Some Superconvergence for a Galerkin Method by Averaging Gradients in One Dimensional Problems

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We consider some superconvergence phenomena followed by the averaging gradients in a Galerkin method for two point boundary value problems using continuous piecewise polynomials. It is shown that several a posteriori methods based on the averaging procedures yield superconvergent approximations to the exact solution and its derivative with one order better rates of convergence than the optimal rates. The special emphasis of the paper is the fact that the superconvergence phenomena only occur in cases using odd degree polynomials. We illustrate some numerical examples which confirm the theoretical results. Furthermore, we also describe the extension of the results to the parabolic problems in a single space variable.

1. Introduction

The averaging gradient was an old practice in finite element methods to improve the accuracy of approximation of derivatives. Krizek & Neittaanmäki [4] first presented a theoretical basis for the technique in case of linear finite element method in R^2. They proved that the averaged gradient is a superconvergent approximation to the exact gradient in L^2-norm sense.

In this paper, we study the various superconvergence phenomena following from the averaging gradient in a Galerkin method for two point boundary value problems using continuous piecewise polynomials of arbitrary degree. Further, we consider the extension of these results to one space dimensional parabolic problems. It is shown that several a posteriori methods based on averaging gradient lead to the approximations which have one order better rates of convergence in L^∞-norm sense than the optimal rates. The special emphasis of the paper is the fact that the superconvergence phenomena by the averaging technique occur only in cases using odd degree polynomials. That is, beyond our expectation, in case of even degree polynomials those phenomena are not observed as clarified later. We also note that each proposition throughout the paper is established without quasi-uniformity condition for the partition of the interval. Moreover, the results obtained here will be extended to the two dimensional elliptic boundary value problems in [9].

In the following section, we describe the two point boundary value problem under consideration and introduce some notations for later use. Then we define the Galerkin approximation for the problem. In 3, it is proved in case of using odd degree polynomials that, by virtue of the property of the Jacobi points, the averaged values of left and right limits of the approximate derivatives at the internal knots converge to exact derivatives with one order higher than the optimal rate. Further, combining the result with the superconvergence of the derivatives at the Gauss points, we consider some simple procedures to provide the globally superconvergent approximations for the exact derivative and the solution itself. Finally, we present some numerical examples which confirm the conclusions obtained in the section. In 4, we extend the results derived in 3 to the parabolic problems in one space dimension.

2. Problem, notations and the Galerkin method

Consider the following two point boundary value problem

$$(2.1)\quad \begin{cases} Ly \equiv -(a(x)y')' + b(x)y' + c(x)y = f(x), & x \in I, \\ y(0) = y(1) = 0, \end{cases}$$

where $I=(0, 1)$. Assume that a, b, $c \in C^1(I)$ and there exist constants α_0, α_1 such that

$$0 < \alpha_0 \leqq a(x) \leqq \alpha_1, \quad x \in I.$$

Further assume that (2.1) has a unique solution for each $f \in C(I)$. Let $\Delta: 0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of I and set $I_i = (x_{i-1}, x_i)$, $h_i = x_i - x_{i-1}$ and $h = \max_{1 \leqq i \leqq N} h_i$. For a positive integer k and $E \subset I$, let $P_k(E)$ denote the set of polynomials on E of degree at most k. Also we define

$$P_k^0(I_i) = \{v \in P_k(I_i): v(x_{i-1}) = v(x_i) = 0\},$$

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$$\mathcal{M}^k_{-1}(\Delta)=\{v: v|_{I_i}\in P_k(I_i),\ 1\leqq i\leqq N\},$$
$$\mathcal{M}^k_0(\Delta)=\{v\in C(I): v\in P_k(I_i),\ 1\leqq i\leqq N,\ v(0)=v(1)=0\}.$$

Here the symbol Δ will be usually omitted. For an integer $m\geqq 0$ and $1\leqq p\leqq\infty$, let

$$W^m_p(E)=\left\{\phi: \frac{d^l\phi}{dx^l}\in L^p(E),\ 0\leqq l\leqq m\right\}$$

and

$$\|\phi\|_{W^m_p(E)}=\sum_{l=0}^m\left\|\frac{d^l\phi}{dx^l}\right\|_{L^p(E)},$$

where $L^p(E)$ denotes usual L^p-space on E. Especially, we denote $W^m_2(E)$ by $H^m(E)$. When $E=I$, the dependency on th interval will be suppressed (*e.g.* $\|\phi\|_{W^m_p}$ means $\|\phi\|_{W^m_p(I)}$). Also, we use the symbol C to denote a generic positive constant independent of h and not necessarily the same in any two places.

