Real Fast Fourier Transform on Quasi-Equidistant Sample Points

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Trigonometric polynomial interpolation of periodic functions with period 2π on equidistant points in the interval $[0, 2\pi)$ is a well-known and effective approximation tool. A standard numerical procedure for implementing this method is based on doubling the number of interpolation points at each step, so that the ordinary Fast Fourier Transform (FFT) technique is applicable. In this paper, a set is called a quasi-equidistant point set when it is the union of equidistant point sets with the same size but with mutually different phases. A fast algorithm is proposed for trigonometric polynomial interpolation on quasi-equidistant sample points for real periodic functions. The proposed algorithm is a generalization of the real FFT, but still requires $n \log_2 n + O(n)$ real arithmetic operations, where n is the number of interpolation points. With the quasi-equidistant point set and the algorithm for the interpolation on them, it is possible to construct an efficient scheme for automatic function approximation in which the rate at which the number of sample points increases is less than 2 and can be arbitrarily close to 1.

1. Introduction

The Fast Fourier Transform (FFT) [1] is a fast and stable algorithm for evaluating the coefficients of the trigonometric polynomial interpolation of a periodic function with period 2π on n equidistant points, $E_n := \{2\pi k/n \mid 0 \le k \le n\}$. Usually n is taken as a power of two and the FFT with base two is used for efficiency. For real periodic functions, it is common to use the real FFT algorithm [2], which takes advantage of the reality of sample values and halves the number of arithmetic operations.

In this paper, we call the operation for evaluating the coefficients of the trigonometric polynomial interpolation of a real periodic function on given sample points the *real discrete Fourier transform*.

Our purpose is to generalize the real FFT algorithm and construct a fast algorithm for the real discrete Fourier transform on non-equidistant sample points of some kind.

A sequence of sample point sets of the FFT with base 2,

$$E_1, E_2, E_4, \cdots, E_{2^k}, \cdots, \tag{1}$$

satisfies $E_{2^k} \subset E_{2^{k+1}}$, $0 \le k$; hence, all sample values can be reused when the number of samples is increased in order to obtain better accuracy. On the other hand, however, the number of sample points is doubled.

Torii, Hasegawa, and Sugiura [3, 4] showed that an efficient algorithm for the interpolation on finite truncations of the Van der Corput sequence can be constructed by using the FFT. In their method, re-usability of sample values is naturally realized when the number of sample points is increased. We introduce their method to construct a sequence of sample point sets for which the rate of increase in the number of sample points is less than 2. It is easy to verify that

$$E_{2^{k+1}} = E_{2^k} \cup \left(E_{2^k} + \frac{\pi}{2^k}\right) = \bigcup_{l=0}^{3} \left(E_{2^{k-1}} + \frac{l\pi}{2^k}\right) = \bigcup_{l=0}^{7} \left(E_{2^{k-2}} + \frac{l\pi}{2^k}\right) = \cdots, \tag{2}$$

where $A + x = \{a + x \mid a \in A\}$ for every set A of real numbers and every real number x. Eq. (2) allowed Torii and his colleagues to insert the new sample point sets

$$E_{2^k} \cup \left(E_{2^{k-1}} + \frac{\pi}{2^k}\right) = \bigcup_{l=0}^{2} \left(E_{2^{k-1}} + \frac{l\pi}{2^k}\right),$$

and

$$E_{2^k} \cup \left(E_{2^{k-2}} + \frac{\pi}{2^k}\right) = \bigcup_{l=0}^{3} \left(E_{2^{k-1}} + \frac{l\pi}{2^{k-1}}\right) \cup \left(E_{2^{k-2}} + \frac{\pi}{2^k}\right),$$

between E_{2^k} and $E_{2^{k+1}}$ in the sequence (1) and made it possible to increase the number of sample points geometrically at an average rate of $\sqrt{2}$ or $\sqrt[3]{2}$. This kind of sequence of sample points was used for numerical integration in papers by Torii and his colleagues, and its efficiency was proved [5, 6, 7].

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We generalize their sample point sets as *the quasi-equidistant point set*

$$R_M(T)=\bigcup_{k=1}^{\kappa}\left(E_M+\frac{\tau_k}{M}\right),\ M\geq 1, \tag{3}$$

with kernel $T=\{\tau_k\}_{k=1}^{\kappa}\subset[0,\ 2\pi),\ \kappa\geq 1$. The set $R_M(T)$ is a union of several equidistant point sets with the same number of points but with different phases. The equidistant point set is a special case when $T=\{0\}$.

