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双対フラクタルとその応用

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Dual Fractals: Theory and Applications

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Abstract This paper presents some definitions and theorems concerning to dual fractals. Among them, *dual-similarity* plays a key-role not only in generating dual fractals but also in handling inter-pattern relations such as pattern recognition or image coding. Dual-similarity is basically defined as a pair of similarity relations between two patterns, from which two mirror operators have been derived. Discussion has also been made on possible applications of dual fractals.
Keywords: dual fractals, dual-similarity, Hutchinson operator, mirror operator.

1. Introduction

Every fractal pattern is characterized by a set of similarity relations defined (or detected) between the entire pattern and its inner sub-patterns. Mathematically this is termed *self-similarity*[1-5]. MRCM (Multiple Reduction Copy Machine) can be regarded as the procedural representation of self-similarity by which a fractal pattern is to be defined as the limit pattern[2-4]. Since self-similarity is basically nothing but a set of inner relations described within *one* fractal pattern, inter-pattern relations or pattern-to-pattern relations has never been discussed in relation to self-similarity. On the other hand, we have many living subjects linking with inter-pattern relations; i.e. pattern recognition, image understanding, computer vision and so on. It follows that a sort of bridge across the gap between self-similarity and inter-pattern relations should be required for evoking new applications of fractal geometry.

2. Contraction Mapping and Dual Fractals

We discuss dual-similarity between two patterns from the view-point of mathematical mapping (or transformation) in the metric space of patterns. The first step towards our discussion is to put some definitions:

Definition 1 Let I be a set of real numbers $\{x \mid 0 \le x \le 1\}$. Let every pattern depicted on the two-dimensional plane $I^2 = \{(x_1, x_2) \mid x_1, x_2 \in I\}$ be defined by a real-valued density function $0 \le f(x_1, x_2) \le 1$. When we write a point on I^2 as $\boldsymbol{x}=(x_1, x_2)$, the density function is written $f(\boldsymbol{x})$. A density function $f(\boldsymbol{x})$ is called a *pattern* and denoted simply as f.

Definition 2 Between any pair of patterns f and g, distance $d(f, g)$ is defined in a space of patterns $\boldsymbol{P}$. Where either Hausdorff distance[2-5] or the supremum distance[4] is employed for $d(f, g)$, corresponding to each of two types of spaces of patterns; i.e. black-and-white and shading patterns. When necessary, we denote two types of spaces as $\boldsymbol{P}_b$ and $\boldsymbol{P}_s$ for black-and-white and shading patterns, respectively.

Definition 3 Let $\boldsymbol{P}$ be a metric space of patterns. Then a transformation T: $\boldsymbol{P} \to \boldsymbol{P}$ is called a *contraction mapping* if there is a constant $0 \le s < 1$ such that

$$d(Tf, Tg) \le s \cdot d(f, g) \quad \text{for any } f, g \in \boldsymbol{P}.$$

Where $d(f, g)$ is the distance between f and g in $\boldsymbol{P}$. Any number s is called a contraction factor of T.

Theorem 1 Let T: $\boldsymbol{P} \to \boldsymbol{P}$ be a contraction mapping on a complete metric space of patterns. Then T possesses exactly one fixed point (pattern) $q \in \boldsymbol{P}$ such that

$$q = T q.$$

Moreover, for any pattern $f \in \boldsymbol{P}$, the sequence $\{T^n f \mid n=1,2,\cdots\}$ converges to q. Symbolically we have

$$\lim_{n \to \infty} T^n f = q.$$

Where $T^n f = TT \cdots Tf$ (operating T on f n times, where $T^0 f = f$ and $T^1 f = Tf$). Fixed point q is called *attractor* of T. (*Fixed Point Theorem*[2-6])

Proof See references[2-6].

Definition 4 Let $T_1, T_2, \cdots, T_n$ be a finite number of contraction mappings on a complete metric space of patterns $\boldsymbol{P}$. Then, for any pattern $f \in \boldsymbol{P}$, a *parallel compound* Tf is denoted by

$$Tf = T_1 f \cup T_2 f \cup \cdots \cup T_n f.$$

For $f \in \boldsymbol{P}_b$, $T_1 f, T_2 f, \cdots, T_n f$ correspond to n sets of points in I^2, respectively. Then Tf implies the set operation $\cup$ (union) for the n sets. For $f \in \boldsymbol{P}_s$, Tf is is defined as follows.

