*Regular Paper*

# On Some Iterative Formulas for Solving Nonlinear Scalar Equations

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First, we show fourth order multipoint iterative formulas which find better approximations to a zero of a function $f(x)$. These formulas require one evaluation of $f(x)$, $f'(x)$ and $f''(x)$ respectively per iteration. Next, we show some third order one-point itrative formulas containing two parameters, and consider the monotonic global convergence of them.

## 1. Introduction

We will study numerical iterative formulas for the computation of the solution of the nonlinear scalar equation

$$f(x)=0 \tag{1}$$

where $f(x)$ is a real function of the real variable $x$. In [1, 2], various types of iterative formulas have been shown. Traub [1, PP. 192-197, PP. 200-204] showed two types of fourth order multipoint iteration functions. Both of them contain two parameters. One of them requires three evaluations of $f'(x)$ and one of $f(x)$ per iteration, the other one evaluation of $f(x)$ and three of $f'(x)$ per iteration. Jarratt [3] showed fourth order multipoint iterative formulas containing one parameter and requiring one evaluation of $f(x)$ and two of $f'(x)$ per iteration. Furthermore, in [6], we showed two types of fourth order multipoint iterative formulas. One of them requires one evaluation of $f'(x)$ and two of $f(x)$ per iteration. The other requires one evaluation of $f(x)$ and two of $f'(x)$ per iteration.

The following one-point iteration functions are well-known [1, p. 90], [2, P. 111]

$$\Phi_{2,0}(x)=x-h-A_2h^2$$

$$\Phi_{1,1}(x)=x-\frac{h}{1-A_2h} \quad \text{(Hally's iteration function)}$$

$$\Phi(x)=x-\frac{h}{\sqrt{1-2A_2h}}$$

(Ostrowski's iteration function)

where $h=\dfrac{f(x)}{f'(x)}$, and $A_2=\dfrac{f''(x)}{2f'(x)}$.

All of them have cubic convergence for all the simple zeros of Eq. (1), and require one evaluation of $f(x)$, $f'(x)$ and $f''(x)$ respectively per iteration.

In this paper, we will study a class of iterative formulas of the type

$$x_{n+1}=\phi(x_n), \quad n=0, 1, 2, \cdots \tag{2}$$

where $\phi(x)=x-hR(X)$, and $X=h\dfrac{f''(x+\alpha h)}{f'(x)}$, $\alpha$ is a parameter, and $R(t)$ is a function of $t$. In 2, we will show that for the class of iterative formulas (2), fourth order formulas can be obtained by suitable choices of the parameter $\alpha$ and the function $R(t)$. In 3, 4 and 5, we will show some third order one-point iterative methods and consider the monotonic global convergence of them to the zeros of $f(x)$ in Eq. (1). In 6, we will give some numerical examples.

## 2. Derivation of Formulas

We assume that $\zeta$ is the simple zero of $f(x)$ in Eq. (1) and that $f(x)$ and $R(t)$ are as smooth as necessary in $x$ and $t$ respectively. Developing $X$ and $R(X)$, we obtain

$$X=2A_2h+6\alpha A_3h^2+12\alpha^2A_4h^3+O(h^4) \tag{3}$$

and

$$R(X)=R(0)+R'(0)X+\frac{1}{2!}R''(0)X^2+\frac{1}{3!}R^{(3)}(0)X^3+O(h^4) \tag{4}$$

where $h\equiv h(x)=\dfrac{f(x)}{f'(x)}$, $X\equiv X(x)=\dfrac{f''(x+\alpha h)}{f'(x)}h$, and

$$A_j\equiv A_j(x)=\frac{f^{(j)}(x)}{j!f'(x)}, \quad j=2, 3, 4.$$

It then follows from (3) and (4) that we obtain

$$\begin{aligned}-hR(X)=&-R(0)h-2R'(0)A_2h^2-2R''(0)A_2^2h^3\\&-6\alpha R'(0)A_3h^3-\frac{4}{3}R^{(3)}(0)A_2^3h^4\\&-12\alpha R''(0)A_2A_3h^4-12\alpha^2R'(0)A_4h^4\\&+O(h^5).\end{aligned} \tag{5}$$

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Since the basic sequence [1, PP. 78-88] is given by

$$E_5(x)=x-h-A_2h^2-2A_2^2h^3+A_3h^3-5A_2^3h^4 +5A_2A_3h^4-A_4h^4,$$

it follows from Eq. (2) and (5) that we obtain

$$\begin{aligned}\phi(x)-E_5(x)=&[1-R(0)]h+[1-2R'(0)]A_2h^2\\&+[2-2R''(0)]A_2^2h^3\\&+[-1-6\alpha R'(0)]A_3h^3\\&+\left[5-\frac{4}{3}R^{(3)}(0)\right]A_2^3h^4\\&+[-5-12\alpha R''(0)]\\&\times A_2A_3h^4+[1-12\alpha^2R'(0)]A_4h^4\\&+O(h^5).\end{aligned}$$