In Sections 2 and 3, we give a fast algorithm for trigonometric polynomial interpolation on quasi-equidistant sample points. Our algorithm is a generalization of the FFT and requires $N\log_2 N+O(N)$ real multiplications, where $N=\kappa M$ is the number of sample points. In Section 4, we construct a sequence of quasi-equidistant sample point sets for which the rate of increase in the number of sample points can be made arbitrarily close to 1 by keeping the reusability of sample values. In Section 5, we show the performance of our algorithm and check the accuracy of the Fourier coefficients.

Our interpolation process also retains other valuable features of interpolation on equidistant points. In our method, the uniform norm of the corresponding interpolation operator, which is called the Lebesgue constant, has a logarithmic rate of growth, and the calculation of the discrete Fourier coefficients is numerically stable [8].

2. Real Discrete Fourier Transform

In this section, we investigate the Lagrangean form of a trigonometric polynomial interpolation of a real periodic function f with period 2π. We express a real trigonometric polynomial g of degree n in the following form:

$$g(t)=\mathrm{Re}\sum_{k=0}^{n}{}''\ c_k e^{ikt},\ c_0\in \boldsymbol{R}, \tag{4}$$

where Σ'' means the summation with the first and the last terms halved. We restrict ourselves to the case in which the number of sample points is even, which is important in an algorithm that we give later. Let U be a set of sample points of size $N=2n$ on the interval $[0,\ 2\pi)$. In order to simplify the descriptions, we define two trigonometric functions with period 4π as follows:

$$s(t):=\sin\frac{t}{2},\ c(t):=\cos\frac{t}{2}. \tag{5}$$

We define the fundamental polynomial of a trigonometric polynomial interpolation on U as follows:

$$w(U,\ \vartheta;\ t):=\left\{\prod_{\varphi\in U-\{\vartheta\}}\frac{s(t-\varphi)}{s(\vartheta-\varphi)}\right\}c(t-\vartheta),\ \vartheta\in U. \tag{6}$$

Since the function $w(U,\ \vartheta;\ t),\ \vartheta\in U$, is the product of an even number of trigonometric functions with the period 4π in Eq. (5), it is a real trigonometric polynomial of degree n. It is easy to verify that

$$w(U,\ \vartheta;\ \vartheta)=1,\ w(U,\ \vartheta;\ \varphi)=0,\ \varphi\in U-\{\vartheta\}. \tag{7}$$

Hence we obtain the following Lagrangean form of the n-th degree real trigonometric polynomial interpolation of f on U:

$$L(U)f(t):=\sum_{\vartheta\in U} w(U,\ \vartheta;\ t)f(\vartheta). \tag{8}$$

For each $\vartheta\in U$, it holds that

$$w(U,\ \vartheta;\ t) =\frac{e^{i(t-\vartheta)/2}+e^{-i(t-\vartheta)/2}}{2}\prod_{\varphi\in U-\{\vartheta\}}\frac{e^{i(t-\varphi)/2}-e^{-i(t-\varphi)/2}}{2is(\vartheta-\varphi)}.$$

and its coefficient $w_{\vartheta,n}$ of e^{int} is expressed as follows:

$$w_{\vartheta,n}=\{\prod_{\varphi\in U-\{\vartheta\}}2s(\vartheta-\varphi)\}^{-1}(-i)^{2n-1}\prod_{\varphi\in U}e^{-i\varphi/2}.$$

Thus

$$w_{\vartheta,n}\in\alpha(U)\boldsymbol{R},\ \vartheta\in U,\ \alpha(U):=(-1)^{n+1}i\prod_{\varphi\in U}e^{-i\varphi/2}. \tag{9}$$

Because $f(\vartheta)\in\mathbf{R}$ for all $\vartheta\in U$ in Eq. (8), the coefficient of e^{int} of the trigonometric polynomial $L(U)f$ belongs to $\alpha(U)\mathbf{R}$. Thus, we obtain

$$L(U)f(t)\in\Pi(U) :=\left\{\mathrm{Re}\sum_{k=0}^{n}{}''\ c_k e^{ikt}\,|\,c_0\in\mathbf{R},\ c_n\in\alpha(U)\mathbf{R}\right\}. \tag{10}$$

Theorem 1

Let f be a periodic function with period 2π and $f_n\in\Pi(U)$. If f_n is an interpolation of f on U, then $f_n=L(U)f$.

