Regular Paper

A Probabilistic Interpretation for Lazy Nonmonotonic Reasoning[1]

KEN SATOH*

This paper presents a formal relationship for probability theory and a class of nonmonotonic reasoning which we call *lazy nonmonotonic reasoning*. In lazy nonmonotonic reasoning, nonmonotonicity emerges only when new added knowledge is contradictory to the previous belief.

In this paper, we consider nonmonotonic reasoning in terms of *consequence relation*. A consequence relation is a binary relation over formulae which expresses that a formula is derivable from another formula under inference rules of a considered system. A consequence relation which has lazy nonmonotonicity is called a *rational consequence relation* studied by Lehmann et al. [8].

We provide a probabilistic semantics which characterizes a rational consequence relation exactly. Then, we show a relationship between propositional circumscription and consequence relation, and apply this semantics to a consequence relation defined by propositional circumscription which has lazy nonmonotonicity.

1. Introduction

This paper is concerned with a formal relationship between nonmonotonic reasoning and probability theory. Nonmonotonic reasoning is a formalization of reasoning when information is incomplete. If someone is forced to make a decision under incomplete information, he uses common sense to supplement lack of information. Common sense can be regarded as a collection of normal results. These normal results are obtained because their probability is very near to certainty. So common sense has a statistical or probabilistic property.

Although there is much research which simulates behavior of nonmonotonic reasoning based on probability theory [15-17], there is no formal relationship between nonmonotonic reasoning and probability theory, as Lifschitz [10] pointed out.

In this paper, we consider nonmonotonic reasoning in terms of *consequence relation* [2, 6-8]. Consequence relation is a binary relation over formulae and expresses that a formula is derivable from another formula under inference rules of the considered system. The researchers consider desired properties in a consequence relation for nonmonotonic reasoning.

Gabbay [2] was the first to consider nonmonotonic reasoning by a consequence relation and Kraus et al. [6] gives a semantics for a consequence relation of nonmonotonic reasoning called *preferential* consequence relation. The semantics is based on an order over possible states which is similar to an order over interpretations in circumscription [12] or Shoham's preference logic [20].

Lehmann et al. [8] define a more restricted consequence relation called *rational* consequence relation and shows that a consequence relation is rational if and only if it is defined by some *ranked* model. A model is ranked if a set of possible states is partitioned into a hierarchical structure, and in a rational consequence relation the previous belief will be kept as long as the new knowledge does not contradict the previous belief. This nonmonotonicity can be said to be *lazy* because only contradictory knowledge can cause a belief revision.

Moreover, they investigate a relationship between Adams' logic [1] (or equivalently, ε-semantics [16]) and rational or preferential entailment in which a conditional assertion is followed by a set of conditional assertions. Although Adams' logic is based on probabilistic semantics, it only considers consistency and entailment for a set of conditional assertions and does not consider probabilistic semantics for a consequence relation. Since nonmonotonic reasoning systems define consequence relations, we must modify Adams' logic to give a probabilistic semantics to these systems.

In this paper, we provide a probabilistic relation which characterizes a rational consequence relation exactly. To do so, we define a *closed consequence relation in the limit*. This property means that there exists a probability function with a positive parameter x such that a conditional probability of a pair of formulae in the consequence relation approaches 1, and a conditional probability of a pair of formulae not in the relation aproaches α except 1, as x approaches 0.

Then, we can show that a consequence relation is closed in the limit if and only if the consequence rela-

*Institute for New Generation Computer Technology, 1-4-28 Mita, Minato-ku, Tokyo 108, Japan.

[1]This paper is an extended version of a paper with the same title that appeared in the Proceedings of the Eighth American Conference on Artificial Intelligence (Menlo Park, CA: The AAAI Press/The MIT Press, 1990),

tion is rational.

We apply this result to giving a probabilistic semantics for circumscription [12], because circumscription has a similar semantics for a rational or preferential consequence relation and circumscription can define a consequence relation each of a pair of which consists of original axiom and derived result. We can show that a consequence relation defined by circumscription is a preferential consequence relation, but not necessarily a rational consequence relation. Especially, we can show that if there are some fixed propositions or if we minimize more than three propositions in parallel, then the consequence relation defined by this circumscription is always non-rational.

However, in some cases, we can separate a set of interpretations into a hierarchy, and so, we can provide a probability function so that a consequence relation defined by the circumscription in these cases is equivalent to a consequence relation defined by the probability function.

2. Consequence Relations and Their Models

In this section, we briefly review a work on consequence relation by Lehmann, Kraus and Magidor [6, 8]. A summary of the work is found in [7].

We consider a propositional language. In the propositional language L, we shall use a set of propositional symbols (finite or infinite). Then, formulae in L are defined as follows.

1. A propositional symbol is a formula.
2. If A and B are formulae, then $\neg A$ and $A \supset B$ are formulae.
3. An expression is a formula only if it satisfies the above conditions.

If A and B are formulae, then $A \wedge B$, $A \vee B$, $A \equiv B$ are abbreviations for $\neg(A \supset \neg B)$, $\neg A \supset B$ and $(A \supset B) \wedge (B \supset A)$, respectively. We use **F** for false and **T** for true.

We use a set of all possible worlds, $\mathscr{U}$, to give a truth value to every propositional symbol. We define a satisfaction relation $\models$ over $\mathscr{U}$ and L as follows. $u \in \mathscr{U}$ satisfies a formula A (written as $u \models A$) if and only if the following conditions are satisfied.

1. If A is a propositional symbol P, then $u(P)$ is true.
2. If A is of the form $\neg B$, then u does not satisfy B (written as $u \not\models B$).
3. If A is of the form $B \supset C$, then $u \models B \supset C$, then either $u \not\models B$ or $u \models C$.

We consider a binary relation over formulae called *consequence relation* $\vdash$ which has some desired property in a considered reasoning system. Intuitively speaking, $A \vdash B$ means that if a state of knowledge is A, then B is derived from A by inference rules defined in a considered reasoning system. The following class of consequence relations is closely related to nonmonotonic reasoning systems.

Definition 1 (Kraus, Lehmann and Magidor) *A consequence relation that satisfies all six properties below is called a* preferential *consequence relation.*

If $A \equiv B$ is a tautology and $A \vdash C$, then $B \vdash C$. **(Left Logical Equivalence)** (1)

If $A \supset B$ is a tautology and $C \vdash A$, then $C \vdash B$. **(Right Weakening)** (2)

$A \vdash A$. **(Reflexivity)** (3)

If $A \vdash B$ and $A \vdash C$, then $A \vdash B \wedge C$. **(And)** (4)

If $A \vdash C$ and $B \vdash C$, then $A \vee B \vdash C$. **(Or)** (5)

If $A \vdash B$ and $A \vdash C$, then $A \wedge B \vdash C$. **(Cautious Monotony)** (6)

A model theory for preferential consequence relations is investigated by Kraus, Lehmann and Magidor [6] as follows.

Definition 2 (Kraus, Lehmann and Magidor) *Let V and U be a set and $V \subseteq U$ and $\prec$ be a strict partial order on U (for any $s \in U$, $\neg(s \prec s)$ and for any s, t, u, if $(s \prec t)$ and $(t \prec u)$, then $(s \prec u)$). We shall say that $t \in V$ is* minimal *in V if and only if there is no $s \in V$, such that $s \prec t$.*

Definition 3 (Kraus, Lehmann and Magidor) *Let $V \subseteq U$. We shall say that V is* smooth *if and only if $\forall t \in V$, either $\exists s$ minimal in V, such that $s \prec t$ or t is itself minimal in V.*

Definition 4 (Kraus, Lehmann and Magidor) *A* preferential *model W is a triple $\langle S, l, \prec \rangle$ where S is a set, the elements of which will be called states, $l: S \mapsto \mathscr{U}$ assigns a world to each state and $\prec$ is a strict partial order on S satisfying the following smoothness condition: for all $A \in L$, the set of states $\hat{A} \stackrel{\text{def}}{=} \{s \mid s \in S, l(s) \models A\}$ is smooth.*

Definition 5 (Kraus, Lehmann and Magidor) *Let W be a preferential model $\langle S, l, \prec \rangle$ and A, B be formulae in L. The consequence relation* defined by *W will be denoted by $\vdash_W$ and is defined by: $A \vdash_W B$ if and only if for any s minimal in $\hat{A}$, $l(s) \models B$.*

There is the following relationship between a preferential consequence relation and a preferential model.

Proposition 1 (Kraus, Lehmann and Magidor) *A binary relation $\vdash$ on L is a preferential consequence relation if and only if it is the consequence relation defined by some preferential model.*

There is an important subclass of preferential consequence relations called *rational consequence relation*.