From now on, we fix an integer r ($\geqq 1$) and simply denote $\mathcal{M}^r_0$ by $\mathcal{M}$. We define the Galerkin approximation $Y\in\mathcal{M}$ to (2.1) by

$$B(Y, v)=(f, v),\quad v\in\mathcal{M}, \tag{2.2}$$

where

$$B(Y, v)\equiv(aY', v')+(bY'+cY, v),$$

and $(\cdot,\cdot)$ implies L^2 inner product on I. The existence and uniqueness of the solution to (2.2) and the error estimates are well known for sufficiently small h (*e.g.* [2], [10] etc.).

We now state some additional definitions used later. The Jacobi polynomial of degree k is defined as

$$J_k(x)=\frac{1}{qw(x)}\frac{d^k}{dx^k}[w(x)(x(1-x))^k], \tag{2.3}$$

where $w(x)=x^\alpha(1-x)^\beta$, α, $\beta>-1$ and q is a constant chosen such that the coefficient of x^k in (2.3) is 1. The Jacobi points σ_j, $1\leqq j\leqq r-1$, on I are determined by the $r-1$ roots of $J_{r-1}(x)=0$ with $w(x)=x(1-x)$. And the Gauss points τ_k, $1\leqq k\leqq r$, are the roots of $J_r(x)=0$ with $w(x)\equiv 1$. Next, the Jacobi and the Gauss points on I_i are defined as the following affine transformations:

$$\text{Jacobi points:}\quad \sigma_{ij}=x_{i-1}+h_i\sigma_j,\quad 1\leqq j\leqq r-1, \tag{2.4}$$
$$\text{Gauss points:}\quad \tau_{ik}=x_{i-1}+h_i\tau_k,\quad 1\leqq k\leqq r. \tag{2.5}$$

3. Superconvergent approximations by averaging gradients

In this section we consider some a posteriori procedures which can be made to provide superconvergent approximations to y and y' at any points on I. Let $H^1_0(I)=H^1(I)\cap\{\phi: \phi(0)=\phi(l)=0\}$. Then for each $g\in H^1_0(I)$, let $Pg\in\mathcal{M}$ be a projection defined by

$$(g'-(Pg)', v')=0,\quad v\in\mathcal{M}. \tag{3.1}$$

We now set $I^*_i=(x_{i-1}, x_{i+1})$, $1\leqq i\leqq N-1$, and for a function ψ which is differentiable on I^*_i except for x_i, define the averaged gradients by

$$D_i\psi=\frac{1}{1+\alpha^r_i}\{\psi'(x_i-)+\alpha^r_i\psi'(x_i+)\},$$

where $a_i=h_i/h_{i+1}$. The following result is essential for our study, so we present a complete proof, though it will be proved in [9] by a slightly different technique.

Lemma 1. If r is odd and $g\in W^{r+2}_\infty(I^*_i)\cap H^1_0(I)$, then

$$|g'(x_i)-D_i(Pg)|\leqq C\bar{h}^{r+1}_i\|g\|_{W^{r+2}_\infty(I^*_i)},$$

where $\bar{h}_i=\max(h_i, h_{i+1})$.

Proof. First, for any $\phi\in W^{r+2}_\infty(I^*_i)$ define $Q_r\phi\in P_r(I_i)\cap P_r(I_{i+1})$ by

$$\begin{cases}(\phi'-(Q_r\phi)', v')_{I_j}=0,\ v\in P^0_r(I_j),\ j=i, i+1,\\ (Q_r\phi)(x_j)=\phi(x_j),\ j=i-1, i, i+1,\end{cases} \tag{3.2}$$

where $(\cdot,\cdot)_{I_j}$ implies L^2 inner product on I_j. Next, we define a linear functional L_i on $W^{r+2}_\infty(I^*_i)$ by

$$L_i(\phi)\equiv\phi'(x_i)-D_i(Q_r\phi). \tag{3.3}$$

It can be easily seen that $P\phi=Q_r\phi$ on I^*_i for each $\phi\in H^1_0(I)$ because of the following well known property (*e.g.* [2]):

$$(P\phi)(x_i)=\phi(x_i),\quad 0\leqq i\leqq N. \tag{3.4}$$

We now verify that $L_i(\phi)=0$ for any $\phi\in P_{r+1}(I^*_i)$. Let ϕ be a fixed polynomial in $P_{r+1}(I^*_i)$. Then, taking into account $\phi=Q_{r+1}\phi$, we have by the result in [6]

$$\begin{cases}(\phi-Q_r\phi)(\sigma_{ij})=0,\quad 1\leqq j\leqq r-1,\\ (\phi-Q_r\phi)(\sigma_{i+1,j})=0,\quad 1\leqq j\leqq r-1.\end{cases} \tag{3.5}$$