(proof)

The linear space $\Pi(U)$ is N-dimensional on $\mathbf{R}$, and N is equal to the number of sample points. From (7) and (9), $w(U,\ \vartheta;\ t),\ \vartheta\in U$, are N linear independent elements of $\Pi(U)$. Hence, they are the basis of $\Pi(U)$. Therefore, $f_n\in\Pi(U)$ can be expressed as a linear combination of the basis $w(U,\ \vartheta;\ t),\ \vartheta\in U$. From the interpolatory condition of f_n, it must be

$$f_n(t)=\sum_{\vartheta\in U}w(U,\ \vartheta;\ t)f(\vartheta).//$$

To show that the definition (8) is some generalization of the ordinary real discrete Fourier expansion, we consider the case of $U=E_N+\tau/N$, where the sample points are equidistant. In this case,

$$\alpha\left(E_N+\frac{\tau}{N}\right)=(-1)^{n+1}i\prod_{l=0}^{N-1}e^{-i\left(\frac{\pi l}{n}+\frac{\tau}{N}\right)/2} =(-1)^{n+1}ie^{-i(n-1/2)\pi-i\tau/2}=e^{-i\tau/2}. \tag{11}$$

The real FFT is a fast algorithm for obtaining the trigonometric polynomial interpolation f_n of a real periodic function f on E_N with the form

$$f_n(t)=\frac{1}{2}a_0+\sum_{k=1}^{n-1}(a_k\cos kt+b_k\sin kt)+\frac{1}{2}a_n\cos nt,$$

where a_k, $0\le k\le n$, b_k, $1\le k\le n-1$ are real numbers. We consider the case $\tau\ne 0$. At the first stage, we interpolate $f(t+\tau/N)$ on E_N by using the real FFT and get the Fourier coefficients a_k, $p\le k\le n$, b_k, $1\le k\le n-1$ of $f_n(t+\tau/N)$. At the second stage, we shift the origin by τ/N and get

$$\begin{aligned}f_n(t)&=\frac{1}{2}a_0+\sum_{k=1}^{n-1}\left(a_k\cos k\left(t-\frac{\tau}{N}\right)\right.\\&\quad\left.+b_k\sin k\left(t-\frac{\tau}{N}\right)\right)+\frac{1}{2}a_n\cos n\left(t-\frac{\tau}{N}\right)\\&=\mathrm{Re}\sum_{k=0}^{n}{}'' c_k e^{ikt},\end{aligned}$$

where $c_0=a_0$, $c_n=a_n e^{-i\tau/2}$, $c_k=(a_k-ib_k)e^{-ik\tau/N}$, $1\le k\le n-1$. Then we obtain $f_n\in\Pi(U)$, and $f_n=L(U)f$ from Theorem 1.

3. Fast Algorithm on Quasi-Equidistant Sample Points

In this section, we construct an algorithm for trigonometric polynomial interpolation on the quasi-equidistant set $R_M(T)$ with a kernel $T=\{\tau_k\}_{k=1}^{\kappa}\subset[0,2\pi)$, $\kappa\ge 1$, defined by Eq. (3).

Let $M\ge 2$ be a number of integer powers of 2 and let $\mu=M/2$. We define

$$\begin{cases}E_{M,k}:=E_M+\dfrac{\tau_k}{M}=\{u_{m,k}\}_{m=0}^{M-1},\\ u_{m,k}:=\dfrac{\pi m}{\mu}+\dfrac{\tau_k}{M},\\ w_{m,k}(t):=w(R_M(T),u_{m,k};t),\\ \qquad\qquad 0\le m\le M-1,\ 1\le k\le\kappa.\end{cases}$$

