Superconvergence Estimates at Jacobi Points of the Collocation-Galerkin Method for Two Point Boundary Value Problems

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In this paper, we consider some error estimates of a collocation-Galerkin method for two point boundary value problems. It is shown that the errors at certain Jacobi points are $O(h^{r+2})$, where h is the maximal size of the partitioned intervals and r is the degree of used polynomials, which is one order higher than the global optimal error. A numerical example which confirms these results is presented.

1. Introduction

The collocation-Galerkin method which was first introduced by Diaz [4] is a mixed technique of a collocation method and a Galerkin method. The scheme enables us to reduce the number of quadratures in the Galerkin method and to obtain higher accuracy than the collocation methods. The optimal global rates of convergence and superconvergence at the mesh points for the scheme have already been established ([1], [4], [9]).

On the other hand, for various Galerkin approximations of the two point boundary value problems, it is known that the approximate solutions are superconvergent at some particular points ([2], [3], [7], [8]). In this paper, we shall prove that the collocation-Galerkin approximation has $O(h^{r+2})$ convergence at certain Jacobi points, where h and r are mesh size and degree of the used approximation piecewise polynomials, respectively. That is one order better than the global rate of convergence. We shall also give a numerical example which illustrate the superconvergence phenomena.

2. Problem and Notations

Consider the following two point boundary value problem.

$$Lu \equiv -u'' + a(x)u' + b(x)u = f,\ x \in I,$$
$$u(0) = u(1) = 0, \tag{1}$$

where $I = (0, 1)$ and assume that, $a, b \in C^1(I)$. We assume that (1) has a unique solution for every $f \in C(I)$.

Let $\Delta: 0 = x_0 < x_1 < \cdots < x_N = 1$, be a partition of I and let $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$ and $h = \max_{1 \leqq i \leqq N} h_i$. For a positive integer k and $E \subset I$, let $P_k(E)$ denote the set of polynomials on E of degree at most k. Furthermore, let

$$P_k^0(I_i) = \{v \in P_k(I_i) \mid v(x_{i-1}) = v(x_i) = 0\},$$
$$\mathscr{M}_0^k(\Delta) = \{v \in C(I) \mid v \in P_k(I_i),\ 1 \leqq i \leqq N,\ v(0) = v(1) = 0\},$$
$$Z_0^k(\Delta) = \{v \in \mathscr{M}_0^k(\Delta) \mid v(x_i) = 0,\ 0 \leqq i \leqq N\}.$$

Here the symbol Δ will be usually omitted. We define for $m \geqq 0$ and $1 \leqq p \leqq \infty$

$$W_p^m(E) = \left\{ \psi \,\middle|\, \frac{d^l \psi}{dx^l} \in L^p(E),\ 0 \leqq l \leqq m \right\}$$

and

$$\|\psi\|_{W_p^m(E)} = \sum_{l=0}^{m} \left\| \frac{d^l \psi}{dx^l} \right\|_{L^p(E)}.$$

Especially, denote $W_2^m(E)$ by $H^m(E)$. Also we define

$$\|\psi\|_{W_p^{-m}(E)} = \sup_{0 \neq g \in W_q^m(E)} \frac{|(\psi, g)_E|}{\|g\|_{W_q^m(E)}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Here $(\cdot, \cdot)_E$ implies the L^2 inner product on E. When $E = I$, we suppress the dependency on the interval. In the following we use C to denote a generic positive constant which is not necessarily the same.

Now we state some well-known inequalities for later use ([4]). Let $\phi \in P_k(I_i)$ for some $k > 0$, then there exists a constant C, independent of h_i, such that

$$\|\phi\|_{L^\infty(I_i)} \leqq C h_i^{-1/p} \|\phi\|_{L^p(I_i)}, \tag{2}$$
$$\|\phi'\|_{L^p(I_i)} \leqq C h_i^{-1} \|\phi\|_{L^p(I_i)}, \tag{3}$$

where $1 \leqq p \leqq \infty$.