Definition 6 (Lehmann and Magidor) *A preferential consequence relation $\vdash$ is said to be* rational *if and only if it satisfies the following condition.*

If $A \vdash C$ and $A \not\vdash \neg B$, then $A \wedge B \vdash C$. **(Rational Monotony)** (7)

Rational monotony was proposed by Makinson as a desired property for a nonmonotonic reasoning system and corresponds to one of the fundamental conditions for minimal change of belief proposed by Gärdenfors [3].

An intuitive meaning of the condition of rational monotony is that the previous conclusion stays in the new belief if the negation of the added information is not in the previous belief.

An alternative view of rational monotony is obtained by the contrapositive form of the above definition:

If $A \vdash C$ and $A \wedge B \nvdash C$, then $A \vdash \neg B$.

This means that if adding B makes the previous belief being retracted, B will be exceptional when the state of knowledge is A.

A model theory for rational consequence relations is investigated by Lehmann and Magidor [8] as follows.

Definition 7 (Lehmann and Magidor) *A* ranked *model W is a preferential model $\langle S, l, \prec \rangle$ for which the strict partial order $\prec$ may be defined in the following way: there is a totally ordered set Ω (the strict order on Ω will be denoted by $<$) and a function $r: S \mapsto \Omega$ such that $s \prec t$ if and only if $r(s) < r(t)$.*

Intuitively speaking, a model is ranked if a set of states is partitioned into a hierarchical structure.

There is the following relationship between a rational consequence relation and a ranked model.

Proposition 2 (Lehmann and Magidor) *A consequence relation is rational if and only if it is defined by some ranked model.*

3. Relationship between Rational Consequence Relation and Closed Consequence Relation in the Limit

From this point, we assume the set of propositional symbols in L is always finite.

Definition 8 *Let L be a propositional language. Then probability function P_x on L with positive parameter x is a function from a set of formulae in L and positive real numbers to real numbers which satisfies the following conditions.*

1. *For any $A \in L$ and for any $x>0$, $0 \le P_x(A) \le 1$.*
2. *For any $x>0$, $P_x(\mathbf{T})=1$.*
3. *For any $A \in L$ and $B \in L$ and for any $x>0$, if $A \wedge B$ is logically false then $P_x(A \vee B)=P_x(A)+P_x(B)$.*

If we ignore a parameter x, the above definition becomes the standard formulation for probability function on L [3]. We introduce a parameter x to express the weight of the probability for every states. Spohn [21] uses a similar probability function to relate his Natural Conditional Functions to probability theory.

Definition 9 *Let A, $B \in L$. We define the conditional probability of B under A, $P_x(B|A)$ as follows.*

$$P_x(B|A)=\begin{cases} 1 & \text{if } P_x(A)=0 \\ \dfrac{P_x(A \wedge B)}{P_x(A)} & \text{otherwise.} \end{cases}$$

Definition 10 *A probability function P_x on L with positive parameter x is said to be* convergent *if and only if for any $A \in L$, there exists α such that*

$$\lim_{x \to 0} P_x(A)=\alpha.$$

Now, we define a consequence relation in terms of the above probability function P_x.

Definition 11 *A consequence relation $\vdash$ is said to be* closed in the limit *if and only if there exists a convergent probability function P_x on L with positive parameter x such that for all $A \in L$ and $B \in L$,*

$$A \vdash B \quad \textit{if and only if} \quad \lim_{x \to 0} P_x(B|A)=1.$$

Intuitively speaking, if a pair, $\langle A, B \rangle$ is included in the closed consequence relation in the limit, then we can let the conditional probability of B under A approach 1 as much as possible and if not, the conditional probability will approach some value except 1. This intuitive meaning will be justified later.

Now, we show the equivalent relationship between a rational consequence relation and a closed consequence relation in the limit.

Theorem 1 *If $\vdash$ is rational then $\vdash$ is closed in the limit.*

Proof:

From Proposition 2, if $\vdash$ is rational, then there exists some ranked model $W=\langle S, l, \prec \rangle$ such that for every pair of formulae A and B, $A \vdash B$ if and only if $A \vdash_W B$. Since the language is logically finite, there exists a finite ranked model with a finite number of ranks. Let the number of ranks be $n (n \ge 1)$. Let η_i be the number of states at the i-th rank (States which are higher in $\prec$ is in a higher rank).

Let a function P_x on L with positive parameter x be defined as follows:[1]

$$P_x(A) \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{n} \eta_i^A * x^{i-1}}{\sum_{i=1}^{n} \eta_i * x^{i-1}}$$

where η_i^A is the number of states at the i-th rank that satisfies A.

This assignment satisfies the following conditions.

1. Every states of the same rank has the same probability.
2. The probability of a state in the $(i+1)$-th rank is x times as much as that of a state in the i-th rank.

Then, P_x is a convergent probability function.

1. For all $A \in L$ and for all $x>0$, since $0 \le \eta_i^A \le \eta_i$ and at least $\eta_1 \ne 0$, $0 \le P_x(A) \le 1$.
2. For all $x>0$, since $\eta_i^{\mathbf{T}}=\eta_i$, $P_x(\mathbf{T})=1$.
3. For all $A \in L$ and $B \in L$ and for all $x>0$, since if $A \wedge B$ is logically false, $\eta_i^{A \vee B}=\eta_i^A+\eta_i^B$, $P_x(A \vee B)=P_x(A)+P_x(B)$.
4. For all A, since $\lim_{x \to 0} P_x(A)=\dfrac{\eta_1^A}{\eta_1}$, $\lim_{x \to 0} P_x(A)$ always exists.

Consider a relation over L, $\vdash'$ defined as follows.

$$A \vdash' B \quad \text{if and only if} \quad \lim_{x \to 0} P_x(B|A)$$

We will show that $\vdash' = \vdash_W$.

If $P_x(A)=0$, there is no state which satisfies A and

[1] This assignment is suggested in [8].

therefore, for any $B \in L$ $A \vdash_w B$. In this case, since $P_x(B|A)=1$, $A \vdash' B$.

Let $P_x(A)$ be not equal to 0. There exists a state which satisfies A. Let $mr(A)$ be the minimum rank where some state satisfies A. Then,

$$\lim_{x\to 0} P_x(B|A)=\lim_{x\to 0}\frac{P_x(A\wedge B)}{P_x(A)}=\frac{\eta^{A\wedge B}_{mr(A)}}{\eta^{A}_{mr(A)}}.$$

If $A \vdash_w B$, then for any s minimal in $\hat{A}$, $l(s) \models A\wedge B$. And any s minimal in $\hat{A}$ is at the minimum rank in ranked model. Therefore, $\eta^{A\wedge B}_{mr(A)}=\eta^{A}_{mr(A)}$, and so,

$$\lim_{x\to 0} P_x(B|A)=1.$$

Thus, $A \vdash' B$.

If $A \nvdash_w B$, then there exists some s minimal in $\hat{A}$, $l(s) \not\models A\wedge B$. Therefore, $\eta^{A\wedge B}_{mr(A)} \neq \eta^{A}_{mr(A)}$, and so,

$$\lim_{x\to 0} P_x(B|A)\neq 1.$$

Thus, $A \nvdash' B$.

Therefore, $\vdash_w$ is closed in the limit. □

Now, we prove the converse of the above theorem.

Lemma 1 *Let P_x be a probability function with positive parameter x. If $P_x(A)\neq 0$, then*

$$\lim_{x\to 0} P_x(B|A)=1 \quad \textit{if and only if} \quad \lim_{x\to 0} P_x(\neg B|A)=0.$$

Proof:

Since if $P_x(A)\neq 0$, $P_x(B|A)=\dfrac{P_x(A\wedge B)}{P_x(A)}$. Then, since

$$1-P_x(B|A)=\frac{P_x(A)-P_x(A\wedge B)}{P_x(A)}=\frac{P_x(A\wedge\neg B)}{P_x(A)},$$

$\lim_{x\to 0} P_x(\neg B|A)=1-\lim_{x\to 0} P_x(B|A)=0.$ □

Lemma 2 *Let P_x be a probability function with positive parameter x. If $A\supset B$ is true and $\lim_{x\to 0} P_x(A\wedge C|B)=1$, then $\lim_{x\to 0} P_x(A\wedge C|A)=1$.*

Proof:

If $P_x(B)=0$, then $P_x(A)=0$ since $P_x(A)\leq P_x(B)$ from $A\supset B$. If $P_x(A)=0$, then the conclusion is always true. Let neither $P_x(A)$ nor $P_x(B)$ be equal to 0. Since $A\supset B$, $P_x(A)\leq P_x(B)$. Thus, since

$$\frac{P_x(A\wedge B\wedge C)}{P_x(B)}\leq\frac{P_x(A\wedge B\wedge C)}{P_x(A)}\leq 1,$$

if $\lim_{x\to 0} P_x(A\wedge C|B)=1$, then $\lim_{x\to 0} P_x(A\wedge C|A)=1$. □

