多様体間の距離とその応用

井宮 淳

千葉大学 工学部 情報工学科

平面上の直線, 2次曲線は, 代数曲線の特別な場合である. 本論文では代数曲線の間に距離を定義する. まず, 最大次数が固定された代数曲線と単位半球面上の点の集合との間の写像が全単射であることを示す. そして, 球面上の2点間の測地線の長さによって代数曲線の間の距離を定義する. また, 球面上の Voronoi 分割を利用して代数曲線の集合をベクトル量子化して曲線の集合を記号に変換する手法を提案する. 代数曲線の間の距離や代数曲線の集合のベクトル量子化によって, ハフ変換を直線や2次曲線の認識機械と解釈することが可能となる.

A Metric of Manifolds and Discrimination of Manifolds

Atsushi IMIYA

Dept. of Information and Computer Sciences, Chiba University
1-33 Yayoi-cho, Inage-ku, Chiba 263, JAPAN
e-mail imiya@ics.tj.chiba-u.ac.jp

This paper defines a distance measure among the elements of the families of algebraic manifolds which are a generalization of the family of lines and the family of conics in the Euclidean plane. The metric between manifolds is introduced by using one-to-one mapping between the positive unit semi-sphere in the Euclidean space and a family of manifolds. Furthermore, by using the proposed metric, I also construct a discrimination algorithm of manifolds. This algorithm classifies curves by nearest-neighbor discrimination on the positive unit semi-sphere. Moreover, the correspondence between curves and the positive unit semi-sphere leads that if the accumulator space for curve detection by the Hough transform is equivalent to the positive unit semi-sphere, one can define a distance in the accumulator space. This accumulator also permits the Hough transform which detects both lines and conics.

1. Introduction

In computer vision, lines, conics, and planes are fundamental data in the construction of objects from measured images [1]. These data are categorized into algebraic manifolds. Thus, for the discrimination of fundamental data and for the estimation of accuracy of reconstructed data in computer vision, metrics among these algebraic manifolds are required. Pattern recognition provides metrics for the discrimination of planar figures and time-varying signals. These discrimination methods are based on the theory of Hilbert space and vector space [2,3]. In the Hilbert space framework of pattern recognition, a pattern is dealt with as a point in space. This embedding of patterns into Hilbert space is possible if patterns are described as functions on an interval. For instance, the Fourier descriptor of a curve is determined as a set of Fourier coefficients of a function of the length of arc of a curve[4]. Embedding of algebraic curves by using the Fourier descriptor is, however, inadequate because some algebraic curves have infinite arc-lengths. Furthermore, an area between two algebraic curves is sometimes infinite. Moreover, some algebraic curves are multivalued functions on the plane. Thus, for the discrimination of manifolds, a new method is required for the embedding of these data into Hilbert space and vector space.

Shapes which are extracted from scenes in the early vision stage provide geometric information such as positions of points, distances among points, and angles between line segments. The protocol from the early vision stage to the intermediate stage is the symbolization through quantization of these numerical raw data obtained in the early vision stage since the intermediate stage manipulates symbolized data. Vector quantization transforms numerical raw data to a finite number of typical data. Thus, by assigning an individual symbol to each typical datum, vector quantization is adopted as the protocol from the early vision stage to the intermediate stage; that is, one can transform geometric raw data to symbol data. A set of generators, which is a set of typical data, divides data space into a finite collection of disjunction sets. This procedure is called Voronoi tessellation, and each disjunction set is called a Voronoi region. Thus, any data in a Voronoi region are transformed to the generator of the region by vector quantization procedure. This quantizatin is called the nearest-neighbor discrimination. However, to derive a Voronoi tessellation and to achieve the nearest-neighbor discrimination in a set of data, a distance measure is required in a data space.

In classical projective geometry, the conics are classified by using distributions of poles of their canonical forms [5]. This classification focuses on clarification of the properties of conics which are invariant under affine transforms in the Euclidean space. For instance, the classification differentiates between parabolas and ellipses. However, a distance measure among ellipses in an image is required. Thus, the classical classification is inadequate for image analysis because of the requirement that curves be directly differentiated by using their forms appearing in images.

This paper defines the distance measure for manifolds such as lines, conics, planar curves, and planes. First, by using basic results of projective geometry, I construct one-to-one mapping between a family of algebraic curves and points on the positive unit semi-sphere. Second, by using spherical geometry, I define the geodesic distance between two points as the distance between two algebraic curves. Furthermore, as a generalization of the definition of the metrics, I also introduce a metric for linear manifolds in higher-dimensional Euclidean space.

I also construct an algorithm for the nearest-neighbor discrimination for lines, conics, planar curves, and manifolds in higher-dimensional Euclidean space since the correspondence permits application of methods of pattern recognition to a family of curves. The algorithm achieves discrimination of elements by computing inner products of vectors between data and generators on the semi-sphere. The most important advantage of the algorithm is that prior determination of discrimination surfaces of Voronoi tessellation is not required. This differs from the classical nearest-neighbor discrimination of patterns which is based on the theory of Hilbert space and statistics.