Hence it follows from Theorem 5-2 [1, PP. 86-87] that for Eq. (2) to be fourth order, the following system of equations must be satisfied:

$$\left.\begin{aligned}1-R(0)=0\\1-2R'(0)=0\\2-2R''(0)=0\\-1-6\alpha R'(0)=0\end{aligned}\right\}. \qquad (6)$$

From the system (6), we obtain

$$R(0)=1,\quad R'(0)=\frac{1}{2},\quad R''(0)=1,\quad \alpha=-\frac{1}{3}. \qquad (7)$$

In the case of (7), it follows from Theorem 5-2 [1, PP. 86-87] that the asymptotic error constant of $\phi(x)$ $C$ is given by

$$C=\lim_{x\to\zeta}\frac{\phi(x)-E_5(x)}{h^4} =\left[5-\frac{4}{3}R^{(3)}(0)\right]\tilde{A}_2^3-\tilde{A}_2\tilde{A}_3+\frac{1}{3}\tilde{A}_4 \qquad (8)$$

where $\tilde{A}_j=A_j(\zeta)$, $j=2, 3, 4$.

Next, we give some forms of $R(X)$ and the asymptotic error constant $C$ which satisfies (7) and (8) respectively.

[I] $R(X)=\frac{1}{2}X^2+\frac{1}{2}X+1,\quad C=5\tilde{A}_2^3-\tilde{A}_2\tilde{A}_3+\frac{1}{3}\tilde{A}_4$

[II] $R(X)=\frac{1}{2}\left(1+\frac{1}{\theta}\right)X+\frac{1}{2}\left(2-\frac{1}{\theta^2}\right)+\frac{1}{2\theta^2(\theta X+1)},$

$$C=(4\theta+5)\tilde{A}_2^3-\tilde{A}_2\tilde{A}_3+\frac{1}{3}\tilde{A}_4 \qquad (9)$$

where $\theta$ is a parameter, and $\theta\neq 0$.

## 3. One-point Iterative Formulas

If $\alpha=0$ in Eq. (2), then we obtain the following iterative formulas

$$x_{n+1}=\phi(x_n),\quad n=0, 1, 2,\cdots \qquad (10)$$

where $\phi(x)=x-hR(X)$, and $X=h\dfrac{f''(x)}{f'(x)}$.

Then, the order of convergence for Eq. (10) is equal to 3 for all simple zeros of Eq. (1), iff $R(0)=1$ and $R'(0)=1/2$. Furthermore, since the basic sequence [1, PP. 78-88] is given by

$$E_4(x)=x-h-A_2h^2-2A_2^2h^3+A_3h^3,$$

the asymptotic error constant of $\phi(x)$ $C$ is given by

$$C=\lim_{x\to\zeta}\frac{\phi(x)-E_4(x)}{h^3}=\{2-2R''(0)\}\tilde{A}_2^2-\tilde{A}_3.$$

Next, we give some examples of $R(X)$ and the asymptotic error constant $C$.

**Example 1.** $R(X)=\dfrac{(\theta+1/2)X+1}{\beta X^2+\theta X+1}$,

$$C=(2+2\theta+4\beta)\tilde{A}_2^2-\tilde{A}_3$$

($\beta$, $\theta$: parameters).

Especially, if $\beta=\theta=0$, then

$$\phi(x)=\Phi_{2,0}(x),\quad C=2\tilde{A}_2^2-\tilde{A}_3.$$

Next, if $\beta=0$ and $\theta=-1/2$, then

$$\phi(x)=\Phi_{1,1}(x),\quad C=\tilde{A}_2^2-\tilde{A}_3.$$

**Example 2.** $R(X)=\dfrac{a+\sqrt{b}}{a+\sqrt{b-\sqrt{b}(a+\sqrt{b})X}}$

$$C=\left[\frac{1}{2}-\frac{a}{2\sqrt{b}}\right]\tilde{A}_2^2-\tilde{A}_3$$

($a$, $b$: parameters).

If $b=1$, then

$$\phi(x)\equiv\lambda(x) =x-\frac{a+1}{a+\sqrt{1-(a+1)X}}h. \text{ [Eq. (2.13) in [5]]}$$

Furthermore, if $a=0$, then

$$\phi(x)=\Phi(x).$$

## 4. Monotonicity of Convergence

Let $f(x)$ be a polynomial of exact degree $r>1$ with only real zeros given by the form

$$f(x)=\prod_{k=1}^{r}(x-\zeta_k) \qquad (11)$$

where $\zeta_k\leqq\zeta_{k+1}$ $(k=1,\cdots,r-1)$.