Lemma 3 *Let P_x be a probability function with positive parameter x. If $A\supset B$ is true and $\lim_{x\to 0} P_x(A\wedge C|A)=0$, then $\lim_{x\to 0} P_x(A\wedge C|B)=0$.*

Proof:

If $P_x(B)=0$, then $P_x(A)=0$ since $P_x(A)\leq P_x(B)$ from $A\supset B$. If $P_x(A)=0$, then the assumption is always false. Let neither $P_x(A)$ nor $P_x(B)$ be equal to 0. Since $A\supset B$, $P_x(A)\leq P_x(B)$. Thus, since

$$0\leq\frac{P_x(A\wedge B\wedge C)}{P_x(B)}\leq\frac{P_x(A\wedge B\wedge C)}{P_x(A)},$$

if $\lim_{x\to 0} P_x(A\wedge C|A)=0$, then $\lim_{x\to 0} P_x(A\wedge C|B)=0$. □

Lemma 4 *Let P_x be a probability function with positive parameter x. If $A\supset B$ is true and $\lim_{x\to 0} P_x(A|C)=1$, then $\lim_{x\to 0} P_x(B|C)=1$.*

Proof:

If $P_x(C)=0$, then the conclusion is always true. Let $P_x(C)$ be not equal to 0. Since $A\supset B$, $P_x(A\wedge C)\leq P_x(B\wedge C)$. Thus, since

$$\frac{P_x(A\wedge C)}{P_x(C)}\leq\frac{P_x(B\wedge C)}{P_x(C)}\leq 1,$$

if $\lim_{x\to 0} P_x(A|C)=1$, then $\lim_{x\to 0} P_x(B|C)=1$. □

Lemma 5 *Let P_x be a probability function with positive parameter x. If $A\supset B$ is true and $\lim_{x\to 0} P_x(B|C)=0$, then $\lim_{x\to 0} P_x(A|C)=0$.*

Proof:

If $P_x(C)=0$, then the assumption is always false. Let $P_x(C)$ be not equal to 0. Since $A\supset B$, $P_x(A\wedge C)\leq P_x(B\wedge C)$. Thus, since

$$0\leq\frac{P_x(A\wedge C)}{P_x(C)}\leq\frac{P_x(B\wedge C)}{P_x(C)},$$

if $\lim_{x\to 0} P_x(B|C)=0$, then $\lim_{x\to 0} P_x(A|C)=0$. □

Lemma 6 *Let P_x be a probability function with positive parameter x. If $\lim_{x\to 0} P_x(B|A)=1$ and $\lim_{x\to 0} P_x(C|A)=1$, then $\lim_{x\to 0} P_x(B\wedge C|A)=1$.*

Proof:

If $P_x(A)=0$, then the conclusion is always true. Let $P_x(A)$ be not equal to 0. Since $\lim_{x\to 0} P_x(B|A)=1$, $\lim_{x\to 0} P_x(\neg B|A)=0$ from Lemma 1. Since $(\neg B\wedge C)\supset\neg B$, $\lim_{x\to 0} P_x(\neg B\wedge C|A)=0$ from Lemma 5. Since $P_x(A\wedge C)-P_x(A\wedge\neg B\wedge C)=P_x(A\wedge B\wedge C)$,

$$\begin{aligned}
&\lim_{x\to 0} P_x(B\wedge C|A)\\
&=\lim_{x\to 0}\frac{P_x(A\wedge B\wedge C)}{P_x(A)}\\
&=\lim_{x\to 0} P_x(C|A)-\lim_{x\to 0} P_x(\neg B\wedge C|A)\\
&=1-0=1.
\end{aligned}$$

□

Lemma 7 *Let P_x be a probability function with positive parameter x. If $\lim_{x\to 0} P_x(C|A)=1$ and $\lim_{x\to 0} P_x(C|B)=1$, then $\lim_{x\to 0} P_x(C|A\vee B)=1$.*

Proof:

If $P_x(A)=0$ or $P_x(B)=0$, then the above statement becomes a tautology. Let neither $P_x(A)$ nor $P_x(B)$ be equal to 0. Since $\lim_{x\to 0} P_x(C|A)=1$, $\lim_{x\to 0} P_x(\neg C|A)=0$ from Lemma 1. Since $P_x(\neg C|A)=P_x(A\wedge\neg C|A)$ and $A\supset(A\vee B)$,

$$\lim_{x\to 0} P_x(A\wedge\neg C|A\vee B)=0 \quad (1)$$

from Lemma 3.

Since $\lim_{x\to 0} P_x(C|B)=1$, $\lim_{x\to 0} P_x(\neg C|B)=0$ from Lemma 1. Since $(\neg A\wedge\neg C)\supset\neg C$, $\lim_{x\to 0} P_x(\neg A\wedge\neg C|B)=0$ from Lemma 5. Since $P_x(\neg A\wedge\neg C|B)=P_x(\neg A\wedge B\wedge\neg C|B)$ and $B\supset(A\vee B)$,

$$\lim_{x\to 0} P_x(\neg A\wedge B\wedge\neg C|A\vee B)=0 \quad (2)$$

from Lemma 3.

Since $P_x(A\vee B)-P_x(A\wedge\neg C)-P_x(\neg A\wedge B\wedge\neg C)=P_x((A\vee B)\wedge C)$,

$$\begin{aligned}
&\lim_{x\to 0} P_x(C|A\vee B)\\
&=\lim_{x\to 0}\frac{P_x((A\vee B)\wedge C)}{P_x(A\vee B)}\\
&=\lim_{x\to 0}\frac{P_x(A\vee B)-P_x((A\vee B)\wedge\neg C)}{P_x(A\vee B)}\\
&=1-\lim_{x\to 0}\frac{P_x(A\wedge\neg C)}{P_x(A\vee B)}\\
&\quad-\lim_{x\to 0}\frac{P_x(\neg A\wedge B\wedge\neg C)}{P_x(A\vee B)}\\
&=1-\lim_{x\to 0}\frac{P_x((A\vee B)\wedge A\wedge\neg C)}{P_x(A\vee B)}\\
&\quad-\lim_{x\to 0}\frac{P_x((A\vee B)\wedge\neg A\wedge B\wedge\neg C)}{P_x(A\vee B)}
\end{aligned}$$

because $((A\vee B)\wedge A\wedge\neg C)\equiv(A\wedge\neg C)$

and $((A\vee B)\wedge\neg A\wedge B\wedge\neg C)\equiv(\neg A\wedge B\wedge\neg C)$

$$\begin{aligned}
&=1-\lim_{x\to 0} P_x(A\wedge\neg C|A\vee B)\\
&\quad-\lim_{x\to 0} P_x(\neg A\wedge B\wedge\neg C|A\vee B)\\
&=1-0-0=1
\end{aligned}$$

from (1) and (2). □

Lemma 8 *Let P_x be a probability function with positive parameter x. If $\lim_{x\to 0} P_x(B|A)=1$ and $\lim_{x\to 0} P_x(C|A)=1$, then $\lim_{x\to 0} P_x(C|A\wedge B)=1$.*

Proof:

If $P_x(A)=0$ or $P_x(B)=0$, then the conclusion is always true because $P_x(A\wedge B)=0$ from $P_x(A\wedge B)\leq P_x(A)$ and $P_x(A\wedge B)\leq P_x(B)$. Let neither $P_x(A)$ nor $P_x(B)$ be equal to 0. Since $\lim_{x\to 0} P_x(B|A)=1$ and $\lim_{x\to 0} P_x(C|A)=1$, $\lim_{x\to 0} P_x(B\wedge C|A)=1$ from Lemma 6. Since $P_x(B\wedge C|A)=P_x(A\wedge B\wedge C|A)$ and $(A\wedge B)\supset A$, $\lim_{x\to 0} P_x(A\wedge B\wedge C|A\wedge B)=1$ from Lemma 2.

Therefore,

$$\lim_{x\to 0} P_x(C|A\wedge B)=\lim_{x\to 0} P_x(A\wedge B\wedge C|A\wedge B)=1. \quad □$$

Lemma 9 *Let P_x be a convergent probability function with positive parameter x. If $\lim_{x\to 0} P_x(C|A)=1$ and $\lim_{x\to 0} P_x(\neg B|A)\neq 1$, then $\lim_{x\to 0} P_x(C|A\wedge B)=1$.*

Proof:

Since $\lim_{x\to 0} P_x(C|A)=1$, $\lim_{x\to 0} P_x(\neg C|A)=0$ from Lemma 1. Since $(B\wedge\neg C)\supset\neg C$,

$$\lim_{x\to 0} P_x(B\wedge\neg C|A)=0 \quad (3)$$

from Lemma 5.