3. Collocation-Galerkin Method

In this section we define, based on Diaz [4], a collocation-Galerkin approximation to (1). In order to do so, first we introduce the Jacobi points. From now on we fix an integer r ($\geqq 2$). Let

$$J_r(x) = \frac{1}{\alpha x(1-x)} \frac{d^{r-1}}{dx^{r-1}} [x^r (1-x)^r],$$

where α is a constant chosen so that the coefficient of x^{r-1} in $J_r(x)$ is 1. $J_r(x)$ has $r-1$ distinct roots ρ_j on I. Then the Jacobi points x_{ij} on the subinterval I_i are

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defined by

$$x_{ij}=x_{i-1}+\rho_j h_i,\quad 1\leqq j\leqq r-1,\quad 1\leqq i\leqq N.$$

It is well-known that there exist $r-1$ positive constants ω_j such that for arbitrary $\phi\in P_{2r-3}(I)$

$$\int_I x(1-x)\phi(x)dx=\sum_{j=1}^{r-1}\omega_j\phi(\rho_j). \tag{4}$$

We now define the collocation-Galerkin approximation to (1) by $U\in\mathcal{M}_0^r$ satisfying the following equations:

$$LU(x_{ij})=f(x_{ij}),\quad 1\leqq j\leqq r-1,\quad 1\leqq i\leqq N, \tag{5}$$

and

$$B(U,v)=(f,v),\quad v\in\mathcal{M}_0^1, \tag{6}$$

where x_{ij} are Jacobi points determined above, $(\cdot,\cdot)$ is L^2 inner product on I and the bilinear form $B(U,v)$ is defined by

$$B(U,v)=(U',v')+(aU'+bU,v).$$

Now, (5) and (6) can be represented as follows in a semidiscrete bilinear form. First, for g and ϕ defined on each I_i and $\phi(x_{i-1})=\phi(x_i)=0$, we define a discrete bilinear form

$$\langle g,\phi\rangle_i=h_i\sum_{j=1}^{r-1}\omega_j\frac{g(x_{ij})\phi(x_{ij})}{\rho_j(1-\rho_j)},$$

and set

$$\langle g,\phi\rangle=\sum_{i=1}^{N}\langle g,\phi\rangle_i.$$

Next, using the unique decomposition $\mathcal{M}_0^r=Z_0^r\oplus\mathcal{M}_0^1$, we define a semidiscrete bilinear form $\mathcal{L}(\cdot,\cdot)$ on $(\bigcap_{i=1}^{N}C^2(I_i))\times\mathcal{M}_0^r$ by

$$\mathcal{L}(g,v)=\langle Lg,v_1\rangle+B(g,v_2),$$

where $v=v_1+v_2$ such that $v_1\in Z_0^r$ and $v_2\in\mathcal{M}_0^1$. Then, it is easily seen that one can rewrite (5) and (6) as the following equivalent form.

$$\mathcal{L}(U,v)=\langle f,v\rangle,\quad v\in\mathcal{M}_0^r. \tag{7}$$

Notice that if u is a solution to (1) then we have

$$\mathcal{L}(u,v)=\langle f,v\rangle,\quad v\in\mathcal{M}_0^r. \tag{8}$$

We now state some already known estimates ([4], [9]). Let u and U be solutions to (1) and (7), respectively. Then, for sufficiently small h,

$$\|u-U\|_{L^2}+h\|u-U\|_{H^1}\leqq Ch^{r+1}\|u\|_{H^{r+1}}, \tag{9}$$

$$\|u-U\|_{L^\infty}\leqq Ch^{r+1}\|u\|_{W_\infty^{r+1}}, \tag{10}$$

$$|(u-U)(x_i)|\leqq Ch^{2r}\|u\|_{W_\infty^{2r}},\quad 1\leqq i\leqq N-1, \tag{11}$$

where C is independent of h.