Then, it has been shown that both Ostrowski's method [2, PP. 110-115] and Laguerre's method [2, PP. 353-362] converge globally and monotonically to the zeros of $f(x)$.

Let $f(x)$ be given by the form

$$f(x)=x^m \exp(a+bx-cx^2)\prod_{k=1}^{\infty}\left(1-\frac{x}{\alpha_k}\right)\times\exp\left(\frac{x}{\alpha_k}\right) \quad (12)$$

where $m$ is non-negative integer, $a$, $b$, $c$ are real, and $c\geqq 0$, $\alpha_k$ are real, and $\Sigma\alpha_k^{-2}<\infty$. Then, it has been shown that both Ostrowski's method [2, PP. 124-126] and Halley's method [4] converge globally and monotonically to the zeros of $f(x)$.

Hansen and Patrick [5] have introduced the methods that form the sequence $x_n$ by the iteration rule:

$$x_{n+1}=\lambda(x_n),\quad n=0,1,2,\cdots,$$

and have shown that if $f(x)$ is a polynomial with real coefficients or if $f(x)$ is given by the form (12), then these methods converge globally and monotnically to the real zeros of $f(x)$ under some assumptions.

Next, we consider the iterative methods for $R(X)$ of $Ex$ 1 in 3 that form the sequence $x_n$ by the iteration rule:

$$x_{n+1}=x_n-K(x_n),\quad n=0,1,2,\cdots \quad (13)$$

where $K(x)\equiv h\dfrac{\left(\theta+\frac{1}{2}\right)X+1}{\beta X^2+\theta X+1}$.

At first, let $f(x)$ be given by the form (11).

Then, it follows from Rolle's theorem that $f'(x)$ has $r-1$ real zeros $\zeta_k'$ such that $\zeta_k\leqq\zeta_k'\leqq\zeta_{k+1}$ $(k=1,\cdots,r-1)$.

Especially, if $\zeta_k<\zeta_{k+1}$ for some $k$, we obtain $\zeta_k<\zeta_k'<\zeta_{k+1}$. For any real $x\neq\zeta_k$, we can define the associated zero $\zeta(x)$ of $f(x)$ to be $\zeta(x)=\zeta_1$ if $x<\zeta_1$, $\zeta(x)=\zeta_r$ if $x>\zeta_r$, $\zeta(x)=\zeta_k$ if $\zeta_k<x<\zeta_k'$, and $\zeta(x)=\zeta_{k+1}$ if $\zeta_k'<x<\zeta_{k+1}$. On the other hand, if $\zeta_k=\zeta_1$ $(k=2,\cdots,r)$, we can define the associated zero $\zeta(x)$ of $f(x)$ to be $\zeta(x)=\zeta_1$ if $x\neq\zeta_1$. Furthermore, $f(x)/f'(x)$ and $x-\zeta(x)$ have the same sign.

Next, taking the logarithmic derivative of (11) and differentiating it, we obtain

$$\frac{f'(x)}{f(x)}=\sum_{k=1}^{r}\frac{1}{x-\zeta_k} \quad (14)$$

$$-\left(\frac{f'(x)}{f(x)}\right)'=\frac{\{f'(x)\}^2-f(x)f''(x)}{\{f(x)\}^2}=\sum_{k=1}^{r}\frac{1}{(x-\zeta_k)^2}. \quad (15)$$

From (15), it follows that

$$1-X=h^2\sum_{k=1}^{r}\frac{1}{(x-\zeta_k)^2}\geqq 0. \quad (16)$$

Next, rearranging $r-1$ zeros except $\zeta(x)$ such that $\zeta_1\leqq\cdots\leqq\zeta_{r-1}$, we obtain

$$1-X-\frac{h^2}{|x-\zeta(x)|^2}=h^2\sum_{k=1}^{r-1}\frac{1}{(x-\zeta_k)^2}. \quad (17)$$

Therefore it follows from (17) that

$$|x-\zeta(x)|>\frac{|h|}{\sqrt{1-X}}. \quad (18)$$

The following lemma is needed below.