Since $\lim_{x\to 0} P_x(\neg B|A)\neq 1$, $\lim_{x\to 0} P_x(B|A)\neq 0$ from Lemma 1. Therefore, since P_x is convergent, there exists α such that

$$\lim_{x\to 0} P_x(B|A)=\alpha\neq 0 \quad (4)$$

Since

$$\begin{aligned}
P_x(\neg C|A\wedge B)&=\frac{P_x(A\wedge B\wedge\neg C)}{P_x(A\wedge B)}\\
&=\frac{P_x(A\wedge B\wedge\neg C)}{P_x(A)} * \frac{P_x(A)}{P_x(A\wedge B)},
\end{aligned}$$

$$\lim_{x\to 0} P_x(\neg C|A\wedge B)=\frac{\lim_{x\to 0} P_x(B\wedge\neg C|A)}{\lim_{x\to 0} P_x(B|A)}=\frac{0}{\alpha}=0$$

from (3) and (4).

Therefore, $\lim_{x\to 0} P_x(C|A\wedge B)=1$ from Lemma 1. □

Theorem 2 *If $\vdash$ is closed in the limit then $\vdash$ is rational.*

Proof:

If $\vdash$ is closed in the limit, then there exists some convergent probability function with positive parameter x such that

$$A\vdash B \text{ if and only if } \lim_{x\to 0} P_x(B|A)=1$$

We show that $\vdash$ satisfies seven properties which every rational consequence relation satisfies.

1. **Left Logical Equivalence:**
   From the definition of probability, it is always valid.
2. **Right Weakening:**
   From Lemma 4, if $A\supset B$ is true and $\lim_{x\to 0} P_x(A|C)=1$, then $\lim_{x\to 0} P_x(B|C)=1$. Therefore, if $A\supset B$ is true and $C\vdash A$, then $C\vdash B$.
3. **Reflexivity:**
   From the definition of probability, it is always valid.
4. **And:**
   From Lemma 6, if $\lim_{x\to 0} P_x(A|B)=1$ and $\lim_{x\to 0} P_x(A|C)=1$, then $\lim_{x\to 0} P_x(A|B\wedge C)=1$. Therefore, if $A\vdash B$ and $A\vdash C$, then $A\vdash B\wedge C$.
5. **Or:**
   From Lemma 7, if $\lim_{x\to 0} P_x(C|A)=1$ and $\lim_{x\to 0} P_x(C|B)=1$, then $\lim_{x\to 0} P_x(C|A\vee B)=1$. Therefore, if $A\vdash C$ and $B\vdash C$, then $A\vee B\vdash C$.

6. **Cautious Monotony:**
From Lemma 8, if $\lim_{x\to 0} P_x(B|A)=1$ and $\lim_{x\to 0} P_x(C|A)=1$, then $\lim_{x\to 0} P_x(C|A\wedge B)=1$. Therefore, if $A\mid\!\sim B$ and $A\mid\!\sim C$, then $A\wedge B\mid\!\sim C$.
7. **Rational Monotony:**
From Lemma 9, if $\lim_{x\to 0} P_x(C|A)=1$ and $\lim_{x\to 0} P_x(\neg B|A)\neq 1$, then $\lim_{x\to 0} P_x(C|A\wedge B)=1$. Therefore, if $A\mid\!\sim C$ and $A\not\mid\!\sim \neg B$, then $A\wedge B\mid\!\sim C$.

From the definition of rational consequence relation, $\mid\!\sim$ is rational. □

Note that the first six properties do not need convergence of probability function. So, in the definition of closed relation in the limit, if we remove the condition of convergence, we can show that the relation is not always rational but still preferential[1].

From Theorem 1 and Theorem 2 we have the following[2].

Theorem 3 *$\mid\!\sim$ is closed in the limit if and only if $\mid\!\sim$ is rational.*

There is another characterization for a closed consequence relation in the limit as follows.

Definition 12 *Let L be a finite propositional language and $\mid\!\sim$ be a consequence relation. $\mid\!\sim$ is said to be ε-*definable *if and only if there exists a function $\lambda:L^2\mapsto[0,1]$ such that*

for all $A, B\in L$, $A\mid\!\sim B$ if and only if $\lambda(A,B)=1$ and
for all $\varepsilon>0$, there exists a probability function P such that
for all $A, B\in L$, $|P(B|A)-\lambda(A,B)|<\varepsilon$.

An ε-definable consequence relation fits our intuitive meaning stated above and as the following theorems show, it is actually equivalent to a closed consequence relation in the limit and therefore, equivalent to a rational consequence relation.

Theorem 4 *$\mid\!\sim$ is closed in the limit if and only if $\mid\!\sim$ is ε-definable.*

Proof:

(1) Suppose $\mid\!\sim$ is closed in the limit. Then there exists a convergent probability function P_x with positive parameter x such that

$$A\mid\!\sim B \quad \text{if and only if} \quad \lim_{x\to 0} P_x(B|A)=1.$$

Let $\lambda:L^2\mapsto[0,1]$ be defined as follows.

$$\lambda(A,B)\stackrel{\text{def}}{=}\lim_{x\to 0} P_x(B|A).$$

Then, for all $A, B\in L$, $A\mid\!\sim B$ if and only if $\lambda(A,B)=1$.

And for all $A, B\in L$ and for all $\varepsilon>0$, there exists $\delta_{\varepsilon,(B|A)}$ such that for all x, if $\delta_{\varepsilon,(B|A)}>x>0$, $|P_x(B|A)-\lambda(A,B)|<\varepsilon$.

Take any arbitrary $\varepsilon>0$. Let δ_ε be the smallest value among the above $\delta_{\varepsilon,(B|A)}$. Let a probability function P for ε be defined as follows:

$$P=P_{\frac{\delta_\varepsilon}{2}}(B|A).$$

Then, for all $A, B\in L$, $|P(B|A)-\lambda(A,B)|<\varepsilon$.

Therefore, $\mid\!\sim$ is ε-definable.

(2) Suppose $\mid\!\sim$ is ε-definable. Then, there exists a function $\lambda:L^2\mapsto[0,1]$ such that
for all $A, B\in L$, $A\mid\!\sim B$ if and only if $\lambda(A,B)=1$ and
for all $\varepsilon>0$, there exists a probability function P such that
for all $A, B\in L$, $|P(B|A)-\lambda(A,B)|<\varepsilon$.

Take any arbitrary $\varepsilon>0$. And let P be the above probability function for ε and define the value at ε for a probability function P_x with positive parameter x as follows.

$$P_\varepsilon(A)\stackrel{\text{def}}{=}P(A).$$

For all $A, B\in L$ and for all $\varepsilon>0$ and for all x, if $\varepsilon>x>0$, then $|P_x(B|A)-\lambda(A,B)|<\varepsilon$.

Then, for all $A, B\in L$ and for all $\varepsilon>0$, there exists δ (δ is any arbitrary value such that $\varepsilon>\delta>0$) such that if $\delta>x>0$ then $|P_x(B|A)-\lambda(A,B)|<\varepsilon$. Therefore, for all $A, B\in L$, $\lim_{x\to 0} P_x(B|A)=\lambda(A,B)$. Thus, P_x is a convergent function, and $\lim_{x\to 0} P_x(B|A)=1$ if and only if $A\mid\!\sim B$.

Therefore, $\mid\!\sim$ is closed in the limit. □

From the equivalence of closed relation in the limit and ε-definable relation, we also have the following theorem.

Theorem 5 *$\mid\!\sim$ is rational if and only if $\mid\!\sim$ is ε-definable.*

Adams [1] and Pearl [16] present a probabilistic treatment of nonmonotonic reasoning called ε-semantics. This treatment is similar to our work in the sense that it gives an infinitesimal analysis for nonmonotonic reasoning. However, we can show that ε-definability implies ε-consistency if we regard a consequence relation as a set of conditional assertion K so that $A\mid\!\sim B$ if and only if $A\Rightarrow B\in K$, where $A\Rightarrow B$ is a conditional assertion.

Definition 13 (Adams)
*A set of conditional assertions K is said to be ε-*consistent *if and only if for all $\varepsilon>0$, there exists a probability function P such that*

$$\textit{if}\quad A\Rightarrow B\in K, \quad \textit{then}\quad P(B|A)\geq 1-\varepsilon.$$

If we regard a consequence relation as a set of conditional assertions K, we can say that Adams considers a probability function P for a pair of formulae in $\mid\!\sim$ so that $P(B|A)\geq 1-\varepsilon$ but does not exclude a probability function P such that $P(B|A)\geq 1-\varepsilon$ even if $A\not\mid\!\sim B$. For example, let L contain only two propositions P and Q, and a set of conditional assertions be $\{P\Rightarrow Q\}$. The set is ε-consistent whereas the set is neither a preferential consequence relation nor a rational consequence relation when we consider the set as a consequence relation.