We obtain the following expression from definition (6):

$$\begin{aligned}w_{m,k}(t)&=\left\{\prod_{j=1,j\ne k}^{\kappa}\prod_{l=0}^{M-1}\frac{s\left(t-\dfrac{\pi}{\mu}l-\dfrac{\tau_j}{M}\right)}{s\left(\dfrac{\pi}{\mu}m+\dfrac{\tau_k}{M}-\dfrac{\pi}{\mu}l-\dfrac{\tau_j}{M}\right)}\right\}\\&\quad\times\left\{\prod_{l=0,l\ne m}^{M-1}\frac{s\left(t-\dfrac{\pi}{\mu}l-\dfrac{\tau_k}{M}\right)}{s\left(\dfrac{\pi}{\mu}m-\dfrac{\pi}{\mu}l\right)}\right\}c\left(t-\frac{\pi}{\mu}m-\frac{\tau_k}{M}\right)\\&=\left\{\prod_{j=1,j\ne k}^{\kappa}\prod_{l=0}^{M-1}\frac{s\left(t-\dfrac{\pi}{\mu}l-\dfrac{\tau_j}{M}\right)}{s\left(\dfrac{\pi}{\mu}m+\dfrac{\tau_k}{M}-\dfrac{\pi}{\mu}l-\dfrac{\tau_j}{M}\right)}\right\}\\&\qquad\times w(E_{M,k},u_{m,k};t).\end{aligned}$$

Therefore, from the formulae

$$\prod_{l=0}^{n-1}\sin\left(x-\frac{\pi}{n}l\right)=\left(-\frac{1}{2}\right)^{n-1}\sin nx,\ n\ge 1,$$

$$\sin(x+\pi m)=(-1)^m\sin x,\ m\in\mathbf{Z},$$

if follows that

$$w_{m,k}(t)=W_k(\mu t)(-1)^{(\kappa-1)m}w(E_{M,k},u_{m,k};t),$$
$$0\le m\le M-1,\ 1\le k\le\kappa, \tag{12}$$

where

$$W_k(t):=\prod_{j=1,j\ne k}^{\kappa}\frac{s(2t-\tau_j)}{s(\tau_k-\tau_j)},\ 1\le k\le\kappa. \tag{13}$$

Finally, we get

$$L(R_M(T))f(t)=\sum_{k=1}^{\kappa}W_k(\mu t)\sum_{m=0}^{M-1}w(E_{M,k},u_{m,k};t)(-1)^{(\kappa-1)m}f(u_{m,k}). \tag{14}$$

If the size κ of T is odd, it follows immediately from Eq. (14) that

$$L(R_M(T))f(t)=\sum_{k=1}^{\kappa}W_k(\mu t)L(E_{M,k})f(t). \tag{15}$$

If κ is even, $\cos(\mu t-\tau_k/2)=(-1)^m$ for every $t=u_{m,k}$, $0\le m\le M-1$, $1\le k\le\kappa$. Hence, it holds that

$$L(R_M(T))f(t)=\sum_{k=1}^{\kappa}W_k(\mu t)L(E_{M,k})\left\{\cos\left(\mu t-\frac{\tau_k}{2}\right)f(t)\right\}. \tag{16}$$

Let

$$L(E_{M,k})f(t)=\mathrm{Re}\sum_{j=0}^{\mu}{}'' C_{k,j}e^{ijt},\ 1\le k\le\kappa, \tag{17}$$

$$L(R_M(T))f(t)=\mathrm{Re}\sum_{l=0}^{\kappa\mu}{}'' c_l e^{ilt}. \tag{18}$$

Because M is a power of 2, the coefficients $\{C_{k,j}\}_{j=0}^{\mu}$, $1\le k\le\kappa$, can be calculated by using the real FFT with base 2. The total number of real multiplications for FFTs is

$$m_{\mathrm{FFT}}=\kappa(M\log_2 M+O(M))=N\log_2 N+O(N). \tag{19}$$

In the next two subsections, we give algorithms for calculating the coefficients c_l, $0\le l\le\kappa\mu$, of $L(R_M(T))f$ from $\{C_{k,j}\}_{j=0}^{\mu}$, $1\le k\le\kappa$.

3.1 Synthesis Algorithm for Odd κ

Since $W_k(t)$, $1\le k\le\kappa$, in Eq. (13) is the product of a real value and an even number of the functions

$$s(2t-\tau_j)=\frac{e^{-i\tau_j/2}}{2i}e^{it}-\frac{e^{i\tau_j/2}}{2i}e^{-it},\ 1\le j\le\kappa, \tag{20}$$

it can be expressed as follows:

$$W_k(t)=w_{k,0}+\sum_{l=1}^{(\kappa-1)/2}(w_{k,l}e^{2ilt}+w_{k,l}e^{-2ilt}),\ 1\le k\le\kappa. \tag{21}$$