4. Superconvergence Estimates at Jacobi Points

In this section, we show that the approximate solution defined by (7) has $O(h^{r+2})$ rates of convergence at specific points (=Jacobi points) on each subinterval. First, for $y\in H^1(I)$, we define an approximation $Y\in P_{r+1}(I_i)$ of y on each I_i by

$$((y-Y)',w')_{I_i}=0,\quad w\in P_{r+1}^0(I_i), \tag{12}$$

and

$$Y(x_{i-1})=y(x_{i-1}),\ Y(x_i)=y(x_i). \tag{13}$$

Then from the well-known result of approximation theory, we have for $1\leqq p\leqq\infty$ and $0\leqq s\leqq r+2$

$$\|y-Y\|_{W_p^s(I_i)}\leqq C_p h_i^{r+2-s}\|y\|_{W_p^{r+2}(I_i)}. \tag{14}$$

Next, let $W\in\mathcal{M}_0^r$ be the collocation-Galerkin interpolation of $y\in(\bigcap_{i=1}^{N}C^2(I_i))\cap\{y(0)=y(1)=0\}$, i.e. W be the solution of the equation:

$$\langle -(y''-W''),v_1\rangle+((y-W)',v_2')=0,\ v\in\mathcal{M}_0^r, \tag{15}$$

where $v=v_1+v_2$ such that $v_1\in Z_0^r$ and $v_2\in\mathcal{M}_0^1$.

The following lemma is essential to our main result.

Lemma 1. Let Y and W be the solutions to (12)–(13) and (15), respectively. If $y\in W_\infty^{r+2}(I_i)$, $1\leqq i\leqq N$, then there exists a constant C, independent of h, such that

$$|(W-Y)(x_{ij})|\leqq Ch_i^{r+2}\|y\|_{W_\infty^{r+2}(I_i)},\quad 1\leqq j\leqq r-1,\quad 1\leqq i\leqq N.$$

Proof. First, for each x_i, $1\leqq i\leqq N$, we take a function $G(x_i,\cdot)\in\mathcal{M}_0^1$ such that

$$G(x_i,\xi)=\begin{cases}(1-x_i)\xi, & 0\leqq\xi\leqq x_i,\\ x_i(1-\xi), & x_i\leqq\xi\leqq 1.\end{cases} \tag{16}$$

Notice that $G(x,\xi)$ is Green's function for the operator $L=-(d^2/dx^2)$. Then, (15) and (16) yield

$$(y-W)(x_i)=((y-W)',\ G_\xi(x_i,\cdot))=0. \tag{17}$$

Next, for each k, $1\leqq k\leqq r-1$, we choose $w_k\in P_r^0(I_i)$ such that

$$w_k''(x_{ij})=\begin{cases}1, & j=k,\\ 0, & j\neq k.\end{cases} \tag{18}$$

Notice that $w_k''\cdot(W-Y)\in P_{2r-1}^0(I_i)$ because of (17). Hence, by (18), (4), (13), and (15) we have

$$\begin{aligned}h_i\cdot\frac{\omega_k}{\rho_k(1-\rho_k)}(W-Y)(x_{ik})&=\langle w_k'',W-Y\rangle_i\\&=(w_k'',W-Y)_{I_i}\\&=((W-Y)'',w_k)_{I_i}\\&=\langle(W-Y)'',w_k\rangle_i\\&=\langle(y-Y)'',w_k\rangle_i.\end{aligned} \tag{19}$$

Thus, we obtain by (14)

$$\begin{aligned}|(W-Y)(x_{ik})|&\leqq Ch_i^{-1}|\langle(y-Y)'',w_k\rangle_i|\\&\leqq C\|y-Y\|_{W_\infty^2(I_i)}\|w_k\|_{L^\infty(I_i)}\\&\leqq Ch_i^{r+2}\|y\|_{W_\infty^{r+2}(I_i)},\end{aligned} \tag{20}$$

where we have used the following estimates:

Using the fact that $w_k(x_{i-1})=w_k(x_i)=0$ and the norm equivalency of finite dimensional spaces, by (18)

$$\begin{aligned}\|w_k\|_{L^\infty(I_i)}&\leqq h_i^{1/2}\|w_k'\|_{L^2(I_i)}\\&\leqq Ch_i\|w_k'\|_{L^\infty(I_i)}\\&\leqq Ch_i^2\|w_k''\|_{L^\infty(I_i)}\end{aligned}$$

$$\le Ch_i^2 \max_{1\le j\le r-1} |w_k''(x_{ij})|$$
$$\le Ch_i^2.$$

Therefore, we can conclude the proof.