This example shows that ε-consistency does not characterize a consequence relation exactly. The follow-

[1]Goldszmidt, Morris and Pearl [4] show that the closed relation in the limit without the condition of convergence exactly characterizes a preferential consequence relation.
[2]Independently, Goldszmidt, Morris and Pearl [4] have obtained a similar result to this theorem, as have Lehmann and Magidor.

ing result shows that ε-definability implies ε-consistency.

Theorem 6 *Suppose a consequence relation $\vdash$ is regarded as a set of conditional assertions. If $\vdash$ is ε-definable, then it is ε-consistent.*

Adams also considers ε-entailment defined as follows.

Definition 14 (Adams)
Let K be a set of conditional assertions and $A \Rightarrow B$ be a conditional assertion. We say $A \Rightarrow B$ is ε-entailed *by K if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every probability function P, if for every $C \Rightarrow D \in K$, $P(D|C) \geq 1-\delta$ then $P(B|A) \geq 1-\varepsilon$.*

Lehmann and Magidor [8] study preferential entailment which is closely related to ε-entailment.

Definition 15 (Lehmann [7])
Let K be a set of conditional assertions and $A \Rightarrow B$ be a conditional assertion. $A \Rightarrow B$ is preferentially entailed *by K if it is satisfied by all preferential models of K. We say the set of all conditional assertions that are preferentially entailed by K is the* preferential closure *of K.*

Proposition 3 (Lehmann [7])
Let K be a set of conditional assertions. Then, the preferential closure of K is a preferential consequence relation. If K is a preferential consequence relation then K is the preferential closure of itself.

Lehmann and Magidor show the following equivalence between ε-entailment and preferentially entailment.

Proposition 4 (Lehmann and Magidor)
For every finite set of conditional assertions K, $A \Rightarrow B$ is ε-entailed by K if and only if $A \Rightarrow B$ is preferentially entailed by K.

Let K be a conditional assertions. Let us call all conditional assertions which are ε-entailed by K the *ε-closure* of K. Then, from Proposition 3 and Proposition 4, we can have one-to-one correspondence between an ε-closure and a preferential consequence relation for a logically finite language. Therefore, from the above discussion and Theorem 5, we can say the following.

Theorem 7 *Suppose a consequence relation $\vdash$ is regarded as a set of conditional assertions. If $\vdash$ is ε-definable, then it is the ε-closure of itself.*

4. Consequence Relation and Circumscription

4.1 Preferential Consequence Relation and Circumscription

Here, we refer circumscription to the following definition. This is a slightly modified version of generalized circumscription [9] as we use $<$ instead of $\leq$.

Definition 16 *Let A be a propositional formula and $\mathbf{P}$ be a tuple of propositions and $\mathbf{p}$ be a tuple of propositional variables. Then Circum $(A; <^{\mathbf{P}})$ is defined as follows:*

$$A(\mathbf{P}) \wedge \neg\exists \mathbf{p}(A(\mathbf{p}) \wedge \mathbf{p} <^{\mathbf{P}} \mathbf{P}),$$

where $A(\mathbf{p})$ is obtained by replacing every proposition of $\mathbf{P}$ in $A(\mathbf{P})$ by every corresponding propositional variable, and $\mathbf{p} <^{\mathbf{P}} \mathbf{P}$ is a formula which includes a tuple of propositions $\mathbf{P}$ and a tuple of propositional variables $\mathbf{p}$ and satisfies the following two conditions:

1. *For any $\mathbf{Q}$, $\mathbf{Q} \not<^{\mathbf{P}} \mathbf{Q}$*
2. *For any $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$, if $\mathbf{Q} <^{\mathbf{P}} \mathbf{R}$ and $\mathbf{R} <^{\mathbf{P}} \mathbf{S}$, then $\mathbf{Q} <^{\mathbf{P}} \mathbf{S}$*

Then, an interpretation order in circumscription is defined as follows. $I_1 <^{\mathbf{P}} I_2$ if and only if $\mathbf{Q} <^{\mathbf{P}} \mathbf{R}$ is true when we replace every proposition Q_i in $\mathbf{Q}$ by interpretation of the corresponding proposition P_i in $\mathbf{P}$ under I_1 and every proposition R_i in $\mathbf{R}$ by interpretation of the corresponding proposition P_i in P under I_2.

Then, we can think of the following preferential model $W = \langle S, l, \prec \rangle$ where a set of interpretations for propositional symbols (in other words, a set of possible worlds, $\mathscr{U}$) is S, and l is an identity function and $\prec$ is a strict partial order $<^{\mathbf{P}}$ over these interpretations. We say the preferential model is *defined by* $<^{\mathbf{P}}$. As Kraus et al. [6] pointed out, if S is finite, the smoothness condition is always satisfied. Here, we consider a finite set of possible worlds, so the smoothness condition is always satisfied.

Then we can have the following relationship between circumscription and preferential consequence relation.

Definition 17 *Let $<^{\mathbf{P}}$ be a strict partial order over interpretations. The consequence relation* defined by $<^{\mathbf{P}}$ *is denoted as $\vdash_{<^{\mathbf{P}}}$ and defined as: $A \vdash_{<^{\mathbf{P}}} B$ if and only if Circum$(A; <^{\mathbf{P}}) \models B$.*

Proposition 5 *Let $<^{\mathbf{P}}$ be a strict partial order over interpretations. The consequence $\vdash_{<^{\mathbf{P}}}$ defined by $<^{\mathbf{P}}$ is a preferential consequence relation.*

Although a consequence relation defined by circumscription is a preferential consequence relation, the converse is not true in general. In propositional circumscription, for any satisfiable formula A, $A \not\vdash_{<^{\mathbf{P}}} \mathbf{F}$[1] (we say $\vdash$ is *proper*), but in preferential consequence relation, this is not always the case.

And since we use an identity function for l in circumscription, there is a preferential consequence relation in a language which can not be represented by circumscription in the same language.

For example, Suppose L contains only two propositions P and Q, and S consists of five states $s_1 \cdots s_5$ which satisfies the following conditions:

1. $l(s_1) \models P \wedge \neg Q$.
2. $l(s_2) \models \neg P \wedge Q$.
3. $l(s_3) \models P \wedge Q$.
4. $l(s_4) \models P \wedge Q$.
5. $l(s_5) \models \neg P \wedge \neg Q$.
6. $s_1 \prec s_3$ and $s_2 \prec s_4$ and there is no other pair which satisfies $\prec$.

Note that s_3 and s_4 are mapped to the same interpretation. Let us consider a consequence relation $\vdash_W$ where

[1] $\mathbf{F}$ is falsity.

$W=\langle S,l,\prec\rangle$. Then, although $P\vee Q\vdash_W(\neg P\wedge Q)\vee(P\wedge\neg Q)$, $P\nvdash_W P\wedge\neg Q$ and $Q\nvdash_W\neg P\wedge Q$. And this relation can not be expressed in circumscription of L because for any order $<^{(P,Q)}$ over interpretations if we have $P\vee Q\vdash_{<(P,Q)}(\neg P\wedge Q)\vee(P\wedge\neg Q)$, then we must have an order between interpretations $\{P, Q\}$ and $\{P, \neg Q\}$ or between interpretations $\{\neg P, Q\}$ and $\{P, Q\}$, that is, $P\vdash_{<(P,Q)}P\wedge\neg Q$ or $Q\vdash_{<(P,Q)}\neg P\wedge Q$. This is because we have the states mapped to the same interpretation.

We say a formula A is *complete* if for every formula B in L, $A\models B$ or $A\models\neg B$. A complete formula corresponds with an interpretation. Then, the following property excludes a preferential consequence relation such that two or more states are mapped to the same interpretation in a corresponding preferential model.

If C is complete and $A\vee B\vdash\neg C$, then $A\vdash\neg C$ or $B\vdash\neg C$[1].

Theorem 8 $\vdash$ *is a proper preferential consequence relation and satisfies the above property if and only if there is some* $<^{\mathbf{P}}$ *such that* $\vdash_{<^{\mathbf{P}}}=\vdash$

Proof:

We can easily show that every consequence relation defined by a circumscription is a proper preferential consequence relation and satisfies the above property. We show the converse. Suppose $\vdash$ is a proper preferential consequence relation and satisfies the above property. Let $\alpha(\mathbf{P})$ and $\beta(\mathbf{P})$ be complete formulae. We construct $\vdash_{<^{\mathbf{P}}}$ as follows. Define $\alpha(\mathbf{P})\prec\beta(\mathbf{P})$ if and only if $\alpha(\mathbf{P})\vee\beta(\mathbf{P})\vdash\alpha(\mathbf{P})$ and $\alpha(\mathbf{P})\neq\beta(\mathbf{P})$. Then $\prec$ is a irreflexive and transitive relation. We can show transitivity by using Lemma 5.5 (21) [6, page 192] stating that for a preferential consequence relation, $\vdash$, if $\alpha\vee\beta\vdash\alpha$ and $\beta\vee\gamma\vdash\beta$ then $\alpha\vee\gamma\vdash\alpha$.