The correspondence between an algebraic curve and a point in the parameter space, which is called the accumulator space, is a fundamental idea for the Hough transform for curve detection from a binary image [6]. one cannot, however, introduce any higher-level operations, such as the discrimination, classification and unification of data, in the accumulator space because the classical accumulator space does not posses any properties of vector space. On the other hand, this paper shows that the accumulator space should be topologically equivalent to the positive unit semi-sphere in a higher-dimensional Euclidean space. Thus, one can define a metric for manifolds by using the metric in Euclidean space. This metric in the new accumulator space permits classification and unification of data in the accumulator space [7].

2. Parametrization of Curves

Let $\mathbf{R}^n$, $n \geq 2$ be n-dimensional Euclidean space. Furthermore, setting

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T \tag{1}$$

as a vector of $\mathbf{R}^n$, where $\cdot^T$ is the transpose of a vector, the inner product of vectors is defined by

$$\boldsymbol{x}^T\boldsymbol{y} = \sum_{i=1}^{n} x_i y_i \tag{2}$$

and the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ by

$$|\boldsymbol{x} - \boldsymbol{y}| = \sqrt{(\boldsymbol{x} - \boldsymbol{y})^T(\boldsymbol{x} - \boldsymbol{y})}. \tag{3}$$

Moreover, for $m < n$ and $\boldsymbol{x} \in \mathbf{R}^m$, by setting

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_m, 0, 0, \cdots, 0)^T, \tag{4}$$

$\boldsymbol{x} \in \mathbf{R}^m$ is embedded in $\mathbf{R}^n$.

Let S^{n-1} be the unit sphere of $\mathbf{R}^n$ consisting of all points $\boldsymbol{x}$ with distance 1 from the origin. For $n = 1$, $S^0 = [-1, 1]$. Furthermore, the positive half-space is defined by

$$\mathbf{R}^n_+ = \{\boldsymbol{x} | x_n > 0\},\ n \geq 1. \tag{5}$$

Now, by setting

$$H^{n-1}_+ = S^{n-1} \bigcap \mathbf{R}^n_+,\ n \geq 1, \tag{6}$$

the positive unit semi-sphere is defined by

$$S^{n-1}_+ = S^{n-2}_+ \bigcup H^{n-1}_+,\ n \geq 1. \tag{7}$$

In this paper, x, y, and z express arguments of polynomials which take values on the real axis $\mathbf{R}$, and $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ express vectors of appropriate dimension of which elements are x, y, z and so on. Furthermore, a, b, and c express coefficients of polynomials and $\boldsymbol{a}$, $\boldsymbol{b}$, and $\mathbf{c}$ express vectors of appropriate dimension of which elements are a, b, c and so on. I call $\boldsymbol{x}$ and $\boldsymbol{a}$ the argument vector and the constant vector, respectively.

In the following, I write POUSS (the POsitive Unit Semi-Sphere) for S^{n-1}_+. I also denote a polynomial of x_1, x_2, $\cdots$, x_n as $P(\boldsymbol{x})$.

In the remainder of this paper with the exception of in section 7, to enable clear discussion I will deal with curves in the two-dimensional Euclidean plane $\mathbf{R}^2$.

Let

$$\mathcal{P}^n = \{p_i(x, y)\}^n_{i=1} \tag{8}$$

be a set of independent monomials of x and y and let

$$\mathcal{A}^n = \{\boldsymbol{a} \mid \boldsymbol{a} = \{a_i\}^n_{i=1}\} \tag{9}$$

be a set of all n-tuple real numbers, where at least one of a_i is nonzero. Then, setting

$$P(x, y) = \sum_{i=1}^{n} a_i p_i(x, y), \tag{10}$$

a set

$$\mathcal{C}_2(\mathcal{P}^n, \mathcal{A}^n) = \{\,(x, y)^T \,|P(x, y) = 0\} \tag{11}$$

defines a family of curves on the plane for $\boldsymbol{a} \in \mathcal{A}^n$. Here, the suffix 2 of $\mathcal{C}_2(\mathcal{P}^n, \mathcal{A}^n)$ indicates a set of algebraic curves of two real arguments.

For families of curves $\mathcal{C}_2(\mathcal{P}^n, \mathcal{A}^n)$ and $\mathcal{C}_2(\mathcal{P}^m, \mathcal{A}^m)$, the following lemma holds.

[Lemma 1] For $m < n$, if $\mathcal{P}^m \subset \mathcal{P}^n$, the relation

$$\mathcal{C}_2(\mathcal{P}^m,\mathcal{A}^m) \subset \mathcal{C}_2(\mathcal{P}^n,\mathcal{A}^n) \quad (12)$$

holds.