Now we describe the main result of this paper.

Theorem 1. Let u and U be the solutions to (1) and (5)–(6), or equivalently (7), respectively. If $u \in W_\infty^{r+2}(I)$, then, for sufficiently small h, there exists a constant C, independent of h, such that

$$|(u-U)(x_{ij})| \le Ch^{r+2}\|u\|_{W_\infty^{r+2}}, \quad 1\le j\le r-1, \quad 1\le i\le N.$$

Proof. Let Y and W be the solutions to (12)–(13) and (15) with $y\equiv u$, respectively. By using the inequality

$$|(u-U)(x_{ij})| \le |(u-Y)(x_{ij})| + |(Y-W)(x_{ij})| + |(W-U)(x_{ij})|,$$

by (14) and Lemma 1, it is enough to prove

$$\|U-W\|_{L^\infty} \le Ch^{r+2}\|u\|_{W_\infty^{r+2}}. \tag{21}$$

Now, consider the following dual problem for each $\psi \in L^1(I)$.

$$\begin{cases} L^*\phi=\psi, & x\in I, \\ \phi(0)=\phi(1)=0, \end{cases} \tag{22}$$

where L^* denotes the formal adjoint operator of L, i.e. $L^*\phi \equiv -\phi''-(a\phi)'+b\phi$. Let $\eta=U-W$ and $\xi=u-W$, then for any $\hat{\phi}\in \mathscr{M}_0^r$ we have by (7) and (8)

$$\begin{aligned}(\eta,\psi)&=(\eta, L^*\phi)\\ &=B(\eta,\phi)\\ &=B(\eta,\phi-\hat{\phi})+B(\eta,\hat{\phi})\\ &=B(\eta,\phi-\hat{\phi})+\{B(\eta,\hat{\phi})-\mathscr{L}(\eta,\hat{\phi})\}+\mathscr{L}(\xi,\hat{\phi}).\end{aligned} \tag{23}$$

We choose $\hat{\phi}$ to be the Galerkin approximation of (22), i.e.

$$B(v, \phi-\hat{\phi})=0, \quad v\in \mathscr{M}_0^r. \tag{24}$$

We now estimate the second and third term of the last right hand side in (23). First, we decompose $\hat{\phi}$ as $\hat{\phi}_1+\hat{\phi}_2$, where $\hat{\phi}_1 \in Z_0^r$ and $\hat{\phi}_2\in \mathscr{M}_0^1$. We note that, by virtue of $\eta''\cdot\hat{\phi}_1 \in P_{2r-1}^0(I_i)$, (4) implies $(\eta'',\hat{\phi}_1)_{I_i}=\langle\eta'',\hat{\phi}_1\rangle_i$. Using (2) and (3) we have

$$\begin{aligned}|B(\eta,\hat{\phi})-\mathscr{L}(\eta,\hat{\phi})| &\le \sum_{i=1}^N |-(\eta'',\hat{\phi}_1)_{I_i}+(a\eta'+b\eta,\hat{\phi}_1)_{I_i}\\ &\quad +\langle\eta'',\hat{\phi}_1\rangle_i-\langle a\eta'+b\eta,\hat{\phi}_1\rangle_i|\\ &\le C\sum_{i=1}^N h_i\|\eta\|_{W_\infty^1(I_i)}\|\hat{\phi}_1\|_{L^\infty(I_i)}\\ &\le C\sum_{i=1}^N h_i^3\|\eta\|_{W_\infty^1(I_i)}\|\hat{\phi}_1''\|_{L^\infty(I_i)}\\ &\le Ch\|\eta\|_{L^\infty}\|\hat{\phi}_1\|_{W_1^2}.\end{aligned}$$

Now from the well known estimates for the Galerkin approximation (e.g. [6])

$$\|\hat{\phi}_1\|_{W_1^2}\le C\|\hat{\phi}\|_{W_1^2}\le C\|\phi\|_{W_1^2}.$$

Thus, noting that $\|\phi\|_{W_1^2}\le C\|\psi\|_{L^1}$, we obtain

$$|B(\eta,\hat{\phi})-\mathscr{L}(\eta,\hat{\phi})|\le Ch\|\eta\|_{L^\infty}\|\psi\|_{L^1}. \tag{25}$$