Suppose we collect all pairs in $\prec$: $\alpha_1(\mathbf{P})\prec\beta_1(\mathbf{P}), \cdots, \alpha_n(\mathbf{P})\prec\beta_n(\mathbf{P})$. Then, $\mathbf{p}<^{\mathbf{P}}\mathbf{P}$ is defined as follows: $(\alpha_1(\mathbf{p})\wedge\beta_1(\mathbf{P}))\vee\cdots\vee(\alpha_n(\mathbf{p})\wedge\beta_n(\mathbf{P}))$. First, we show if $A\vdash B$ then $A\vdash_{<^{\mathbf{P}}}B$. Suppose $A\vdash B$. We collect all complete formulae $C_1,\cdots, C_n$ which do not imply B. Then, we can write B as $\neg C_1\wedge\neg C_2\wedge\cdots\wedge\neg C_n$.

If $A\models\neg C_i$ then $A\vdash_{<^{\mathbf{P}}}\neg C_i$.

Otherwise, that is, in the case of $C_i\models A$, we can write A as $(A_1\vee C_i)\vee(D_1\vee C_i)$ where $A_1\models\neg D_1\wedge\neg C_i$ and D_1 is a complete formula which is not equivalent to C_i. Then, from the above property, $(A_1\vee C_i)\vdash\neg C_i$ or $(D_1\vee C_i)\vdash\neg C_i$.

If $(D_1\vee C_i)\vdash\neg C_i$, then we stop this process. Otherwise, we continue this process until we find D_k such that $(D_k\vee C_i)\vdash\neg C_i$. This process will stop because A can be represented as a finite disjunction of complete formulae.

Then we can write A as $D_k\vee C_i\vee E_1\vee\cdots\vee E_m$ where $D_k\vee C_i\vdash\neg C_i$ and E_i is a complete formula such that $E_i\models A\wedge\neg C_i$. Then, from the construction of $<^{\mathbf{P}}$, $D_k\vee C_i\vdash_{<^{\mathbf{P}}}\neg C_i$. And since $E_i\models\neg C_i$, $E_i\vdash_{<^{\mathbf{P}}}\neg C_i$. Then the fifth property of preferential consequence relation, $A\vdash_{<^{\mathbf{P}}}\neg C_i$.

Therefore, $A\vdash_{<^{\mathbf{P}}}\neg C_1\wedge\cdots\wedge\neg C_n$, that is, $A\vdash_{<^{\mathbf{P}}}B$.

Now, we show if $A\vdash_{<^{\mathbf{P}}}B$ then $A\vdash B$. Suppose $A\vdash_{<^{\mathbf{P}}}B$. We collect all complete formulae $C_1,\cdots, C_n$ which do not imply B. Then, we can write B as $\neg C_1\wedge\neg C_2\wedge\cdots\wedge\neg C_n$.

If $A\models\neg C_i$ then $A\vdash\neg C_i$.

Otherwise, that is, in the case of $C_i\models A$, since *Circum* $(A; <^{\mathbf{P}})\not\models C_i$, there exists a complete formula D such that $D\models A$ and $D\prec C_i$. Then, from the construction of $<^{\mathbf{P}}$, $(D\vee C_i)\vdash\neg C_i$.

Then we can write A as $D\vee C_i\vee E_1\vee\cdots\vee E_m$ where $D\vee C_i\vdash\neg C_i$ and E_i is a complete formula such that $E_i\models A\wedge\neg C_i$. Then the fifth property of preferential consequence relation, $A\vdash\neg C_i$.

Therefore, $A\vdash\neg C_1\wedge\cdots\wedge\neg C_n$, that is, $A\vdash B$. □

4.2 Rational Consequence Relation and Circumscription

Unfortunately, although a consequence relation defined by circumscription is always preferential, it is not always rational. We show it by using the following lemma.

Lemma 10 *Let S be a set and* $\prec$ *be a strict partical order. The following are equivalent.*

1. *There is a totally ordered set* Ω *whose total order is denoted as* $<$ *and a function* $r:S\mapsto\Omega$ *such that* $s\prec t$ *if and only if* $r(s)<r(t)$.
2. *For all* $s\in S$, *for all* $t\in S$ *and for all* $u\in S$, *if* $s\prec t$ *then either* $s\prec u$ *or* $u\prec t$.

Proof:

Suppose 1.

Then, there is a totally ordered set Ω and a function r from S to Ω such that $s\prec t$ if and only if $r(s)<r(t)$. Suppose $s\prec t$. Then, for all u, $r(s)<r(u)$ or $r(u)<r(t)$ because Ω is a totally ordered set. Therefore, $s\prec u$ or $u\prec t$.

Suppose 2.

We define a binary relation $\sim$ over S as follows:

$$s\sim t \overset{\text{def}}{=} \neg(s\prec t)\wedge\neg(t\prec s).$$

Then, we can show $\sim$ is an equivalence relation. Reflexivity and symmetry can be proved easily. We prove transitivity. Suppose $(s\sim t)\wedge(t\sim u)$. Then,

$$\neg(s\prec t)\wedge\neg(t\prec s)\wedge\neg(t\prec u)\wedge\neg(u\prec t).$$

From 2, if $\neg(s\prec t)\wedge\neg(t\prec u)$ then $\neg(s\prec u)$ and if $\neg(u\prec t)\wedge\neg(t\prec s)$ then $\neg(u\prec s)$. Therefore, $s\sim u$.

Suppose Ω is $S/\sim$ and a binary relation $<$ over Ω is defined as follows:

$$x<y \quad \text{if and only if} \quad \exists s\exists t((s\in x)\wedge(t\in y)\wedge(s\prec t)).$$

Then, $<$ is a total order.

Let r be a function from S to Ω be defined as follows.

$$r(s)=x \quad \text{such that} \quad s\in x.$$

Then $s\prec t$ if and only if $r(s)<r(t)$. □

[1] This property corresponds with (R8) in [5].

Now, we give a class of circumscription whose consequence relation is not rational.

Theorem 9

1. *If a tuple of propositions,* **P** *does not contain all propositions in L and for any non-trivial partial order* $<^{\mathbf{P}}$ *(there are some interpretations, I and J such that* $J<^{\mathbf{P}}I$*), the consequence relation defined by* $<^{\mathbf{P}}$ *is always non-rational.*
2. *If* **P** *contains all propositions in L, then a consequence relation defined by minimizing one or two propositions in parallel is rational.*
3. *Even if* **P** *contains all propositions in L, a consequence relation defined by minimizing more than three propositions in parallel is always non-rational.*

Proof:

1. Since $<^{\mathbf{P}}$ is non-trivial, there exist some interpretations, I and J such that $J<^{\mathbf{P}}I$. And there exists some proposition P which is not in **P**. Let K be a truth assignment which is the same as J except the assignment of P. Then since $J<^{\mathbf{P}}I$, the assignment of P in I is the same as in J from the definition of $<^{\mathbf{P}}$. Then, K is different from J and I in the assignment of P. Therefore, $\neg(J<^{\mathbf{P}}K)$ and $\neg(K<^{\mathbf{P}}I)$. From Lemma 10, the preferential model defined by $<^{\mathbf{P}}$ is not ranked. Therefore, the consequence relation defined by $<^{\mathbf{P}}$ is not rational from Proposition 2.
2. We can easily check that a preferential model defined by minimizing one or two propositions is ranked.
3. Let **P** contain the following minimized propositions, P, Q and R. And let the following three interpretations I, J and K satisfy the following conditions:
   (a) Every assignments are the same except assignments for P, Q and R.
   (b) $I \models \neg P \wedge \neg Q \wedge R$, $J \models P \wedge Q \wedge \neg R$ and $K \models \neg P \wedge Q \wedge R$.

   Then $I<^{\mathbf{P}}K$, but $\neg(I<^{\mathbf{P}}J)$ and $\neg(J<^{\mathbf{P}}K)$. From Lemma 10, $<^{\mathbf{P}}$ is not ranked. Therefore, a consequence relation defined by minimizing more than three propositions is not rational from Proposition 2. □

Although rational monotony corresponds with one of fundamental conditions for minimal change of belief proposed by Gärdenfors [3], there are several examples in commonsense reasoning which correspond with the third case of Theorem 9. For example, consider the following axiom[1].