(*Proof*) By setting

$$\mathcal{A}^{m,n} = \{\boldsymbol{a} \mid a_i = 0,\ 1 \le i \le n-m,\ \boldsymbol{a} \in \mathcal{A}^m\}, \quad (13)$$

the relation

$$\mathcal{C}_2(\mathcal{P}^m,\mathcal{A}^m) = \mathcal{C}_2(\mathcal{P}^m,\mathcal{A}^{m,n}). \quad (14)$$

holds. Since

$$\mathcal{C}_2(\mathcal{P}^m,\mathcal{A}^{m,n}) \subset \mathcal{C}_2(\mathcal{P}^n,\mathcal{A}^n), \quad (15)$$

I obtain the lemma. (*Q.E.D.*)

Now, I show some examples of a set of monomials and algebraic curves defined by these monomials.

Example 1. If elements of $\mathcal{P}^3$ are

$$p_1(x, y) = 1,\ p_2(x, y) = x, p_3(x,y) = y, \quad (16)$$

I obtain

$$P(x, y) = ax + by + c. \quad (17)$$

Then, $\mathcal{C}_2(\mathcal{P}^3,\mathcal{A}^3)$ determines the family of lines in the Euclidean plane if one of a and b is not zero and $c \ge 0$.

Example 2. If elements of $\mathcal{P}^6$ are

$$\begin{array}{ll} p_1(x, y) = 1, & p_2(x, y) = x, \\ p_3(x,y) = y, & p_4(x, y) = x^2, \\ p_5(x, y) = xy, & p_6(x,y) = y^2, \end{array} \quad (18)$$

I obtain

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f. \quad (19)$$

Then, $\mathcal{C}_2(\mathcal{P}^6,\mathcal{A}^6)$ determines the family of conics in the Euclidean plane if at least one of a, b, and c, and f are not zero.

Example 3. $\mathcal{P}^3$ which is defined by the monomials of eq.(16) is a subset of $\mathcal{P}^6$ which is defined by the monomials of eq. (18). Thus, if at least one of a, b, c, d, and e is not zero, $\mathcal{C}_2(\mathcal{P}^6,\mathcal{A}^6)$ defines the family of all lines and conics.

The following lemma holds.

[Lemma 2] An element of $\mathcal{A}^n$ corresponds to a point in the n-dimensional vector space.

From lemma 2, I define a coefficient vector of $P(x, y)$ as

$$\boldsymbol{a} = (a_1, a_2, \cdots, a_n)^T. \quad (20)$$

For a positive real value λ, $\lambda P(x, y) = 0$ and $-\lambda P(x, y) = 0$ define the same curve. Conversely, once a point $\boldsymbol{a}$ on S^{n-1}_+ is fixed, I can obtain a curve of $\mathcal{C}_2(\mathcal{P}^n,\mathcal{A}^n)$. This leads to the following theorem.

[Theorem 1] There is one-to-one mapping between $\mathcal{C}^n_2(\mathcal{P}^n,\mathcal{A}^n)$ and S^{n-1}_+.

I write the correspondence between a family of curves and POUSS and between a curve and a point on POUSS as

$$\Re(\mathcal{C}^n_2(\mathcal{P}^n,\mathcal{A}^n)) = S^{n-1}_+ \quad (21)$$

and

$$\wp(\boldsymbol{c}) = \boldsymbol{a}, \quad (22)$$

where $\boldsymbol{c} \in \mathcal{C}^n_2(\mathcal{P}^n,\mathcal{A}^n))$ and $\boldsymbol{a} \in S^{n-1}_+$, respectively. The Hough transform determines a point on POUSS from samples on a curve and detects curves by using this one-to-one correspondence. Figure 1 illustrates the correspondence between a line in the plane and a point on S^2_+.

From lemmas 1 and 2, theorem 1, and the property

$$S^{m-1}_+ \subset S^{n-1}_+, \quad (23)$$

for $m < n$, the following theorem holds.

[Theorem 2] For $m < n$, the relation

$$\Re(\mathcal{C}_2(\mathcal{P}^m,\mathcal{A}^m)) \subset \Re(\mathcal{C}_2(\mathcal{P}^n,\mathcal{A}^n)), \quad (24)$$

holds if $\mathcal{P}^m \subset \mathcal{P}^n$.

3. Metrics of Curves

In this section, I define a metric among elements of $\mathcal{C}_2(\mathcal{P}^n,\mathcal{A}^n)$ by using theorem 1 and properties of the spherical geometry.

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ on S^{n-1},

$$d(\boldsymbol{a}, \boldsymbol{b}) = \cos^{-1}(\boldsymbol{a}^T\boldsymbol{b}) \quad (25)$$

is the great-circle distance between $\boldsymbol{a}$ and $\boldsymbol{b}$ which coincides with the geodesic distance between $\boldsymbol{a}$ and $\boldsymbol{b}$ on S^{n-1} [5,8]. Thus, a metric between two algebraic curves $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ is defined by

$$\begin{aligned} D(\boldsymbol{c}_1, \boldsymbol{c}_2) &= d(\wp(\boldsymbol{c}_1), \wp(\boldsymbol{c}_2)) \\ &= \cos^{-1}(\boldsymbol{a}_1^T\boldsymbol{a}_1), \end{aligned} \quad (26)$$

where $\wp(\boldsymbol{c}_1) = \boldsymbol{a}_1$ and $\wp(\boldsymbol{c}_2) = \boldsymbol{a}_2$.