**Lamma 1.** If $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2\leqq\beta\leqq-\frac{1}{2}\left(\theta+\frac{1}{2}\right)$, then

$$\frac{1}{1-\frac{1}{2}X}\leqq\frac{\left(\theta+\frac{1}{2}\right)X+1}{\beta X^2+\theta X+1}\leqq\frac{1}{\sqrt{1-X}}\qquad (X<1). \quad (19)$$

**Proof.** It now follows from the assumption that we obtain $-\frac{3}{2}\leqq\theta\leqq-\frac{1}{2}$. Hence $\left(\theta+\frac{1}{2}\right)X+1=-\left(\theta+\frac{1}{2}\right)(1-X)+\theta+\frac{3}{2}>0$. Now, putting $g(X)=\beta X^2+\theta X+1$, we obtain $g'(X)=2\beta X+\theta<2\beta+\theta\leqq-\frac{1}{2}$. On the other hand, since $g(1)=\beta+\theta+1\geqq\frac{1}{2}\left(\theta+\frac{3}{2}\right)^2\geqq 0$, we obtain $g(X)>0$. Therefore, we obtain

$$\frac{\left(\theta+\frac{1}{2}\right)X+1}{\beta X^2+\theta X+1}\geqq\frac{\left(\theta+\frac{1}{2}\right)X+1}{-\frac{1}{2}\left(\theta+\frac{1}{2}\right)X^2+\theta X+1}=\frac{1}{1-\frac{1}{2}X}.$$

In addition, using elementary inequality $ab\leqq\frac{1}{2}(a^2+b^2)$, we obtain

$$\left\{\left(\theta+\frac{1}{2}\right)X+1\right\}\sqrt{1-X}\leqq\frac{1}{2}\left[\left\{\left(\theta+\frac{1}{2}\right)X+1\right\}^2+1-X\right].$$

Thus

$$\left\{\left(\theta+\frac{1}{2}\right)X+1\right\}\sqrt{1-X}\leqq\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2X^2+\theta X+1.$$

Since $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2\leqq\beta$, we obtain

$$\frac{\left(\theta+\frac{1}{2}\right)+1}{\beta X^2+\theta X+1}\leqq\frac{1}{\sqrt{1-X}}.$$

Lemma 1 is completely proved.

It now follows from (18) and Lemma 1 that

$$|x-\zeta(x)|>|K(x)|. \quad (20)$$

Then, on the monotonic convergence of the rule (13) to $\zeta$ we obtain:

**Theorem 1.** Let $f(x)$ be given by the form (11). Then, if

$$\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2 \leqq \beta \leqq -\frac{1}{2}\left(\theta+\frac{1}{2}\right)$$

and if we take the real starting value in (13) $x_0$ such that $f(x_0)f'(x_0) \neq 0$, the sequence $x_n$ in (13) converges monotonically to $\zeta(x_0)$.

**Proof.** Assume that we take $x_0 > \zeta(x_0)$ such that

$$f(x_0)/f'(x_0) > 0, \quad K(x_0) > 0.$$

Then, using (13) and (20), we obtain

$$x_0 > x_1 > \zeta(x_0),$$

and by the definition of $\zeta(x)$, $\zeta(x_0) = \zeta(x_1)$. Furthermore, by repetition of the same argument, we obtain

$$x_0 > x_1 > x_2 > \cdots > \zeta(x_0).$$

Therefore, it follows that the sequence $x_n$ converges monotonically to a limit $\zeta$:

$$x_n \downarrow \zeta \geqq \zeta(x_0).$$

It now follows from (13) that $\lim_{x_n \to \zeta} K(x_n) = 0$. On the other hand, it follows from (14) and (17) that

$$1 - X(x_n) = \frac{1 + \{x_n - \zeta(x_0)\}^2 \sum_{k=1}^{r-1} \frac{1}{(x_n - \zeta_k)^2}}{\left[1 + \{x_n - \zeta(x_0)\} \sum_{k=1}^{r-1} \frac{1}{x_n - \zeta_k}\right]^2}.$$

Therefore, $-\infty < \lim_{x_n \to \zeta} X(x_n) < 1$. Hence, it follows from (19) that $\lim_{x_n \to \zeta} R(X(x_n)) > 0$. Thus, $\lim_{x_n \to \zeta} f(x_n)/f'(x_n) = 0$ and hence $f(\zeta) = 0$, that is $\zeta = \zeta(x_0)$. In the same way, if we take $x_0 < \zeta(x_0)$ such that $f(x_0)/f'(x_0) < 0$, $K(x_0) < 0$, we can show that the sequence $x_n$ converges monotonically to $\zeta(x_0)$. Theorem 1 is proved.