Substituting Eqs. (17) and (21) into Eq. (15), we get

$$\begin{cases} c_{2l\mu+j} = \sum_{k=1}^{\kappa} w_{k,l}\, C_{k,j},\ 0\le j\le \mu-1,\ 0\le l\le(\kappa-1)/2, \\ c_{2l\mu+\mu} = \frac{1}{2}\sum_{k=1}^{\kappa} (w_{k,l}\, C_{k,\mu} + w_{k,l+1}\, \overline{C_{k,\mu}}), \\ \qquad\qquad 0\le l\le(\kappa-3)/2, \\ c_{2l\mu-j} = \sum_{k=1}^{\kappa} w_{k,l}\, \overline{C_{k,j}},\ 1\le j\le\mu-1,\ 1\le l\le(\kappa-1)/2, \\ c_{\kappa\mu} = \sum_{k=1}^{\kappa} w_{k,(\kappa-1)/2}\, C_{k,\mu}. \end{cases} \tag{22}$$

3.2 Synthesis Algorithm for Even κ

Since $W_k(t)$, $1\le k\le\kappa$, defined in Eq. (13) is the product of a real value and an odd number of $s(2t-\tau_j)$, $1\le j\le\kappa$, defined by Eq. (20), it can be expressed as follows:

$$W_k(t) = \sum_{l=0}^{\kappa/2-1} (w_{k,l}\, e^{i(2l+1)t} + w_{k,l}\, e^{-i(2l+1)t}),\ 1\le k\le\kappa. \tag{23}$$

We denote

$$\alpha_k := e^{-i\tau_k/2} = \alpha(E_{M,k}),\ 1\le k\le\kappa. \tag{24}$$

From the equalities

$$\begin{cases} \cos\left(\mu t-\frac{\tau_k}{2}\right) = \frac{1}{2}\alpha_k e^{i\mu t} + \frac{1}{2}\overline{\alpha_k} e^{-i\mu t},\ 1\le k\le\kappa, \\ e^{\pm iMt} = \alpha_k^{\mp 2},\ t\in e_{M,k}, \end{cases}$$

it holds that

$$\begin{aligned} \cos\left(\mu t-\frac{\tau_k}{2}\right) e^{ijt} &= \frac{1}{2}\alpha_k e^{i(j+\mu)t} + \frac{1}{2}\overline{\alpha_k} e^{i(j-\mu)t} \\ &= \frac{1}{2}\alpha_k \alpha_k^{-2} e^{i(j-\mu)t} + \frac{1}{2}\overline{\alpha_k} e^{i(j-\mu)t} \\ &= \overline{\alpha_k} e^{i(j-\mu)t},\ 0\le j\le\mu, \end{aligned}$$

for all $t\in E_{M,k}$, $1\le k\le\kappa$.

Taking the complex conjugate of the most left-hand side and the most right-hand side of the above equalities, we also obtain

$$\cos\left(\mu t-\frac{\tau_k}{2}\right) e^{-ijt} = \alpha_k e^{i(\mu-j)t},\ 0\le j\le\mu.$$

From these equalities and the expression (17), it holds that

$$\begin{aligned} \cos\left(\mu t-\frac{\tau_k}{2}\right) f(t) &= \cos\left(\mu t-\frac{\tau_k}{2}\right) L(E_{M,k}) f(t) \\ &= \mathrm{Re} \sum_{j=0}^{\mu}{}'' \alpha_k \overline{C_{k,\mu-j}}\, e^{ijt}, \\ &\qquad t\in E_{M,k},\ 1\le k\le\kappa. \end{aligned}$$

Since $C_{k,0}\in\mathbf{R}$, the coefficient $\alpha_k\overline{C_{k,0}}$ of $e^{i\mu t}$ of the most right-hand side of the above equality belongs to $\alpha_k\mathbf{R}$. Hence, the series of the most right-hand side belongs to $\Pi(E_{M,k})$. From Theorem 1, we get

$$L(E_{M,k})\left\{\cos\left(\mu t-\frac{\tau_k}{2}\right) f(t)\right\} = \mathrm{Re} \sum_{j=0}^{\mu}{}'' \alpha_k \overline{C_{k,\mu-j}}\, e^{ijt},\ 1\le k\le\kappa. \tag{25}$$