If, for $|\epsilon_1| = 1$ and $|\epsilon_2| = 1$,

$$\epsilon_1 a_n > 0, \epsilon_1 b_n > 0. \tag{27}$$

hold, $\epsilon_1 \boldsymbol{a}/|\boldsymbol{a}|$ and $\epsilon_2 \boldsymbol{b}/|\boldsymbol{b}|$ are elements of S_+^{n-1}. Thus, for coefficient vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ which are not elements of S_+^{n-1}, I define the metric as

$$D(\boldsymbol{c}_1,\ \boldsymbol{c}_2) = \cos^{-1} \frac{\epsilon \boldsymbol{a}^T \boldsymbol{b}}{|\boldsymbol{a}| \cdot |\boldsymbol{b}|}, \tag{28}$$

where $\epsilon = \epsilon_1 \epsilon_2$.

Setting $\boldsymbol{A}_i$ and $\boldsymbol{a}_i$ to be appropriate fixed 2×2 real matrices and 2-dimensional constant vectors, respectively, the even-order polynomials and the odd-order polynomials are expressed by

$$p^{2n}(x, y) = \sum_{k=1}^{n} \left(\prod_{n=1}^{k} \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} \right), \tag{29}$$

and

$$p^{2n+1}(x, y) = \begin{cases} \sum_{k=1}^{n} \left\{ \left(\prod_{i=1}^{k} (\boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x}) \right) \cdot (\boldsymbol{a}_k^T \boldsymbol{x}) \right\}, & n \geq 1 \\ (\boldsymbol{a}_i^T \boldsymbol{x}), & n = 0, \end{cases} \tag{30}$$

respectively, because

$$q^{2k}(x, y) = \prod_{i=1}^{k} \boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x} \tag{31}$$

and

$$q^{2k+1}(x, y) = \left(\prod_{i=1}^{k} (\boldsymbol{x}^T \boldsymbol{A}_i \boldsymbol{x}) \right) \cdot (\boldsymbol{a}_k^T \boldsymbol{x}) \tag{32}$$

contain all $2k$th-order monomials and all $(2k+1)$th-order monomials, respectively. Thus, for positive real numbers c and f, lines and conics are expressed by

$$\boldsymbol{l} = \{(x, y)^T | \boldsymbol{a}^T \boldsymbol{x} + c = 0\}, \tag{33}$$

where $\boldsymbol{a} = (a, b)^T$ such that $\boldsymbol{a} \neq \boldsymbol{o}$, and

$$\boldsymbol{c} = \{(x, y)^T | P(x, y) = 0\}, \tag{34}$$

where

$$P(x, y) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + f, \tag{35}$$

for

$$\boldsymbol{A} = \begin{pmatrix} a, & \frac{b}{\sqrt{2}} \\ \frac{b}{\sqrt{2}}, & c \end{pmatrix} \tag{36}$$

and

$$\boldsymbol{b} = (d, e)^T. \tag{37}$$

Here, I show some examples of the expressions of the metrics among curves.

Example 4a. For $a_i c_i \neq 0$, by setting $\boldsymbol{a}_i = (a_i, -1)^T$, lines in $\mathbf{R}^2$ are expressed by

$$\boldsymbol{l}_i = \{(x, y)^T | y = a_i x + c_i\}. \tag{38}$$

Now, the distance between two lines $\boldsymbol{l}_i$ and $\boldsymbol{l}_j$ is defined by

$$D(\boldsymbol{l}_i, \boldsymbol{l}_j) = \cos^{-1} \frac{\epsilon(a_i a_j + c_i c_j + 1)}{\sqrt{(a_i^2 + c_i^2 + 1)(a_j^2 + c_j^2 + 1)}}. \tag{39}$$

Example 4b. If lines are expressed by

$$\boldsymbol{l}_i = \{(x, y)^T | x \cos\theta_i + y \sin\theta_i = r_i\}, \tag{40}$$

where $0 \leq \theta_i < 2\pi$ and $r_i \geq 0$, the distance between two lines $\boldsymbol{l}_i$ and $\boldsymbol{l}_j$ is defined by

$$D(\boldsymbol{l}_i, \boldsymbol{l}_j) = \cos^{-1} \frac{\cos(\theta_i - \theta_j) + r_i r_j}{\sqrt{(1 + r_i^2)(1 + r_j^2)}}. \tag{41}$$

Example 5. By setting

$$c(i, j) = \frac{\epsilon((\boldsymbol{A}_i, \boldsymbol{A}_j) + \boldsymbol{b}_i^T \boldsymbol{b}_j + f_i f_j)}{\sqrt{(\|\boldsymbol{A}_i\|^2 + |\boldsymbol{b}_i|^2 + f_i^2)(\|\boldsymbol{A}_j\|^2 + |\boldsymbol{b}_j|^2 + f_j^2)}}, \tag{42}$$

the distance among conics is defined by

$$D(\boldsymbol{c}_i,\ \boldsymbol{c}_j) = \cos^{-1} c(i, j). \tag{43}$$

For the notations of $(\boldsymbol{A}_i, \boldsymbol{A}_i)$ and $\|\boldsymbol{A}_i\|$, see appendix A.