Next, let $f(x)$ be given by the form (12). Then if the real $x_0$ is neither less nor greater than all $\alpha_k$, by repetition of the same argument as the above, we can establish the monotonic convergence of the rule (13) (see [4]). More precisely:

**Theorem 2.** Let $f(x)$ be given by the form (12). Then, if

$$\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2 \leqq \beta \leqq -\frac{1}{2}\left(\theta+\frac{1}{2}\right)$$

and if we take the real $x_0$ such that $f(x_0)f'(x_0) \neq 0$ and $x_0$ is neither less nor greater than all $\alpha_k$, the sequence $x_n$ in (13) converges monotonically to $\zeta(x_0)$.

We have established the monotonic global convergence of the rule (13) under the same assumptions as those on Ostrowski's method for the starting values $x_0$.

**Remarks.** (a) Let $f(x)$ be a polynomial of exact degree $r > 1$ with real coefficients, let $\zeta$ be an arbitrary real zero of $f(x)$, and let $r-1$ zeros of $f(x)$ except $\zeta$ be $\zeta_k$ $(k = 1, 2, \cdots, r-1)$. Then, in the case where $f(x)$ has complex zeros, if $\sum_{k=1}^{r-1} \frac{h^2}{(x-\zeta_k)^2}$ is positive, it follows from (15) that $\frac{f'(x)}{f(x)}$ is monotonically decreasing and from (17) that (18) holds. Therefore, also in the case where $f(x)$ has complex zeros, if $\sum_{k=1}^{r-1} \frac{h^2}{(x-\zeta_k)^2}$ is positive, then it can be seen that Theorem 1 holds.

(b) Before long, we will present the results on the global convergence of iterative formulas for $R(X)$ of *Ex* 2 in 3.

## 5. Modification for Multiple Zeros

In the case where $\zeta$ is a zero of $f(x)$ of multiplicity $m > 1$, we consider the order of convergence for Eq. (10).

Then, we obtain

$$f(x) = (x-\zeta)^m g(x), \quad g(\zeta) \neq 0,$$
$$f'(x) = m(x-\zeta)^{m-1} g(x) + (x-\zeta)^m g'(x),$$
$$f''(x) = m(m-1)(x-\zeta)^{m-2} g(x) + 2m(x-\zeta)^{m-1} g'(x) + (x-\zeta)^m g''(x).$$

From these formulas, we obtain

$$h \equiv \frac{f(x)}{f'(x)} = \frac{(x-\zeta)g}{mg + (x-\zeta)g'}$$

$$X \equiv h \frac{f''(x)}{f'(x)} = \frac{m(m-1)g^2 + 2m(x-\zeta)gg' + (x-\zeta)^2 gg''}{\{mg + (x-\zeta)g'\}^2}$$

where $g \equiv g(x)$, $g' \equiv g'(x)$, $g'' \equiv g''(x)$.

Since $\frac{h}{x-\zeta} \to \frac{1}{m}$, $X \to 1 - \frac{1}{m}$ $(x \to \zeta)$, we obtain

$$\lim_{x \to \zeta} \frac{\phi(x) - \zeta}{x - \zeta} = \lim_{x \to \zeta} \frac{x - hR(X) - \zeta}{x - \zeta} = 1 - \frac{1}{m} R\left(1 - \frac{1}{m}\right).$$

Therefore, if $R\left(1 - \frac{1}{m}\right) \neq m$, then the convergence of $x_n$ to $\zeta$ is only linear. Also in this case, we can still obtain cubic convergence of $x_n$ to $\zeta$ by the iterative formulas

$$x_{n+1} = \phi_m(x_n), \quad n = 0, 1, \cdots \tag{21}$$

where $\phi_m(x) = x - mhR(1 - m + mX)$.

Then, the asymptotic error constant of $\phi_m(x)$ $C_m$ is given by

$$C_m = \lim_{x \to \zeta} \frac{\phi(x) - \zeta}{(x-\zeta)^3} = \frac{1}{m^2(m+1)^2} \left\{\frac{3}{2} + \frac{m}{2} - 2R''(0)\right\} \times \left\{\frac{f^{(m+1)}(\zeta)}{f^{(m)}(\zeta)}\right\}^2 - \frac{1}{m(m+1)(m+2)}$$

$$\times\frac{f^{(m+2)}(\zeta)}{f^{(m)}(\zeta)}. \tag{22}$$

To prove that the above contents hold, it suffices to prove that (22) holds. Here, it follows from (21) that

$$\lim_{x\to\zeta}\frac{\phi_m(x)-\zeta}{(x-\zeta)^3}=\lim_{x\to\zeta}\frac{x-mhR(1-m+mX)-\zeta}{(x-\zeta)^3}.$$

Then, since

$$R(1-m+mX)=R(0)+(1-m+mX)R'(0)+\frac{1}{2}(1-m+mX)^2R''(0)+\cdots,$$

$R(0)=1$ and $R'(0)=\frac{1}{2}$, we obtain

$$R(1-m+mX)=1+\frac{1}{2}(1-m+mX)+\frac{1}{2}(1-m+mX)^2R''(0)+\cdots. \tag{23}$$

Furthermore, we obtain

$$1-m+mX=\frac{2mgg'k+\{(1-m)g'^2+mgg''\}k^2}{(mg+kg')^2} \tag{24}$$

where $k\equiv x-\zeta$.