Hence, we have with constants α and β

$$\phi(x)=\alpha(x-\sigma_{i1})\cdots(x-\sigma_{i,r-1})(x-x_{i-1})(x-x_i)+(Q_r\phi)(x),\qquad x\in I_i, \tag{3.6}$$

and

$$\phi(x)=\beta(x-\sigma_{i+1,1})\cdots(x-\sigma_{i+1,r-1})(x-x_i)(x-x_{i+1})+(Q_r\phi)(x),\qquad x\in I_{i+1}. \tag{3.7}$$

But we have $\alpha=\beta$, for $\phi(x)$ is a single polynomial globally on I^*_i. Thus, differentiating (3.6), (3.7) and using (2.4), (2.5) we obtain

$$\begin{aligned}\phi'(x_i-)&=a(x_i-\sigma_{i1})\cdots(x_i-\sigma_{i,r-1})\times(x_i-x_{i-1})+(Q_r\phi)'(x_i-)\\&=ah^r_i(1-\sigma_1)\cdots(1-\sigma_{r-1})+(Q_r\phi)'(x_i-)\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\phi'(x_i+)&=a(x_i-\sigma_{i+1,1})\cdots(x_i-\sigma_{i+1,r-1})\times(x_i-x_{i+1})+(Q_r\phi)'(x_i+)\\&=(-1)^r ah^r_{i+1}\sigma_1\cdots\sigma_{r-1}+(Q_r\phi)'(x_i+).\end{aligned} \tag{3.9}$$

By virtue of the form of Jacobi polynomial, the following relation holds:

$$(1-\sigma_1)\cdots(1-\sigma_{r-1})=\sigma_1\cdots\sigma_{r-1}.$$

Hence, for an odd integer r, (3.8) and (3.9) yield

$$\begin{aligned}\phi'(x_i)&=\frac{1}{h_i^r+h_{i+1}^r}\{h_{i+1}^r(Q_r\phi)'(x_i-)+h_i^r(Q_r\phi)'(x_i+)\}\\&=\frac{1}{1+\alpha_i^r}\{(Q_r\phi)'(x_i-)+\alpha_i^r(Q_r\phi)'(x_i+)\}\\&=D_i(Q_r\phi).\end{aligned}$$

Thus, we have $L_i(\phi)=0$ for arbitrary $\phi\in P_{r+1}(I_i^*)$. Therefore, using the Peano Kernel theorem, we can conclude the proof.

We now obtain the following superconvergence estimates at internal nodal points.

Theorem 1. Let r be an odd integer ($\geqq 1$). And let y and Y be solutions to (2.1) and (2.2), respectively. If $y\in W_\infty^{r+2}(I)$, then for sufficiently small h,

$$\max_{1\leqq i\leqq N-1}|y'(x_i)-D_iY|\leqq Ch^{r+1}\|y\|_{W_\infty^{r+2}}.$$

Proof. From the definition of D_i, observe that

$$\begin{aligned}|y'(x_i)-D_iY|&\leqq|y'(x_i)-D_i(Py)|+|D_i(Py-Y)|\\&\leqq|y'(x_i)-D_i(Py)|+\|(Py)'-Y'\|_{L^\infty}.\end{aligned}$$

Here, by the proof of Theorem 2 in [7], we have

$$\|(Py)'-Y'\|_{L^\infty}\leqq Ch^{r+1}\|y\|_{W_\infty^{r+1}}.$$

Thus, combining this with Lemma 1 we obtain immediately the desired estimates.