Substituting Eqs. (17) and (25) into Eq. (16), we get

$$\begin{cases} c_0 = \sum_{k=1}^{\kappa} \mathrm{Re}(w_{k,0}\,\overline{\alpha_k}) C_{k,0}, \\ c_{2l\mu+j} = \sum_{k=1}^{\kappa} w_{k,l}\,\overline{\alpha_k}\, C_{k,j},\ 1\le j\le\mu,\ 0\le l\le\kappa/2-1, \\ c_{2l\mu-j} = \sum_{k=1}^{\kappa} w_{k,l}\,\alpha_k\, \overline{C_{k,j}},\ 0\le j\le\mu-1,\ 1\le l\le\kappa/2, \\ c_{2l\mu} = \frac{1}{2}\sum_{k=1}^{\kappa} (w_{k,l}\,\overline{\alpha_k} + w_{k,l+1}\,\alpha_k) C_{k,0},\ 1\le l\le\kappa/2-1, \\ c_{\kappa\mu} = \sum_{k=1}^{\kappa} w_{k,\kappa/2-1}\,\alpha_k\, C_{k,0}. \end{cases} \tag{26}$$

Let $N=\kappa M$ be the number of sample points. It takes $\kappa N/2+O(1)$ complex multiplications to calculate the right-hand sides of (22). It also takes $\kappa N/2+O(1)$ complex multiplications to calculate the right-hand sides of (26), if we supply the values

$$w_{k,l}\,\alpha_k,\ w_{k,l}\,\overline{\alpha_k},\ 0\le l\le\mu,\ 1\le k\le\kappa/2-1,$$

in advance. Therefore, the number of real multiplications m_{syn} needed to calculate c_l, $0\le l\le\kappa\mu$, from $\{C_{k,j}\}_{j=0}^{\mu}$, $1\le k\le\kappa$, is not affected by the parity of κ and

$$m_{\mathrm{syn}} = 2\kappa N + O(1). \tag{27}$$

From Eqs. (27) and (19), the total number of real multiplications m_{total} is

$$m_{\mathrm{total}} = m_{\mathrm{FFT}} + m_{\mathrm{syn}} = N\log_2 N + O(N). \tag{28}$$

This is almost equal to the total number of real multiplications for the real FFT on the usual equidistant sample points.

Let us call the process for calculating the coefficients of the trigonometric polynomial interpolation on N sample points the "N-point transform." In our algorithm, the $\kappa 2^m$-point transform for calculating the coefficients of $L(R_{2^m}(T))f$ is reduced through synthesis rule (22) or (26) to the κ 2^m-point transforms that calculate the coefficients of $L(E_{2^m}+\tau_k/2^m)f$, $1\le k\le\kappa$. In the algorithm mentioned above, we use the ordinary real FFT with base 2 for these 2^m-point transforms. From equality (2), it holds that

$$\begin{aligned} E_{2^m}+\frac{\tau}{2^m} &= \left(E_{2^{m-1}}+\frac{\tau}{2^m}\right)\cup\left(E_{2^{m-1}}+\frac{\tau}{2^m}+\frac{\pi}{2^{m-1}}\right) \\ &= R_{2^{m-1}}\left(\left\{\frac{\tau}{2},\ \frac{\tau}{2}+\pi\right\}\right). \end{aligned}$$

Thus, each 2^m-point transform is also reduced to two

2^{m-1}-point transforms through synthesis (26). By repeated application of this reduction, the $\kappa 2^m$-point transform is finally reduced to $\kappa 2^{m-1}$ 2-point transforms. In other words, the $\kappa 2^m$-point transform is synthesized from the $\kappa 2^{m-1}$ 2-point transforms in the opposite way to the above reduction process. When $\kappa=1$ and $T=\{0\}$, our algorithm agrees with the ordinary real FFT with base 2 described, for example, by Swarztrauber [9].

4. Sequence of Reusable Sample Point Sets

In this section, we consider an infinite sequence of sample point sets $\{U_n\}_{n\geq 0}$ on the interval $[0, 2\pi)$ and trigonometric polynomial interpolation on these points.

If a sequence $\{U_n\}_{n\geq 0}$ satisfies

$$U_0 \subset U_1 \subset \cdots \subset U_n \subset \cdots, \tag{29}$$

all sample values can be reused without waste when we increase the number of the sample points along this sequence. We say that a sequence of sample point sets that satisfies condition (29) is *reusable*.

We can construct a reusable sequence in which all the elements are equidistant point sets, as in sequence (1), but the rate of increase in the number of sample points must be greater than or equal to 2. This rate is especially large when we apply sequence (1) to multi-dimensional interpolation problems as a direct product.