Hence, it follows from (23) and (24) that

$$x-mhR(1-m+mX)-\zeta=\left[\frac{m(m+3)gg'^2-m^2g^2g''}{2(mg+kg')^3}-\frac{2m^3g^3g'^2}{(mg+kg')^5}R''(0)\right]k^3+O(k^4).$$

Therefore

$$\lim_{x\to\zeta}\frac{\phi_m(x)-\zeta}{(x-\zeta)^3}=\frac{\{g'(\zeta)\}^2}{m^2g^2(\zeta)}\left\{\frac{3}{2}+\frac{m}{2}-2R''(0)\right\}-\frac{g''(\zeta)}{2mg(\zeta)}.$$

Finally, since

$$g(\zeta)=\frac{f^{(m)}(\zeta)}{m!},\quad g'(\zeta)=\frac{f^{(m+1)}(\zeta)}{(m+1)!},$$

and $g''(\zeta)=\frac{2f^{(m+2)}(\zeta)}{(m+2)!}$,

we obtain (22), and the above contents are proved.

Next, we consider the case of *Ex* 1 in 3.

Then, if $\frac{1}{2}\left(\theta+\frac{1}{2}\right)^2\leqq\beta\leqq-\frac{1}{2}\left(\theta+\frac{1}{2}\right)$, it follows from (19) that

$$R\left(1-\frac{1}{m}\right)<\frac{1}{\sqrt{1-\left(1-\frac{1}{m}\right)}}=\sqrt{m}<m.$$

Therefore, the convergence of $x_n$ in (13) to $\zeta$ is only linear. Modifying (13) by (21), we obtain

$$x_{n+1}=x_n-mh_nR(1-m+mX_n),\quad n=0,1,\cdots \tag{25}$$

where $h_n\equiv h(x_n)$, $X_n\equiv X(x_n)$, and $R(X)=\dfrac{\left(\theta+\frac{1}{2}\right)X+1}{\beta X^2+\theta X+1}$.

Then, it follows from (22) that the asymptotic error constant $C_m$ is given by

$$C_m=\frac{1}{m^2(m+1)^2}\left(\frac{3}{2}+\frac{m}{2}+2\theta+4\beta\right)\left\{\frac{f^{(m+1)}(\zeta)}{f^{(m)}(\zeta)}\right\}^2-\frac{1}{m(m+1)(m+2)}\cdot\frac{f^{(m+2)}(\zeta)}{f^{(m)}(\zeta)}. \tag{26}$$

On the monotonicity of convergence of the rule (25), we obtain:

**Theorem 3.** Let $f(x)$ be given by the form (11). Then, under the assumptions of Theorem 1, the sequence $x_n$ in (25) converges monotonically to $\zeta(x_0)$. Furthermore, we obtain:

**Theorem 4.** Let $f(x)$ be given by the form (12). Then, under the assumptions of Theorem 2, the sequence $x_n$ in (25) converges monotonically to $\zeta(x_0)$.

In the case where $\zeta$ is a zero of multiplicity $m>1$ and $r>m$, we can rearrange $r-m$ zeros except $\zeta$ such that $\zeta_1\leqq\cdots\leqq\zeta_{r-m}$. Then, (16) can be expressed in the form

$$1-X=\begin{cases}\dfrac{mh^2}{(x-\zeta)^2}+\displaystyle\sum_{k=1}^{r-m}\frac{h^2}{(x-\zeta_k)^2} & (r>m)\\[2ex] \dfrac{mh^2}{(x-\zeta)^2} & (r=m).\end{cases} \tag{27}$$

Therefore, it follows from (27) that

$$|x-\zeta(x)|\geqq\frac{\sqrt{m}\,|h|}{\sqrt{1-X}}. \tag{28}$$

Furthermore, since $1-m+mX<1$, using (19), we obtain

$$\frac{\sqrt{m}}{\sqrt{1-X}}\geqq mR(1-m+mX). \tag{29}$$

Hence, it follows from (28) and (29) that

$$|x-\zeta(x)|\geqq m|h|R(1-m+mX). \tag{30}$$

Consequently, these Theorems can be obtained in the way similar to the proof of Theorem 1 and Theorem 2.