Moreover, it is also known in [7], particularly Theorem 2, that for each Gauss point $\tau_{ik}\in I_i$, $1\leqq k\leqq r$,

(3.10) $$|(y'-Y')(\tau_{ik})|\leqq Ch^{r+1}(\|y\|_{W_\infty^{r+1}}+\|y\|_{W_\infty^{r+2}(I_i)}).$$

Using this estimates and Theorem 1, we can provide superconvergent approximations to y' and y on I. That is, first choose $DY\in\mathfrak{M}_{-1}^r$ satisfying on each I_i, $1\leqq i\leqq N$,

(3.11) $$\begin{cases}DY(x_i)=D_iY,\\DY(\tau_{ik})=Y'(\tau_{ik}),\quad 1\leqq k\leqq r.\end{cases}$$

When $i=N$ replace the first equation above by $DY(x_{N-1})=D_{N-1}Y$. Next, we define $\tilde{Y}\in\mathfrak{M}_{-1}^{r+1}$ by

(3.12) $$\tilde{Y}(x)=\int_{x_{i-1}}^x DY(\xi)d\xi+Y(x_{i-1})\qquad\text{for } x\in I_i.$$

Then, DY and $\tilde{Y}$ converge to y' and y with one order better than the optimal rate of convergence, respectively.

Theorem 2. Assume the hypotheses of Theorem 1. And let Dy and $\tilde{Y}$ be functions defined by (3.11) and (3.12), respectively. Then, for sufficiently small h.

(3.13) $$\|y'-DY\|_{L^\infty}\leqq Ch^{r+1}\|y\|_{W_\infty^{r+2}}\qquad\text{for}\quad r\geqq1,$$

(3.14) $$\|y-\tilde{Y}\|_{L^\infty}\leqq Ch^{r+2}\|y\|_{W_\infty^{r+2}}\qquad\text{for}\quad r\geqq3.$$

Proof. We define $Z\in\mathfrak{M}_0^{r+1}$ by

$$(y'-Z',v')=0,\qquad v\in\mathfrak{M}_0^{r+1}.$$

Now observe that

(3.15) $$\|y'-DY\|_{L^\infty}\leqq\|y'-Z'\|_{L^\infty}+\|Z'-DY\|_{L^\infty}.$$

Since all norms in the finite dimensional space $P_r(I_i)$ are equivalent with each other, we have

$$\begin{aligned}&\|Z'-DY\|_{L^\infty(I_i)}\\&\leqq C\{|(Z'-DY)(x_i)|+\sum_{j=1}^r|(Z'-DY)(x_{ij})|\}\\&\leqq C\{|(y'-DY)(x_i)|+\sum_{j=1}^r|(y'-DY)(x_{ij})|\}\\&\qquad\qquad+C\|y'-Z'\|_{L^\infty(I_i)}\\&\leqq Ch^{r+1}\|y\|_{W_\infty^{r+2}},\end{aligned}$$

where we have used (3.11), (3.10), Theorem 1 and the well known estimates for $y'-Z'$. Combining this with (3.15), we obtain the estimates (3.13).

Next, by (3.12) for arbitrary $x\in I_i$, we have

(3.16) $$y(x)-\tilde{Y}(x)=\int_{x_{i-1}}^x(y'(\xi)-DY(\xi))d\xi+y(x_{i-1})-Y(x_{i-1}).$$

Note that, from the well known superconvergence property at the mesh points (*e.g.* [2]),

$$|y(x_{i-1})-Y(x_{i-1})|\leqq Ch^{2r}\|y\|_{H^{r+1}}.$$

Therefore, (3.14) follows immediately by (3.16) and (3.13).

Now, we present some numerical examples to demonstrate the results obtained in this section.

Problem:

(3.17) $$\begin{cases}-y''+3y'-2y=(-2x+3)e^x,\quad x\in I,\\y(0)=y(1)=0.\end{cases}$$

The exact solution to (3.17):

$$y(x)=x(1-x)e^x.$$

We solved (3.17) numerically using the approximation scheme (2.2) with uniform partition and computed various errors for several r and N. We illustrate these values in Table 1 ~ 3. Meanings of the symbols in these tables are as follows:

$$\text{LEFT}=\max_{1\leqq i\leqq N-1}\{|(y'-Y')(x_i-)|\},$$

$$\text{RIGHT}=\max_{1\leqq i\leqq N-1}\{|(y'-Y')(x_i+)|\},$$

$$\text{MEAN}=\max_{1\leqq i\leqq N-1}\left\{\left|y'(x_i)-\frac{1}{2}(Y'(x_i-)+Y'(x_i+))\right|\right\},$$

$$\text{OTHGRAD}=\max_{1\leqq i\leqq N}\left\{\left|(y'-Y')\left(x_i-\frac{2}{3}h\right)\right|\right\},$$

$$\text{NEWGRAD}=\max_{1\leqq i\leqq N}\left\{\left|(y'-DY)(x_i-\frac{2}{3}h)\right|\right\},$$

$$\text{OTHAPPR} = \max_{1 \le i \le N} \left\{ \left| (y-Y)\left(x_i - \frac{2}{3}h\right) \right| \right\},$$

$$\text{NEWAPPR} = \max_{1 \le i \le N} \left\{ \left| (y-\tilde{Y})\left(x_i - \frac{2}{3}h\right) \right| \right\}.$$