Example 6. The distance between a line

$$\boldsymbol{l} = \{\boldsymbol{x} | \boldsymbol{a}^T \boldsymbol{x} + c = 0\}, \tag{44}$$

and a conic

$$c = \{x | x^T A x + b^T x + f = 0\}, \quad (45)$$

is obtained by

$$D(c, l) = \cos^{-1} m(i,j), \quad (46)$$

where

$$m(i,j) = \frac{\epsilon(a^T b + cf)}{\sqrt{(\|A\|^2 + |b|^2 + f^2)(|a|^2 + c^2)}}. \quad (47)$$

4. Vector Quantization of Manifolds

For a set of finite points $\mathcal{G} = \{a_i\}_{i=1}^N$ on S_+^{n-1}, a set

$$V(a_i) = \{a | d(a, a_i) \leq d(a, a_j), i \neq j\} \quad (48)$$

is called the Voronoi region of a_i. Each $V(a_i)$ is a convex spherical polygon[5,8]. Here, if $V(a_i)$ and $V(a_j)$ have common points, then these common points lie on a common edge of the spherical polygons $V(a_i)$ and $V(a_j)$. Furthermore, a collection of sets which is defined by eq. (48) divides S_+^{n-1} into finite sets; that is, the relation

$$S_+^{n-1} = \bigcup_{i=1}^{N} V(a_i). \quad (49)$$

holds. This is called the Voronoi tessellation on S_+^{n-1}. For points a and b on S_+^{n-1}, since

$$d(a, b) = \cos^{-1}(a^T b) \quad (50)$$

the following procedure derives a set of generators $\mathcal{G}$ from a finite set $\mathcal{D}$ on S_+^{n-1}.

Generator Generation

for *a randomly selected point* $a \in \mathcal{D}$,
  $a_1 := a$, $\mathcal{G} := \{a_1\}$, $\mathcal{D} := \mathcal{D} \setminus \{a_1\}$
  $i := 2$, $0 < \phi < \pi$,
  while $\mathcal{D} \neq \emptyset$ **do**
    begin
      for *a randomly selected point* $a \in \mathcal{D}$
      if $a_i^T a < \cos\phi$, **then** $a_i := a$, $\mathcal{G} := \mathcal{G} \bigcup \{a_i\}$,
        else $\mathcal{G} := \mathcal{G}$
      $\mathcal{D} := \mathcal{D} \setminus \{a_i\}$, $i := i + 1$.
    end

The distance between two nearist generators a_i and a_j which are produced by this algorithm is longer then ϕ. For a set of generators the following theorem holds.

[Theorem 3] By setting

$$v_{ij} = a_i - a_j, \quad (51)$$

if $v_{ij}^T a > 0$ for a fixed i, a is a element of $a \in V(a_i)$.

Theorem 3 implies that the quantization of vectors on S_+^{n-1} through the nearest-neighbor discrimination [2,3] is achieved by the transform Q from S_+^{n-1} onto $\mathcal{G}$ such that

$$Q(a) = a_i \text{ if } v_{ij}^T a > 0, \text{ for } i \neq j. \quad (52)$$

v_{ij} is required only for $i > j$ because

$$v_{ij} = -v_{ji}. \quad (53)$$

Consequently, the total number of v_{ij} required for the discrimination is $N(N-1)/2$. The domain complexity and the time complexity to compute $v_{ij}^T a$ are equivalent to those of the inner product. Thus, by setting the domain complexity and the time complexity to be d_i and t_i, respectively, the domain complexity and time complexity of the discrimination of curves by Q are $d_i KN(N-1)/2$ and $t_i KTN(N-1)/2$ for K curves, respectivly.

5. Classification of Curves

Let four curves c_1, c_2, c_3, and c_4 be

$$\begin{aligned} c_1 &= \{(x,y)^T | x^3; y^2 - x^2 = 0\}, & (54) \\ c_2 &= \{(x,y)^T | x^3 + y^3 - x^2 + xy^2 = 0\}, & (55) \\ c_3 &= \{(x,y)^T | x^3 - 3xy + y^3 = 0\}, & (56) \end{aligned}$$

and

$$c_4 = \{(x,y)^T | x^5 - 2xy + y^5 = 0\}, \quad (57)$$

respectively. Three curves c_1, c_2, and c_3 are algebraic curves of the third order, and curve c_4 is an algebraic curve of the fifth order. Thus, according to the theory of algebraic geometry, a set of curves is classified into two classes as

$$\{c_1, c_2, c_3, c_4\} = \{c_1, c_2, c_3\} \bigcup \{c_4\}. \quad (58)$$

However, forms of these curves lead to a separation

$$\{c_1, c_2, c_3, c_4\} = \{c_1, c_2\} \bigcup \{c_3, c_4\}. \quad (59)$$

Figure 2 shows these four algebraic curves.