$A_1 \stackrel{\text{def}}{=}$
$((Japanese \wedge \neg Ab1) \supset \neg Big) \wedge$
$((Hockey\text{-}player \wedge \neg Ab2) \supset Strong) \wedge$
$((Professor \wedge \neg Ab3) \supset \neg Strong) \wedge$
$(Strong \supset Big) \wedge$
$Japanese \wedge Hockey\text{-}player \wedge Professor.$

(If a man is a Japanese, he is normally not big, and if a man is a hockey player, he is normally strong, and if a man is a professor, he is normally not strong, and if a man is strong, he is big, and the man is a Japanese professor who plays hockey.)

If we minimize $Ab1$, $Ab2$ and $Ab3$ in parallel with every proposition allowed to vary and consider the consequence relation $\vdash$ defined by this minimization, then we can show the following.

$$A_1 \vdash (\neg Big \wedge \neg Strong) \vee (Big \wedge Strong),$$

and

$$A_1 \nvdash \neg Big.$$

However,

$$A_1 \wedge Big \nvdash (\neg Big \wedge \neg Strong) \vee (Big \wedge Strong).$$

So, this case does not satisfy rational monotony.

Another example is a closed world assumption. In this case, we minimize all propositions and so, we do not have rational monotony if a number of propositions is more than three.

Note that if we minimize more than three propositions in a *prioritized circumscription*, there is a case where rational monotony is obtained. For example, if we minimize $Ab1$ prior to $Ab2$ and $Ab3$ in the above Japanese-professor-playing-hockey example, rational monotony is obtained.

So, one may argue that a rational consequence relation is not practically *rational*. However, what we would like to say here is *not* whether it is rational or not, but that circumscription in general does not have the probabilistic semantics which we have defined so far and that if an order defined by circumscription is ranked, then it has a probabilistic *rationale*.

4.3 Probabilistic Interpretation for Lazy Circumscription

In this subsection, we consider the following kind of circumscription.

Definition 18 *Circumscription* $<^{\mathbf{P}}$ *is* lazy *if the preferential model defined by* $<^{\mathbf{P}}$ *is ranked.*

We can show that a consequence relation $\vdash$ is proper and rational if and only if there is some $<^{\mathbf{P}}$ of lazy circumscription such that $\vdash_{<^{\mathbf{P}}} = \vdash$. If a circumscription $<^{\mathbf{P}}$ is lazy, the consequence relation $\vdash_{<^{\mathbf{P}}}$ is rational. That is, for all formulae, A, B and C, if $A \vdash_{<^{\mathbf{P}}} C$ and $A \nvdash_{<^{\mathbf{P}}} \neg B$ then $A \wedge B \vdash_{<^{\mathbf{P}}} C$. This means that in lazy circumscription, belief revision does not occur if the added information is consistent with the current belief.

And, if a circumsciption is lazy, we can attach a probability function used in the proof of Theorem 1 because the preferential model defined by $<^{\mathbf{P}}$ is ranked. In this case, we consider a set of interpretations $\mathcal{U}$ as a set of states.

Example 1.

Let a set of propositions be $\{P, Q\}$. Then, $\mathcal{U}$ consists of four possible worlds:

$$\{\langle \neg P, \neg Q \rangle, \langle P, \neg Q \rangle, \langle \neg P, Q \rangle, \langle P, Q \rangle\}.$$

Suppose we minimize P and Q in parallel. We denote

[1] This example was suggested by David Poole.

the strict partial order relation by this minimization as $<^{(P,Q)}$. Then the consequence relation defined by $<^{(P,Q)}$ is as follows:

$A(P,Q) \vdash_{<^{(P,Q)}} B(P,Q)$ if and only if
$A(P,Q) \wedge \neg \exists p \exists q (A(p,q) \wedge ((p,q) < (P,Q)))$
$\models B(P,Q)$,

where $(p,q) < (P,Q)$ is the following abbreviation:

$$(p,q) < (P,Q) \stackrel{\text{def}}{=} (p \supset P) \wedge (q \supset Q) \wedge \neg((P \supset p) \wedge (Q \supset q)).$$

The preferential model defined by $<^{(P,Q)}$ is ranked (Fig. 1). In the figure, a lower interpretation is more preferable than an upper interpretation. In probabilistic semantics, we regard this order as an order of probability. This means that a lower interpretation is more probable than an upper interpretation. Moreover, we make the probability function of an interpretation in $(i+1)$-th rank be x times as much as that of an interpretation in i-th rank so that we can ignore less probable interpretation as x approaches 0.

Let η_i be a number of interpretations in i-th rank and η_i^A be a number of interpretations satisfying A in i-th rank. From Fig. 1, $\eta_1=1$, $\eta_2=2$ and $\eta_3=1$.

Let a probability function P_x with a positive parameter x be defined as follows.

$$P_x(A) \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{3} \eta_i^A * x^{i-1}}{\sum_{i=1}^{3} \eta_i * x^{i-1}} = \frac{\eta_1^A + \eta_2^A * x + \eta_3^A * x^2}{1+2x+x^2}$$

Then, this function is convergent and

$$\lim_{x \to 0} P_x(B|A) = \begin{cases} 1 & \text{if } P_x(A)=0 \\ \dfrac{\eta_{mr(A)}^{A \wedge B}}{\eta_{mr(A)}^{A}} & \text{otherwise} \end{cases}$$

where $mr(A)$ is the minimum rank where some interpretation satisfies A.

Intuitively, making x approach 0 means that we consider only the most probable interpretations which satisfy A and the fact that $P_x(B|A)$ approaches 1 means that in all the most probable interpretations which satisfy A, B is extremely probable. This is a probabilistic semantics for lazy circumscription.

Let $\vdash$ be a consequence relation as follows.

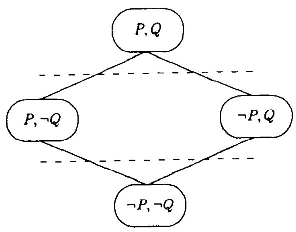

Fig. 1 Strict Partial Order by Minimizing P and Q.

$$A \vdash B \text{ if and only if } \lim_{x \to 0} P_x(B|A) = 1$$

Let us check if $P \vee Q \vdash \neg P \vee \neg Q$.

Since $\langle P \wedge \neg Q \rangle \models P \vee Q$, $mr(P \vee Q) = 2$. And since $\eta_2^{P \vee Q} = 2$ and $\eta_2^{(P \vee Q) \wedge (\neg P \vee \neg Q)} = 2$,

$$\lim_{x \to 0} P_x(\neg P \vee \neg Q | P \vee Q) = \frac{\eta_2^{(P \vee Q) \wedge (\neg P \vee \neg Q)}}{\eta_2^{P \vee Q}} = 1.$$

Therefore, $P \vee Q \vdash \neg P \vee \neg Q$. This corresponds with the result of $P \vee Q \vdash_{<^{(P,Q)}} \neg P \vee \neg Q$. However, suppose we check if $P \vee Q \vdash P \wedge \neg Q$.

Since $\eta_2^{(P \vee Q) \wedge (P \wedge \neg Q)} = \eta_2^{P \wedge \neg Q} = 1$,

$$\lim_{x \to 0} P_x(P \wedge \neg Q | P \vee Q) = \frac{\eta_2^{P \wedge \neg Q}}{\eta_2^{P \vee Q}} \neq 1.$$

Therefore, $P \vee Q \not\vdash P \wedge \neg Q$. This corresponds with the result of $P \vee Q \not\vdash_{<^{(P,Q)}} P \wedge \neg Q$. Actually, $\vdash$ is equivalent to $\vdash_{<^{(P,Q)}}$ from Theorem 1.

Example 2.

Another example is the "flying bird and non-flying penguin" example.

Suppose that we consider a set of proposition $\{B, P, F\}$ where B expresses "bird", and P expresses "penguin" and F expresses "flying", and we maximize $P \supset \neg F$ prior to $B \supset F$. We denote the strict partial order relation by this maximization as $<^{(B,P,F)}$. Then the consequence relation defined by $<^{(B,P,F)}$ is as follows:

$A(B,P,F) \vdash_{<^{(B,P,F)}} B(B,P,F)$ if and only if
$A(B,P,F) \wedge \neg \exists b \exists p \exists f (A(b,p,f) \wedge (b,p,f) < (B, P, F)) \models B(B,P,F)$,

where $(b,p,f) < (B,P,F)$ is the following abbreviation:

$$\begin{aligned} &(b,p,f) < (B,P,F) \stackrel{\text{def}}{=} \\ &((P \supset \neg F) \supset (p \supset \neg f)) \wedge \\ &(((P \supset \neg F) \equiv (p \supset \neg f)) \supset ((B \supset F) \supset (b \supset f))) \wedge \\ &\neg(((p \supset \neg f) \supset (P \supset \neg F)) \wedge \\ &\quad (((p \supset \neg f) \equiv (P \supset \neg F)) \supset ((b \supset f) \supset (B \supset F)))). \end{aligned}$$

We use prioritized formula circumscription [13] to obtain the above formula. We minimize both $\neg(B \supset F)$ and $\neg(P \supset \neg F)$ but minimize $\neg(P \supset \neg F)$ at higher priority than $\neg(B \supset F)$. This corresponds with maximizing $(P \supset \neg F)$ prior to $(B \supset F)$.