## 6. Numerical Examples

If $\theta=-1$ in Eq. (9), then the iterative formula takes the particularly simple form:

$$x_{n+1}=x_n-\frac{1}{2}h_n-\frac{1}{2}\frac{f(x_n)}{f'(x_n)-f''\left(x_n-\frac{1}{3}h_n\right)h_n},\quad n=0,1,2,\cdots \tag{31}$$

Table 1

$x_0 = y_0 = z_0 = s_0 = t_0$
$= 0.4500000Q+02$

| $n$ | $x_n$ |
|---|---|
| 1 | 0.3922423595774759797681206516554 8Q+02 |
| 2 | 0.34362151299985694239163865068215Q+02 |
| 3 | 0.30299874418934075176564424202332Q+02 |
| 4 | 0.26943454481265801778595163229070Q+02 |
| 5 | 0.24216901668675174001188907041319Q+02 |
| 6 | 0.22060625082605398194537720247626Q+02 |
| 7 | 0.20430109587432181461381629668151Q+02 |
| 8 | 0.19294115168350793813134603664262Q+02 |
| 9 | 0.18626469732301582712154215179234Q+02 |
| 10 | 0.18362859526537913964360352337751Q+02 |
| 11 | 0.18321475601477172079158370432756Q+02 |
| 12 | 0.18320508596465555520111739480423Q+02 |
| 13 | 0.18320508075688923925266501619449Q+02 |
| 14 | 0.18320508075688772935274463427751Q+02 |
| 15 | 0.18320508075688772935274463415059Q+02 |
| 16 | 0.18320508075688772935274463415059Q+02 |

| $n$ | $y_n$ |
|---|---|
| 1 | 0.33462197369676993581779543575938Q+02 |
| 2 | 0.25764519736767293694032640934885Q+02 |
| 3 | 0.21037777952415490150767098673454Q+02 |
| 4 | 0.18760520612840935324935857761278Q+02 |
| 5 | 0.18323016641738299956404946174075Q+02 |
| 6 | 0.18320508075693518511403504582957Q+02 |
| 7 | 0.18320508075688772935274463415059Q+02 |
| 8 | 0.18320508075688772935274463415059Q+02 |

| $n$ | $z_n$ |
|---|---|
| 1 | 0.30367557800614883376701069519220Q+02 |
| 2 | 0.22421368909221744024174907807517Q+02 |
| 3 | 0.19029969543358684655890890915843Q+02 |
| 4 | 0.18334107868928654323580269513679Q+02 |
| 5 | 0.18320508215874156062542605811349Q+02 |
| 6 | 0.18320508075688772935274618144752Q+02 |
| 7 | 0.18320508075688772935274463415059Q+02 |
| 8 | 0.18320508075688772935274463415059Q+02 |

| $n$ | $s_n$ |
|---|---|
| 1 | 0.35005654722129833510739928804150Q+02 |
| 2 | 0.27860805750046702580438485947232Q+02 |
| 3 | 0.22984416626802894887291243502721Q+02 |
| 4 | 0.19985583969267830347605144828327Q+02 |
| 5 | 0.18588574905097711517327680474687Q+02 |
| 6 | 0.18323648409347100278463460110394Q+02 |
| 7 | 0.18320508082206973057748223226465Q+02 |
| 8 | 0.18320508075688772935274463473531Q+02 |
| 9 | 0.18320508075688772935274463415059Q+02 |
| 10 | 0.18320508075688772935274463415059Q+02 |

| $n$ | $t_n$ |
|---|---|
| 1 | 0.33923577641170267546185150685597Q+02 |
| 2 | 0.26404634692834795010559515028977Q+02 |
| 3 | 0.21648884997184653763969338846021Q+02 |
| 4 | 0.19131638502788661679966339354616Q+02 |
| 5 | 0.18360718543302134493849087080215Q+02 |
| 6 | 0.18320516506931216815745515941698Q+02 |
| 7 | 0.18320508075688773015377343085697Q+02 |
| 8 | 0.18320508075688772935274463415059Q+02 |
| 9 | 0.18320508075688772935274463415059Q+02 |