The coefficient vectors of four curves are elements of S^{20}_{+},

$$a_1 = \frac{1}{\sqrt{3}}(o_3, -1, 0, 1, 1, 0, 0, 0, o_3, 0, o_5, 0)^T, \quad (60)$$

$$a_2 = \frac{1}{2}(o_3, -1, 0, 1, 1, 0, 1, 0, o_3, 0, o_4, 0)^T, \quad (61)$$

$$a_3 = \frac{1}{\sqrt{10}}(o_3, 0, -3, 0, 1, 0, 0, 1, o_3, 0, o_4, 0)^T, \quad (62)$$

and

$$a_4 = \frac{1}{\sqrt{6}}(o_3, 0, -2, 0, 0, 0, 0, 0, o_3, 1, o_4, 1)^T, \quad (63)$$

where o_n is defined by

$$o_n = \underbrace{0, 0, \cdots, 0}_{n \text{ elements}}. \quad (64)$$

Table 1 shows distances between pairs of these four curves.

Now, by setting $\phi = \pi/2$, the procedure Generator Generation yields

$$\mathcal{G} = \{a_1, a_2\}. \quad (65)$$

This leads to the tessellation

$$S^{20}_{+} = V(a_1) \bigcup V(a_4). \quad (66)$$

Moreover, since

$$v_{14} = (o_3, -1/\sqrt{3}, 2/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{3}, 0, 0, 0, o_3, -1/\sqrt{6}, o_4, -1/\sqrt{6})^T, \quad (67)$$

the relations

$$v^T_{14}a_2 > 0, \; v^T_{14}a_2 < 0. \quad (68)$$

hold. This mathematically leads to the separation of eq. (59). Thus, the curve discrimination which is discussed in the previous sections derives a natural result for shape analysis directly based on forms appearing in images.

6. Classification of Lines

6.1. Metric of Planar Lines

Since the Euclidean motion is expressed by

$$y = Rx + t \quad (69)$$

by using a rotation matrix R and a translation vector t, the distance between a line l and its rotation r is obtained by

$$D(l, r) = \frac{\epsilon[\{R^T a\}^T a + (a^T t + c)c]}{\sqrt{(|a|^2 + (a^T t + c)^2)(|a|^2 + c^2)}}. \quad (70)$$

Equation (70) implies the following theorem by setting $c = 0$ and $t = o$.

[Theorem 4] The distance between two lines which pass through the origin is defined by the smallest angle between them.

(*Proof*) For a rotation matrix

$$R = \begin{pmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{pmatrix}, \quad (71)$$

it is sufficient to set $0 \leq \theta < \pi$ to express rotations of lines in the Euclidean plane. Thus,

$$D(l, r) = \cos^{-1}\cos\theta, \quad (72)$$

holds. (*Q.E.D.*)

Example *4b* also implies the following corollary.

[Corollary 4.1] If $c \ll 1$, the distance between two lines is approximated by the smallest angle between them.

6.2. Vector Quantization of Planar Lines

Equation (70) implies that the rotation and the translation transform a to Ra and c to $(a^T t + c)$, respectively. On the positive semi-sphere, the rotation and the translation transform a point along a parallel of latitude and along a circle of longitude, respectively. Figure 3 illustrates a motion of a point on POUSS corresponding to the Euclidean motion of a line in the plane. The motion of points on POUSS caused by Euclidean motion suggests the quantization of latitude and longitude for vector quantization of a collection of lines.

By setting

$$0 = c_0 < c_1 < c_2 < \cdots < c_{M-1} < c_M = 1, \quad (73)$$

let intervals on the c-axis be

$$I_k = [c_{k-1}, c_k],\ 1 \le k \le M. \quad (74)$$

Furthermore, for vector $\boldsymbol{a} = (a,b,c)^T$ on S^2_+, $\overline{\boldsymbol{a}}$ is defined by

$$\overline{\boldsymbol{a}} = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 0\right)^T. \quad (75)$$

The set N,

$$N = \{\boldsymbol{a} | \boldsymbol{a} = (a,b,c)^T,\ c \in I_M\} \quad (76)$$

is the north cap. Points in the north cap correspond to lines which are far from the origin in the plane.

