisfies L1 because of its combinatorial nature, only showing the series of lemmas for the proof. It is worthwhile, however, to note that the proofs are by no means trivial because

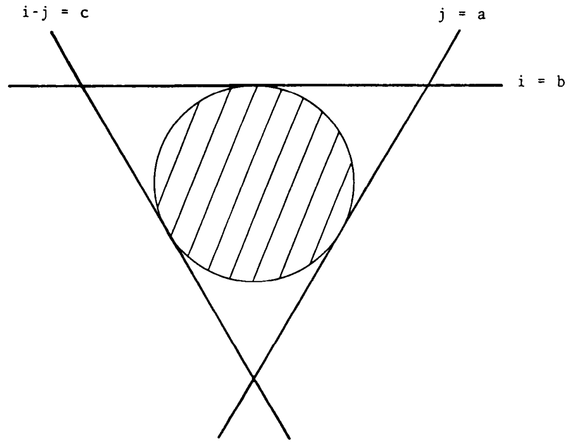


Fig. 3 Surrounding triangle.

operation $L_0$ is not monotonous, i.e., 1-setting of 0-points as well as 0-setting of 1-points occur.

*Lemma 1*

Given a component $C$, let $xyz$ be a path which satisfies the conditions,

$$x \in C^*, y \in C, z \in C \cup C^*, \text{ and}$$
$$\text{if } x \in C^+ \text{ and } y \in C^- \text{ then } y \in \alpha(x).$$

Then there exists a path of the form $xw_n \cdots w_1 z$ ($n=0$, 1, or 2) such that

$$w_i \in C^*, \text{ and}$$
$$\text{if } w_1 \in C^+ \text{ and } z \in C^- \text{ then } z \in \alpha(w_1).$$

(Only the case $y \in C^-$ is non-trivial.)

*Lemma 2*

Given a non-singular component $C$, $C^*$ is a connected component.

(Paths in $C$ which connect points of $C^*$ can be micro-converted into those in $C^*$ using Lemma 1.)

*Lemma 3*

If $S_1$ and $S_2$ are non-singular, different components, $S_1^*$ and $S_2^*$ are non-adjacent, disjoint components.

(The contradiction $S_1 = S_2$ will be derived if this lemma is supposed not to hold.)

*Lemma 4*

Adjacency of a non-singular 1-component $C$ and a non-singular 0-component $D$ is preserved between $C^*$ and $D^*$. (We use the fact proved in [4] that either $C$ or $D$ surrounds the other component. The proof of this lemma is made using an extremal point of surrounded component.)

From Lemma 1 to 4, we can show the correctness of the following proposition.

*Proposition 3*

$L_0$ satisfies the subcondition L1 of L-condition.

## 5. Conclusion

With conditions N1 to N4 and Propositions 1 to 3, we can complete the proof of the following theorem.

*Theorem*

Operation $L_0$ defined by the majority function of 3-argument is the one and only one parallel operation on the hexagonal lattice, which satisfies L-condition.

*Remarks*

The hexagonal lattice has the property that the connectivity is defined simply and uniquely. On the square lattice, on the other hand, we usually have to handle two types of connectivity separately for 0- and 1-components. That is one reason why the combinatorial proofs on the square lattice is more complicated than on the hexagonal lattice. Since L-operation was first defined on the square lattice, our next objective will be to attack L-operation on a square lattice. The majority function will also play an important role in this case. The 4-argumentness, however, will present other kind of problems in the proof.

### References

1. LEVIALDI, S. On Shrinking Binary Picture Patterns, *Comm. ACM* **15**, 1, (1972), 7–10.
2. STEFANELLI, R. and ROSENFELD, A. Some parallel Thinning Algorithms for Digital Pictures, *J. ACM* **18**, (1971), 255–264.
3. ROSENFELD, A. Connectivity in Digital Pictures, *J. ACM* **17**, (1970), 146–160.
4. ROSENFELD, A. Adjacency in Digital Pictures, *Inf. Control* **26**, (1974), 24–33.
5. ROSENFELD, A. A Characterization of Parallel Thinning Algorithms, *Inf. Control* **29**, (1975), 286–291.

(Received June 9, 1981)