By using quasi-equidistant point sets, it is possible to construct a reusable series in which the rate of increase in the number of sample points is arbitrarily close to 1.

Theorem 2

Let $\{T_n\}_{0\leq n\leq \nu-1}$ be a sequence of finite sets on the interval $[0, 2\pi)$ that satisfies

$$T_0 \subset T_1 \subset \cdots T_{\nu-1} \subset R_2(T_0). \tag{30}$$

Then, the sequence

$$U_{m\nu+k} := R_{2^m}(T_k), \quad 0\leq k\leq \nu-1, \; 0\leq m, \tag{31}$$

is reusable and the average rate of increase in the number of sample points is $\sqrt[\nu]{2}$.

(proof)

From (3), for every point set A, B on $[0, 2\pi)$, it holds that

$$A\subset B \Rightarrow R_M(A)\subset R_M(B), \quad M\geq 1. \tag{32}$$

The reusability of sequence (31) is trivial from this statement. The average rate of increase in the number of sample points is $\sqrt[\nu]{2}$, because the number of sample points of $U_{k+\nu}$ is twice the number of sample points of U_k.

The algorithm in Section 3 can be applied to the interpolation on U_n, $n\geq 0$.

We can construct infinitely many sequences by using Theorem 2. The sequence in Section 1 proposed by Torii and his colleagues is a typical example.

Let T_0 be an equidistant set

$$T_0 = E_p, \quad p\geq 2; \tag{33}$$

then $T_0 \subset E_{2p} = R_2(T_0)$ is satisfied and every sequence $E_p = T_0 \subset T_1 \subset \cdots T_{\nu-1} \subset E_{2p}$ can be used as a kernel of sequence (31). In practice, we must choose sequence (30) while taking account of the stability and accuracy of trigonometric interpolation on U_n, $n\geq 0$.

5. Numerical Experiments

Let $p=3$ in the expression (33) and let

$$\tau_1=0,\; \tau_2=\frac{2\pi}{3},\; \tau_3=\frac{4\pi}{3},\; \tau_4=\frac{\pi}{3},\; \tau_5=\frac{5\pi}{3}, \tag{34}$$

$$T_n=\{\tau_1, \cdots, \tau_{n+3}\}, \quad 0\leq n\leq 2. \tag{35}$$

Then T_n, $0\leq n\leq \nu-1$, $\nu=3$, satisfies condition (30). Therefore, the sequence of sample point sets $\{U_n\}_{n\geq 0}$ defined by sequence (31) has reusability, and the average rate of increase in the number of sample points is $\sqrt[3]{2}$. We give explicit expressions of $W_k(t)$, $1\leq k\leq \kappa=n+3$ for $n=0, 1, 2$, which are deduced by the product formulae of sines and cosines.

⟨a⟩$T=T_0 := \{\tau_1, \tau_2, \tau_3\}$

$$\begin{cases} W_1(t)=(1+2\cos 2t)/3, \\ W_2(t)=\left(1+2\cos\left(2t-\dfrac{2\pi}{3}\right)\right)/3, \\ W_3(t)=\left(1+2\cos\left(2t+\dfrac{2\pi}{3}\right)\right)/3, \end{cases} \tag{36}$$

⟨b⟩$T=T_1 := \{\tau_1, \tau_2, \tau_3, \tau_4\}$

$$\begin{cases} W_1(t)=\left(2\cos t+2\cos\left(3t+\dfrac{\pi}{3}\right)\right)/3, \\ W_2(t)=\left(2\cos\left(t-\dfrac{\pi}{3}\right)+2\cos\left(3t+\dfrac{2\pi}{3}\right)\right)/3, \\ W_3(t)=\left(2\cos\left(t-\dfrac{2\pi}{3}\right)+2\cos 3t\right)/3, \\ W_4(t)=\cos\left(3t-\dfrac{\pi}{2}\right). \end{cases} \tag{37}$$