For a set of points $\{\overline{\boldsymbol{a}_i}\}_{i=1}^M$,

$$E_{ij} = \{\boldsymbol{a} | d(\overline{\boldsymbol{a}}, \overline{\boldsymbol{a}_i}) \le d(\overline{\boldsymbol{a}}, \overline{\boldsymbol{a}_k}), c \in I_j\} \quad (77)$$

where $k \neq i$ and $j \le M-1$ define a collection of sets which are encircled by two parallels of latitude and two circles of longitude. Furthermore, if E_{ij} and $E_{i'j'}$ have common points, then these points lie in a common edge of E_{ij} and $E_{i'j'}$. Moreover,

$$S^2_+ = (\bigcup_{i=1}^{M}\bigcup_{j=1}^{M-1} E_{ij}) \bigcup N. \quad (78)$$

Consequently, similarly to Voronoi tessellation, a transform

$$E(\boldsymbol{a}) = \boldsymbol{a}_{ij} \text{ if } \boldsymbol{v}_{ii'}^T \overline{\boldsymbol{a}} > 0, \text{ for } i \neq i' \text{ and } c \in I_j \quad (79)$$

achieves the vector quantization of data, where I set

$$c_{ij} = \frac{1}{2}(c_i + c_{i+1}). \quad (80)$$

A sorting procedure achieves the discrimination of lines by using parameter c.

6.3. Approximation of Accumulator Space

If c_{K-1} is small, a truncated semi-sphere

$$C = S^2_+ \setminus N \quad (81)$$

is approximated by a cylinder[5,9]. Furthermore, one can define a one-to-one mapping from points on the surface of a cylinder to points in a rectangle by setting

$$\theta = \begin{cases} \tan^{-1}\frac{a}{b}, & \text{if } a \neq 0 \\ \pi/2, & \text{if } a = 0 \text{ and } b > 0 \\ 3\pi/2, & \text{if } a = 0 \text{ and } b < 0 \end{cases} \quad (82)$$

This permits expression of the accumulator space of the Hough transform by a data structure which is topologically equivalent to a rectangle. This property mathematically clarifies that the classical accumulator space approximates POUSS if one consider lines which are distributed around the origin of the plane.

7. Generalization to Higher Dimensions

Let $\boldsymbol{a}_i$ be unit vector in $\mathbf{R}^n$. Then, a first-order polynomial

$$P_i(\boldsymbol{x}) = \boldsymbol{a}_i^T \boldsymbol{x} + a_{n+1}^i \quad (83)$$

defines a plane

$$\boldsymbol{p}_i = \{\boldsymbol{x} \,|\, P_i(\boldsymbol{x}) = 0\} \quad (84)$$

in $\mathbf{R}^n$. Thus, according to discussions in previous sections, a distance between $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ is defined by

$$D(\boldsymbol{p}_1, \boldsymbol{p}_2) = \cos^{-1}\frac{\epsilon(\boldsymbol{a}_i^T\boldsymbol{a}_j + a_{n+1}^i a_{n+1}^j)}{\sqrt{(1+(a_{n+1}^i)^2)\cdot(1+(a_{n+1}^j)^2)}}. \quad (85)$$

If $a_{n+1}^i = 0$, $\boldsymbol{p}_i$ passes through the origin of $\mathbf{R}^n$ and defines a linear subspace of $\mathbf{R}^n$. The angle between $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ is defined by

$$\angle \boldsymbol{p}_i O \boldsymbol{p}_j = \min_{\boldsymbol{e}_i, \boldsymbol{e}_j}\{\cos^{-1}(\boldsymbol{e}_i^T \boldsymbol{e}_j)\}, \quad (86)$$

for unit vectors $\boldsymbol{e}_i \in \boldsymbol{p}_i$ and $\boldsymbol{e}_j \in \boldsymbol{p}_j$. Consequently, the relation

$$\min_{\boldsymbol{e}_i, \boldsymbol{e}_j}\{\cos^{-1}(\boldsymbol{e}_i^T \boldsymbol{e}_j)\} = \cos^{-1}(\boldsymbol{a}^T \boldsymbol{b}) \quad (87)$$

holds. Therefore, the definition of the distance measure between manifolds coincides with the angles between two subspaces if the manifolds are subspaces.

8. Conclusions

I defined a metric among the elements of the families of algebraic manifolds which are a generalization of lines and conics in the Euclidean plane. The distance between manifolds is introduced by using one-to-one mapping between the unit semi-sphere in the Euclidean space and a family of manifolds. Furthermore, by using the proposed metric, I proposed a discrimination algorithm of manifolds. This algorithm achieves the discrimination of curves through nearest-neighbor discrimination on the unit semi-sphere. Since S_{+}^{n-1} is a subspace of $\mathbf{R}^n$, for the determination of generators, one can apply any method developed for pattern recognition such as the principal component analysis.

Moreover, I clarified that a metric is defined in the accumulator spaces if accumulator space for curve detection is equivalent to the unit semi-sphere. This accumulator also permits the Hough transform which detects both lines and conics. Results of map projections [9] allow the derivation of an approximate metric for the proposed metric on a rectangular accumulator space which is appropriate for practical implementation.

Part of this research was carried out under Grants-in-Aid for research from the Ministry of Education, Science, and Culture of Japan.