Table 1. Improvement of errors by averaging (for $r=3$)

N	LEFT	RIGHT	MEAN
8	0.5204E-3	0.6509E-3	0.6520E-4
16	0.7591E-4	0.8483E-4	0.4461E-5
32	0.1024E-4	0.1083E-4	0.2918E-6
64	0.1331E-5	0.1368E-5	0.1866E-7
Order≈	$O(h^3)$	$O(h^3)$	$O(h^4)$

Table 2. Non-improvement of errors by averaging (for $r=2$)

N	LEFT	RIGHT	MEAN
8	0.2237E-1	0.2896E-1	0.2567E-1
16	0.6682E-2	0.7592E-2	0.7137E-2
32	0.1824E-2	0.1944E-2	0.1884E-2
64	0.4765E-3	0.4918E-3	0.4814E-3
Order≈	$O(h^2)$	$O(h^2)$	$O(h^2)$

Table 3. Improvements of errors by procedures (3.11) and (3.12) (for $r=3$)

N	OTHGRAD	NEWGRAD	OTHAPPR	NEWAPPR
8	0.2191E-3	0.1946E-4	0.1763E-5	0.3620E-6
16	0.3146E-4	0.1286E-5	0.1231E-6	0.1248E-7
32	0.4210E-5	0.8261E-7	0.8109E-8	0.4094E-9
64	0.5445E-6	0.5233E-8	0.5201E-9	0.1310E-10
Order ≈	$O(h^3)$ ⟶	$O(h^4)$	$O(h^4)$ ⟶	$O(h^5)$

Table 1 and 3 confirm the superconvergence estimates asserted in Theorem 1 and 2, respectively. On the contrary, Table 2 shows that one cannot expect to improve the accuracy by the averaging techniques in case we adopt even degree polynomials.

4. Extension to the parabolic case

The results derived in previous section are naturally extended to the diffusion problems in one space dimension. We now mention the outline for the parabolic case. The main technique which plays an essential role for the proof of the superconvergence estimates is the quasi-projection method ([3]). The method can be applied to the present case in the similar manner to that of [7].

Now, consider parabolic problem

$$(4.1)\quad \begin{cases} \rho(x)\dfrac{\partial u}{\partial t} = \mathcal{L}u \equiv \dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial u}{\partial x}\right) + b(x)\dfrac{\partial u}{\partial x} + c(x)u \\ \qquad + \phi(x,t), \quad (x,t) \in I \times J, \\ u(0,t) = u(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in I, \end{cases}$$

where $J=(0, T)$. We assume that there exist positive constants ρ_0, ρ_1, α_0, α_1 such that

$$0 < \rho_0 \le \rho(x) \le \rho_1, \qquad x \in I,$$
$$0 < \alpha_0 \le a(x) \le \alpha_1, \qquad x \in I,$$

and that ρ, a, b, c and ϕ are sufficiently smooth.

The semidiscrete Galerkin approximation to (4.1) is defined by a map $U: J \to \mathcal{M}$ satisfying

$$(4.2\text{-i})\quad \left(\rho\frac{\partial U}{\partial t}, v\right) + B(U, v) = (\phi, v) \qquad v \in \mathcal{M},\ t \in J,$$

and

$$(4.2\text{-ii})\quad B(u_0 - U(0), v) = 0, \qquad v \in \mathcal{M},\ t = 0.$$

Here, the bilinear form $B(\cdot, \cdot)$ is the same as in 2, that is,

$$B(\phi, \psi) = (a\phi_x, \psi_x) + (b\phi_x + c\phi, \psi).$$

The averaging procedure to obtain superconvergent approximations for u_x of $O(h^{r+1})$ at internal spatial knots is also identically defined as before, i.e.

$$(4.3)\ D_iU(\cdot, t) = \frac{1}{1+\alpha_i^r}\left\{\frac{\partial U}{\partial x}(x_i -, t) + \alpha_i^r \frac{\partial U}{\partial x}(x_i +, t)\right\},$$
$$1 \le i \le N-1,$$

where $\alpha_i = h_i / h_{i+1}$.