Next, we will estimate the third term in (23). Using the above decomposition of $\hat{\phi}$, by (15) we have

$$\mathscr{L}(\xi,\hat{\phi})=\sum_{i=1}^N \langle a\xi'+b\xi,\hat{\phi}_1\rangle_i+(a\xi'+b\xi,\hat{\phi}_2). \tag{26}$$

Here, the first term is estimated, by the results in [9], as follows:

$$\begin{aligned}\left|\sum_{i=1}^N \langle a\xi'+b\xi,\hat{\phi}_1\rangle_i\right| &\le C\sum_{i=1}^N h_i\|\xi\|_{W_\infty^1(I_i)}\|\hat{\phi}_1\|_{L^\infty(I_i)}\\ &\le C\|\xi\|_{W_\infty^1}\sum_{i=1}^N h_i^2\|\hat{\phi}_1''\|_{L^1(I_i)}\\ &\le Ch^{r+2}\|u\|_{W_\infty^{r+1}}\|\phi\|_{W_1^2}\\ &\le Ch^{r+2}\|u\|_{W_\infty^{r+1}}\|\psi\|_{L^1}.\end{aligned} \tag{27}$$

Now taking notice of (17), using the similar argument to that of Theorem 2 in [9], we can easily obtain the following negative norm estimate:

$$\|\xi\|_{W_\infty^{-1}(I_i)}\le Ch_i^{r+2}\|u\|_{W_\infty^{r+2}(I_i)}.$$

Therefore, we have

$$\begin{aligned}(a\xi'+b\xi,\hat{\phi}_2)&=(\xi,(a\hat{\phi}_2)'+b\hat{\phi}_2)\\ &\le C\sum_{i=1}^N \|\xi\|_{W_\infty^{-1}(I_i)}\|\hat{\phi}_2\|_{W_1^2(I_i)}\\ &\le C\sum_{i=1}^N h_i^{r+2}\|u\|_{W_\infty^{r+2}(I_i)}\|\hat{\phi}\|_{W_1^2(I_i)}\\ &\le Ch^{r+2}\|u\|_{W_\infty^{r+2}}\|\phi\|_{W_1^2}\\ &\le Ch^{r+2}\|u\|_{W_\infty^{r+2}}\|\psi\|_{L^1}.\end{aligned} \tag{28}$$

From (26) to (28) we have

$$|\mathscr{L}(\xi,\hat{\phi})|\le Ch^{r+2}\|u\|_{W_\infty^{r+2}}\|\psi\|_{L^1}. \tag{29}$$

Therefore, combining (25) and (29) with (23), for sufficiently small h, we obtain the estimate (21), which completes the proof.

5. A Numerical Example

The two point boundary value problem studied is

$$\begin{cases} -u''+100u=-100, & x\in I, \\ u(0)=u(1)=0. \end{cases} \tag{30}$$

The exact solution is

$$u(x)=\frac{1-e^{-10}}{e^{10}-e^{-10}}e^{10x}+\frac{e^{10}-1}{e^{10}-e^{-10}}e^{-10x}-1.$$

We adopted $r=3$ i.e. piecewise cubic polynomials and divided I into $N=32$ equal subintervals. In this case, the roots of Jacobi polynomial $J_3(x)$ are $(1\pm 1/\sqrt{5})/2$. Thus, there exist two Jacobi points on each subinterval I_i $(1\le i\le 32)$. Table 1 shows the results of our numerical experiments.

Table 1. Errors of the approximate solution.

	Jacobi points $(x_{i-1}+\rho_j h)$	Mesh points (x_i)	Mid points $(x_i+\frac{1}{2}h)$
Maximum:	0.1352E-6	0.5681E-9	0.4242E-5
Minimum:	0.6892E-10	0.1038E-9	0.6754E-7
Mean:	0.2954E-7	0.3084E-9	0.9333E-6

One can say that these results show the validity of the corresponding error estimates of Theorem 1, (11) and (10), respectively.

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