References

[1] Kanatani, K.: *Geometric Computation for Machine Vision,* Oxford University Press; Oxford, 1993.

[2] Schalkoff, R.: *Pattern Recognition: Statistical, Structural and Neural Approaches,* John Wiley & Sons; New York, 1992.

[3] Oja, E.: *Subspace Methods of Pattern Recognition,* John Wiley & Sons; New York, 1983.

[4] Ballard, D. and Brown, Ch. M.: *Computer Vision,* Prentice-Hall; New Jersey, 1982.

[5] Berger, M.: *Geometry I, II.* Springer-Verlag; Berlin, 1987.

[6] Sanz, J. L. C., Hinkle, E. B., and Jain, A. K.: *Radon and Projection Transform-Based Computer Vision.* Springer-Verlag; Berlin, 1988.

[7] Oja, E., Xu, L., and Kultanen, P.: "Curve detection by an extended self-organization map and the related RHT method," Proceedings of International Neural Network Conference, Vol. 1, pp.27-30, 1990.

[8] Okabe, A., Boots, B., and Sugihara, K.: *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams.* John Wiley & Sons; New York, 1992.

[9] Milnor, J.: "A problem in cartography," The American Mathematical Monthly, Vol. 76, pp.1101-1112, 1969.

Appendix A

For $N \times N$ real matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, the matrix inner product is defined by

$$(\boldsymbol{A}, \boldsymbol{B}) = \sum_{m=1}^{N} \sum_{n=1}^{N} a_{mn} b_{mn},$$

where a_{mn} and b_{mn} are the mn-th elements of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Furthermore, the matrix norm is defined by

$$\|\boldsymbol{A}\| = \sqrt{(\boldsymbol{A}, \boldsymbol{A})}.$$

By setting that $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are the i-th column vectors of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, if I define vectors

$$\begin{aligned} \boldsymbol{a} &= (\boldsymbol{a}_1^T, \boldsymbol{a}_2^T, \cdots, \boldsymbol{a}_N^T)^T \\ \boldsymbol{b} &= (\boldsymbol{b}_1^T, \boldsymbol{b}_2^T, \cdots, \boldsymbol{b}_N^T)^T, \end{aligned}$$

the matrix inner product of $\boldsymbol{A}$ and $\boldsymbol{B}$ coincides with the ordinary inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$ in the N^2-dimensional real vector space.

Figure Legends

Figure.1 A line in the plane corresponds to a point on the positive unit semi-sphere.

Figure.2 From up to bottom $x^3 + y^2 - x^2 = 0$, $x^3 + y^3 - x^2 + xy^2 = 0$, $x^3 - 3xy + y^3 = 0$, and $x^5 - 2xy + y^5 = 0$.

Figure.3 Rotation and translation of a line in the Euclidean plane transform a point on the unit semi-sphere along a parallel of latitude and along a circle of longitude.

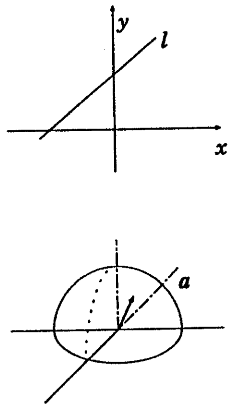

Figure.1 A Line and The Positive Unit Semi-Sphere.

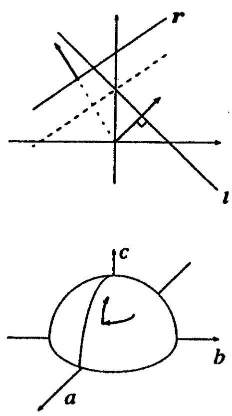

Figure.3 Transformation on The Positive Unit Semi-Sphere.

Table 1. Distances of Curves

$D(\mathbf{c}_i, \mathbf{c}_j)$	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$	$\mathbf{c}_4$
$\mathbf{c}_1$	0	$\pi/6$	1.387	$\pi/2$
$\mathbf{c}_2$	$\pi/6$	0	1.412	$\pi/2$
$\mathbf{c}_3$	1.387	1.412	0	0.685
$\mathbf{c}_4$	$\pi/2$	$\pi/2$	0.685	0

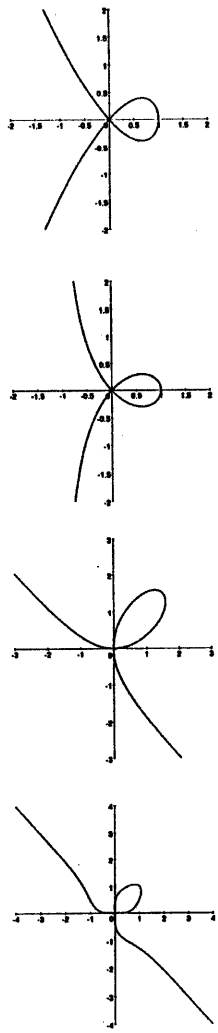

Figure.2 Four Algebraic Curves.