Now, before describing the superconvergence theorem, we introduce some additional definitions. When X is a normed space with norm $\|\cdot\|_X$, for a map $\phi: J \to X$, we say $\phi \in W_p^k(J:X)$ if $\|\phi(t)\|_X$ belongs to $W_p^k(J)$. Then the norm in $W_p^k(J:X)$ is defined by

$$\|\phi\|_{W_p^k(J:X)} = \sum_{j=0}^{k} \left\| \frac{d^j\phi}{dt^j} \right\|_{L^p(J:X)},$$

where

$$\|\psi\|_{L^p(J:X)} = \begin{cases} \left[\displaystyle\int_J \|\psi(t)\|_X^p dt\right]^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess.sup}_{t \in J} \|\psi(t)\|_X, & p = \infty. \end{cases}$$

We can now state the following superconvergence estimates at the spatial nodal points.

Theorem 3. Let u and U be solutions to (4.1) and (4.2), respectively. If r is odd and $u \in W_\infty^1(J: W_\infty^{r+2})$, then, for sufficiently small h,

$$\left|\frac{\partial u}{\partial x}(x_i, t) - D_i u(\cdot, t)\right| \leqq Ch^{r+1}\|u\|_{W^1_\infty(J:W^{r+2}_\infty)}, \qquad t \in J.$$

Proof. We define the so-called elliptic projection W: $J \to \mathfrak{M}$, which allows us to obtain the desired estimates, by

$$\text{(4.4)} \qquad B(u-W, v) = 0, \qquad v \in \mathfrak{M},\ t \in J.$$

Next, we introduce a quasi-projection $Z: J \to \mathfrak{M}$ by

$$\text{(4.5)} \qquad B(Z, v) = \left(\rho \frac{\partial}{\partial t}(u-W), v\right), \qquad v \in \mathfrak{M},\ t \in J.$$

Further, we set

$$\text{(4.6)} \qquad E = W - U + Z.$$

Then, for $1 \leqq i \leqq N-1$ by Theorem 1

$$\text{(4.7)} \quad \left|\frac{\partial u}{\partial x}(x_i, t) - D_i W(\cdot, t)\right| \leqq Ch^{r+1}\|u(\cdot, t)\|_{W^{r+2}_\infty}, \quad t \in J,$$

On the other hand, by the estimates in [7] we have

$$\text{(4.8)} \qquad \|D_i E(\cdot, t)\|_{L^\infty} \leqq \|E_x(\cdot, t)\|_{L^\infty} \leqq Ch^{r+1} \times \|u\|_{W^1_\infty}(J:H^{r+1})$$

and

$$\text{(4.9)} \qquad \|D_i Z(\cdot, t)\|_{L^\infty} \leqq Ch^{r+1}\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(J:H^{r+1})}.$$

Thus, the proof is now completed by (4.7)-(4.9) and the triangle inequality.

Moreover, the global superconvergent approximations for u_x and u are also constructed by the similar procedures to (3.11) and (3.12), respectively. That is, define $DU: J \to \mathfrak{M}^r_{-1}$ satisfy on each I_i, $1 \leqq i \leqq N$,

$$\text{(4.10)} \qquad \begin{cases} DU(x_i, t) = D_i U(\cdot, t), & t \in J, \\ DU(\tau_{ik}, t) = U_x(\tau_{ik}, t), & 1 \leqq k \leqq r,\ t \in J, \end{cases}$$

where τ_{ik} is defined by (2.5) and when $i = N$ replaces i by $N-1$ in the first equation. Further $\tilde{U}: J \to \mathfrak{M}^{r+1}_{-1}$ is defined by

$$\text{(4.11)} \quad \tilde{U}(x, t) = \int_{x_{i-1}}^{x} DU(\xi, t)d\xi + U(x_{i-1}, t), x \in I_i,\ t \in J.$$

By similar arguments to those used in the previous section and the estimates obtained in [7] we get the following results.

Theorem 4. Assume the hypotheses of Theorem 3 and let DU and $\tilde{U}$ be maps defined by (4.10) and (4.11), respectively. Then, for sufficiently small h,

$$\left\|\frac{\partial u}{\partial x} - DU\right\|_{L^\infty(I \times J)} \leqq Ch^{r+1}\|u\|_{W^1_\infty(J:W^{r+2}_\infty)} \qquad \text{for} \quad r \geqq 1,$$

$$\|u - \tilde{U}\|_{L^\infty(I \times J)} \leqq Ch^{r+2}\|u\|_{W^1_\infty(J:W^{r+2}_\infty)} \qquad \text{for} \quad r \geqq 3.$$

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