Then, the consequence relation defined by $<^{(B,P,F)}$ is rational, because the preferential model by $<^{(B,P,F)}$ is ranked (Fig. 2).

Let η_i be a number of interpretations in i-th rank and η_i^A be a number of interpretations satisfying A in i-th rank. From Fig. 2, $\eta_1=4$, $\eta_2=2$ and $\eta_3=2$.

Let a probability function P_x with a positive parameter x be defined as follows.

$$P_x(A) \stackrel{\text{def}}{=} \frac{\sum_{i=1}^{3} \eta_i^A * x^{i-1}}{\sum_{i=1}^{3} \eta_i * x^{i-1}} = \frac{\eta_1^A + \eta_2^A * x + \eta_3^A * x^2}{4+2x+2x^2}$$

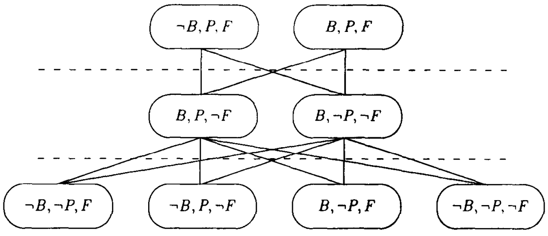

Fig. 2 Strict Partial Order for Flying Bird and Non-flying Penguin.

Then, this function is convergent and

$$\lim_{x\to 0} P_x(B|A) = \begin{cases} 1 & \text{if } P_x(A)=0 \\ \dfrac{\eta^{A\wedge B}_{mr(A)}}{\eta^{A}_{mr(A)}} & \text{otherwise} \end{cases}$$

Let $\mid\!\sim$ be a consequence relation as follows.

$$A \mid\!\sim B \quad \text{if and only if} \quad \lim_{x\to 0} P_x(B|A)=1$$

Let us check if $B\wedge(P\supset B)\mid\!\sim F$.

Since $\langle B, \neg P, F\rangle \models B\wedge(P\supset B)$, $mr(B\wedge(P\supset B))=1$. And since $\eta_1^{B\wedge(P\supset B)}=\eta_1^{B}=1$ and $\eta_1^{B\wedge(P\supset B)\wedge F}=\eta_1^{B\wedge F}=1$,

$$\lim_{x\to 0} P_x(F|B\wedge(P\supset B))=\lim_{x\to 0} P_x(F|B)=\frac{\eta_2^{B\wedge F}}{\eta_2^{B}}=1.$$

Therefore, $B\wedge(P\supset B)\mid\!\sim F$. This corresponds with the result of $B\wedge(P\supset B)\mid\!\sim_{<(B,P,F)} F$.

And, suppose we check if $P\wedge(P\supset B)\mid\!\sim \neg F$.

Since $\langle B, P, \neg F\rangle \models P\wedge(P\supset B)$, $mr(P\wedge(P\supset B))=2$. And since $\eta_2^{P\wedge(P\supset B)}=\eta_2^{P\wedge B}=1$ and $\eta_2^{P\wedge(P\supset B)\wedge\neg F}=\eta_2^{P\wedge B\wedge\neg F}=1$,

$$\lim_{x\to 0} P_x(\neg F|P\wedge(P\supset B))=\lim_{x\to 0} P_x(\neg F|P\wedge B) =\frac{\eta_2^{P\wedge B\wedge\neg F}}{\eta_2^{P\wedge B}}=1.$$

Therefore, $P\wedge(P\supset B)\mid\!\sim \neg F$. This corresponds with the result of $P\wedge(P\supset B)\mid\!\sim_{<(B,P,F)} \neg F$.

5. Conclusion

We propose a probabilistic semantics called a closed consequence relation in the limit for lazy nonmonotonic reasoning and show that a consequence relation is closed in the limit if and only if it is rational. Then, we apply our result to giving a probabilistic semantics for a class of circumscription which has lazy nonmonotonicity.

We think we need to do the following research.

1. Lazy circumscription is defined in terms of order over interpretations. But we do not have a syntactical characterization for lazy circumscription. We would like to know which form of a formula characterizes a lazy circumscription.
2. We would like to know a probabilistic semantics which exactly characterizes a class of consequence relations defined by the whole class of circumscription.
3. We can not apply our result to Default Logic [18] or Autoepistemic Logic [14] because a consequence relation defined by these logics is not even preferential as shown in [11]. We must extend our results to apply them to these logics.

Acknowledgements

I would like to thank Kurt Konolige, Moises Goldszmidt, Johan van Benthem and anonymous referees for instructive comments in an earlier version of this paper and especially, David Poole for pointing out an error of Definition 12 and providing an example which does not satisfy rational monotony, also Daniel Lehmann for pointing out an error of Definition 9 in an earlier version of the paper.

References

1. Adams, E. The Logic of Conditionals, D. Reidel, Dordrecht (1975).
2. Gabbay, D. Theoretical Foundations for Non-monotonic Reasoning in Expert Systems, *Logics and Models of Concurrent Systems*, K. R. Apt ed., Springer-Verlag (1985), 439–457.
3. Gärdenfors, P. Knowledge in Flux: Modeling the Dynamics of Epistemic States, MIT Press (1988).
4. Goldszmidt, M., Morris, P. and Pearl, J. A Maximum Entropy Approach to Nonmonotonic Reasoning, *Proc. of AAAI-90* (1990), 646–652.
5. Katsuno, H. and Mendelzon, A. O. Propositional Knowledge Base Revision and Minimal Change, Dept. of Computer Science, University of Toronto, Canada, Technical Report, KRR-TR-90-3 (1990).
6. Kraus, S., Lehmann, D. and Magidor, M. Nonmonotonic Reasoning, Preferential Models and Cumulative Logics, *Arftificial Intelligence*, **44** (1990), 167–207.
7. Lehmann, D. What Does a Conditional Knowledge Base Entail?, *Proc. of KR-89* (1989), 212–222.
8. Lehmann, D. and Magidor, M. Rational Logics and Their Models: a Study in Cumulative Logics, Dept. of Computer Science, Hebrew University, Jerusalem, Israel, Technical Report, #TR-88-16 (1988).
9. Lifschitz, V. Some Results on Circumscription, *Proc. of Nonmonotonic Reasoning Workshop* (1984), 151–164.
10. Lifschitz, V. Open Problems on the Border of Logic and AI, *Unpublished Manuscript* (1989).
11. Makinson, D. General Theory of Cumulative Inference, *Proc. of 2nd International Workshop on Non-monotonic Reasoning, Lecture Notes in Artificial Intelligence 346* (1988), 1–18.
12. McCarthy, J. Circumscription - a Form of Non-monotonic Reasoning, *Artificial Intelligence*, **13** (1980), 27–39.
13. McCarthy, J. Applications of Circumscription to Formalizing Common-sense Knowledge, *Artificial Intelligence*, **28** (1986), 89–116.
14. Moore, R. C. Semantical Considerations on Nonmonotonic Logic, *Artificial Intelligence*, **25** (1985), 75–94.
15. Neufeld, E. and Poole, D. Probabilistic Semantics and Defaults, *Proc. of 4th AAAI Workshop on Uncertainty in AI* (1988), 275–281.
16. Pearl, J. Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference, Morgan Kaufmann (1988).
17. Pearl, J. Probabilistic Semantics for Nonmonotonic Reasoning: A Survey, *Proc. of KR-89* (1989), 505–516.
18. Reiter, R. A Logic for Default Reasoning, *Artificial Intelligence*, **13** (1980), 81–132.
19. Satoh, K. A Probabilistic Interpretation for Lazy Nonmonotonic Reasoning, *Proc. of AAAI-90* (1990), 659–664.
20. Shoham, Y. Reasoning about Change: Time and Causation from the Standpoint of Artificial Intelligence, MIT Press (1988).
21. Spohn, W. A General Non-Probabilistic Theory of Inductive Reasoning, *Proc. of 4th AAAI Workshop on Uncertainty in AI* (1988), 315–322.

(Received April 18, 1991; revised July 23, 1991)