⟨c⟩$T=T_2 := \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$

$$\begin{cases} W_1(t)=(1+2\cos 4t)/3, \\ W_2(t)=\left(1+\sqrt{3}\cos\left(2t-\dfrac{5\pi}{6}\right)+\cos\left(4t+\dfrac{\pi}{3}\right)\right)/3, \\ W_3(t)=\left(1+\sqrt{3}\cos\left(2t+\dfrac{5\pi}{6}\right)+\cos\left(4t-\dfrac{\pi}{3}\right)\right)/3, \\ W_4(t)=\left(\sqrt{3}\cos\left(2t-\dfrac{\pi}{6}\right)+\sqrt{3}\cos\left(4t-\dfrac{5\pi}{6}\right)\right)/3, \\ W_5(t)=\left(\sqrt{3}\cos\left(2t+\dfrac{\pi}{6}\right)+\sqrt{3}\cos\left(4t+\dfrac{5\pi}{6}\right)\right)/3, \end{cases} \tag{38}$$

When $T=T_0$, T_2, the size $\kappa=3$, 5 of T is odd. We can

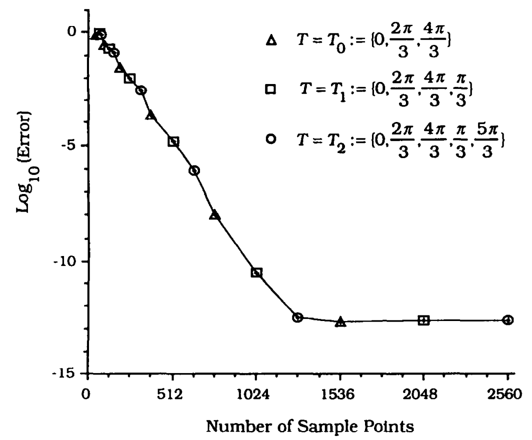

Fig. 1 Relative error of real discrete Fourier coefficients on quasi-equidistant sample point sets for the function $f(t)=\frac{1+2a\sin t-a^2}{1-2a\cos t+a^2}$, $a=0.95$.

calculate the coefficient $w_{k,l}$, $0\le l\le(\kappa-1)/2$, $1\le k\le\kappa$, from expressions (36) and (38) and the formula

$$\cos(kt+\alpha)=\frac{e^{i\alpha}e^{ikt}+e^{-i\alpha}e^{-ikt}}{2}.$$

When $T=T_1$, the size $\kappa=4$ of T is even. We can calculate the coefficient $w_{k,l}$, $0\le l\le\kappa/2-1$, $1\le k\le\kappa$, from expression (37), and α_k, $1\le k\le\kappa$, from Eq. (24).

The coefficients of $L(R_M(T))f$ are synthesized from $L(E_{M,k})f$, $1\le k\le\kappa$, by expression (22) or (26) according to the parity of κ.

The number of real multiplications for the synthesis m_{syn} is less than that shown in expression (27), because of the speciality of T. Corresponding to the number of sample points N, it becomes

$$m_{syn}=\begin{cases}\frac{4}{3}N+O(1),\ T=T_0,\\ \frac{7}{4}N+O(1),\ T=T_1,\\ \frac{7}{5}N+O(1),\ T=T_2.\end{cases}\tag{39}$$

In expression (39), we did not count multiplications by 2 and 0.5 because of the speciality of binary computers.

We show the numerical results of calculating the coefficient of $L(R_M(T))f$, $T=T_0$, T_1, T_2, $M=2^m$, $4\le m\le 9$. We use the following test function:

$$f(t):=\frac{1+2a\sin t-a^2}{1-2a\cos t+a^2}=1+\mathrm{Re}\sum_{n=1}^{\infty}(1-i)a^n e^{int},\ a=0.95.\tag{40}$$

In Fig. 1, the horizontal axis represents the number of sample points, and the vertical axis common logarithms of the error ε. The error ε is the relative error given by the absolute sum of errors of calculated coefficients divided by 27.870... which is the absolute sum of all coefficients of f.

When the number of sample points $N=48,\ldots,3\times2^m,\ldots,1536$, $T=T_0$ and $R_{2^m}(T)$ is an equidistant point set $E_{3\cdot2^m}$. When $N=64,\ldots,4\times2^m,\ldots,2048$, and $N=80,\ldots,5\times2^m,\ldots,2560$, $T=T_1$ and T_2 respectively. In the latter two cases, $R_{2^m}(T)$ is a quasi-equidistant point set but not equidistant.

Figure 1 shows that the common logarithm of the error is decreasing linearly until it reaches the round-off error bound around 10^{-13}. In the non-equidistant cases in which $T=T_1$ or T_2, both accuracy and stability are almost held.

The numerical experiment was done in FORTRAN77 with double precision on the FACOM M780 at Nagoya University's Computing Center.

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