Table 2

$x_0 = y_0 = z_0 = s_0 = t_0 \simeq$ -0.90000000Q+01

| $n$ | $x_n$ |
|---|---|
| 1 | 0.1100000000000000000000000000000000Q+02 |
| 2 | -0.9000000000000000000000000000000000Q+01 |
| ⋮ | oscillating |

| $n$ | $y_n$ |
|---|---|
| 1 | 0.5968143354902936784469885515181 7Q+00 |
| 2 | 0.1000071431902170550800481284730 3Q+01 |
| 3 | 0.9999999999999999999999988178930 39Q+00 |
| 4 | 0.1000000000000000000000000000000000Q+01 |

| $n$ | $z_n$ |
|---|---|
| 1 | -0.6199439831943980392941411838679 5Q+01 |
| 2 | -0.3937452115963157401587709314087 4Q+01 |
| 3 | -0.3475888530825995975105438317077 3Q+01 |
| 4 | -0.3472135957223194181775264640234 5Q+01 |
| 5 | -0.3472135954999579392818347337925 7Q+01 |
| 6 | -0.3472135954999579392818347337462 6Q+01 |
| 7 | -0.3472135954999579392818347337462 6Q+01 |

| $n$ | $s_n$ |
|---|---|
| 1 | -0.8230769230769230769230769230769 2Q+01 |
| 2 | -0.6195043400832304353006307529445 0Q+01 |
| 3 | -0.3935730293951189310412506590573 8Q+01 |
| 4 | -0.3477614181771167324557487366146 3Q+01 |
| 5 | -0.3472135967172515830586578271345 7Q+01 |
| 6 | -0.3472135954999579392818347471525 4Q+01 |
| 7 | -0.3472135954999579392818347337462 6Q+01 |
| 8 | -0.3472135954999579392818347337462 6Q+01 |

| $n$ | $t_n$ |
|---|---|
| 1 | -0.7613607188703465982028241335044 9Q+01 |
| 2 | -0.4929375154656013849175736624163 6Q+01 |
| 3 | -0.3569033200536399776514934028940 9Q+01 |
| 4 | -0.3472186418383711020644634049278 0Q+01 |
| 5 | -0.3472135954999586874643531896885 6Q+01 |
| 6 | -0.3472135954999579392818347337462 6Q+01 |
| 7 | -0.3472135954999579392818347337462 6Q+01 |

※ $1+10\sqrt{3} =$ 18.32050807568877293527446341 5……

$1-2\sqrt{5} =$ -3.47213595499957939281834733 7……

where $h_n \equiv h(x_n)$.
Furthermore, if $\theta = -\frac{3}{4}$, $\beta = \frac{1}{16}$ in the formulas (13), then we obtain the iterative formula:

$$x_{n+1} = x_n - \frac{1 - \frac{1}{4} X_n}{\frac{1}{16} X_n^2 - \frac{3}{4} X_n + 1} h_n, \qquad n = 0, 1, 2, \cdots \tag{32}$$

where $X_n \equiv X(x_n)$.

Next, we give numerical examples by the following five iterative methods:

Newton's method: $x_{n+1} = x_n - h(x_n)$
The method by (31): $y_{n+1} = \phi(y_n)$
Ostrowski's method: $z_{n+1} = \Phi(z_n)$
Halley's method: $s_{n+1} = \Phi_{1,1}(s_n)$
The method by (32): $t_{n+1} = \phi(t_n)$, $n = 0, 1, 2, \cdots$.

Here, we consider the polynomial

$$f(x) = x^7 - 7x^6 - 499x^5 + 2565x^4 + 64835x^3 - 204821x^2 - 992593x + 1130519.$$

The exact zeros of $f(x)$ are $1 \pm 10\sqrt{3}$, $1 \pm 10\sqrt{2}$, $1 \pm 2\sqrt{5}$, and 1.

Let $\zeta$ be an arbitrary zero of $f(x)$. Then, it follows from Corollary 6·4$K$ [7, P. 457] that we obtain

$$|\zeta| < 2 \max(7, 499^{1/2}, 2565^{1/3}, 64835^{1/4}, 204821^{1/5}, 992593^{1/6}, 1130519^{1/7}) = 2 \cdot 499^{1/2} < 45.$$

Therefore, we take $|x_0| = |y_0| = |z_0| = |s_0| = |t_0| \leqq 45$.

In Table 1, we obtain $\zeta < z_n < t_n < s_n$ $(n \geqq 1)$. In Table 2, we obtain $\zeta > z_n > t_n > s_n$ $(n \geqq 1)$.

In general, let $z_0$ be an arbitrary real number, and let us take $t_0 = s_0 = z_0$. Then, it follows from (19) that we obtain $\zeta < z_1 < t_1 < s_1$ if $h(z_0) > 0$, and $\zeta > z_1 > t_1 > s_1$ if $h(z_0) < 0$. But, theoretically it cannot be concluded that we obtain $\zeta < z_n < t_n < s_n$ $(n \geqq 2)$ if $h(z_0) > 0$, and $\zeta > z_n > t_n > s_n$ $(n \geqq 2)$ if $h(z_0) < 0$.

From Table 2, it can be seen that the monotonicity of convergence for the method by (31) does not hold. All calculations are made with quadruple precesion.

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