With each contraction mapping $T_i\,(i=0,1,\cdots,n)$ on P_s, let a transformed pattern $T_i f$ be given by a density function defined over a range $R_i \subset I^2$. In addition, suppose that the following conditions hold for n ranges $R_1, R_2, \cdots, R_n$:

$$R_i \cap R_j = \emptyset \quad \text{for } i \neq j \quad (i, j = 0, 1, \cdots, n)$$

$$R_1 \cup R_2 \cup \cdots \cup R_n \subseteq I^2$$

Then the parallel compound is defined by such a sum as

$$Tf(x)=T_1f(x)+T_2f(x)+ \cdots +T_nf(x).$$

The parallel compound can be regarded as a transformed pattern given by a composite mapping on a metric space so that we have a common concise notation such as

$$T=T_1 \cup T_2 \cup \cdots \cup T_n . \quad \textit{(Hutchinson operator[5])}$$

Theorem 2 Let $T_1, T_2, \cdots, T_n$ be a finite number of contraction mappings on a complete metric space of patterns $\boldsymbol{P}$ with contraction factors $s_1, s_2, \cdots, s_n$, respectively. Then $T=T_1 \cup T_2 \cup \cdots \cup T_n$ denoted in (11) is also a contraction mapping on $\boldsymbol{P}$ with contraction factor $s=max\{ s_1, s_2, \cdots, s_n \}$.

Proof See Hutchinson's proof in case of $\boldsymbol{P}=\boldsymbol{P}_b$ [5]. On the other hand, when $\boldsymbol{P}=\boldsymbol{P}_s$, Fisher made a proof in terms of *z-contraction* with the supremum distance [4].

Theorem 3 Let T be a contraction mapping with contraction factor s on a complete metric space of patterns $\boldsymbol{P}$. Then we have

$$d(f, q) \leq \frac{1}{1-s} d(f, T f).$$

Where q is attractor of T. (*Collage Theorem*[2])

Proof See Barnsley's proof[2] or other references[3,4].

Definition 5 Let $T_1, T_2, \cdots, T_n$ be a finite number of contraction mappings on a complete metric space of patterns $\boldsymbol{P}$. Then, for any pattern $f \in \boldsymbol{P}$, a *serial* (or *cascaded*) *compound* Tf is defined by

$$T f =T_nT_{n-1} \cdots T_2T_1 f.$$

Here, we write T such a composite mapping as

$$T =T_n \cdots T_2T_1 .$$

Theorem 4 Let $T_1, T_2, \cdots, T_n$ be a finite number of contraction mappings on a complete metric space of patterns $\boldsymbol{P}$ with contraction factors $s_1, s_2, \cdots, s_n$, respectively. Then a composite mapping $T=T_n \cdots T_2T_1$ is also a contraction mapping on $\boldsymbol{P}$ with contraction factor $s=s_1 s_2 \cdots s_n$.

Proof For any f and $g \in \boldsymbol{P}$,

$$
\begin{aligned}
d(Tf, Tg) &= d(T_n(T_{n-1} \cdots T_2 T_1 f), T_n(T_{n-1} \cdots T_2 T_1 g)) \\
&\le s_n\, d(T_{n-1}(\cdots T_2 T_1 f), T_{n-1}(\cdots T_2 T_1 g)) \\
&\ \vdots \\
&\le s_n s_{n-1} \cdots s_2\, d(T_1 f, T_1 g) \\
&\le s_n s_{n-1} \cdots s_1\, d(f, g) = s\, d(f, g)
\end{aligned}
$$

Q.E.D.

Definition 6 Let $\boldsymbol{P}$ be a complete metric space of patterns. Then between given f, $g \in \boldsymbol{P}$, *dual-similarity* holds if there exists a pair of contraction mappings T and U on $\boldsymbol{P}$ such that

$$g = Tf \quad \text{and} \quad f = Ug.$$

Immediately we have

$$g = TUg \quad \text{and} \quad f = UTf.$$

We call dual composite mappings TU and UT *mirror operators*.

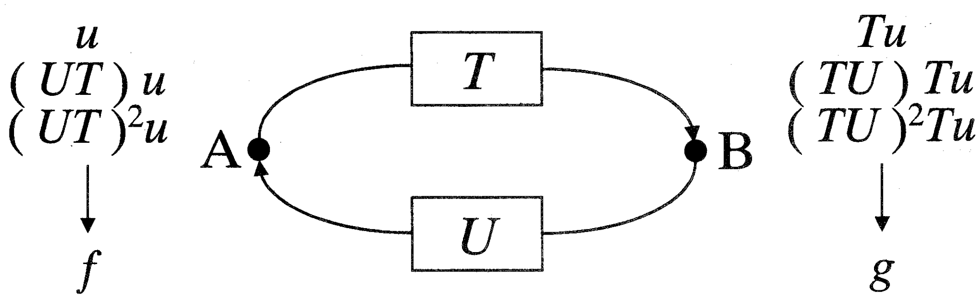


Figure 1: Feedback loop.

Note that mirror operators ($TU \neq UT$) are also contraction mappings from Theorem 4. Then g and f are attractors of TU and UT, respectively. By Theorem 2, a sequence $\{(TU)^n u \mid u \in \boldsymbol{P}\}$ converges to attractor g. Another sequence $\{(UT)^n v \mid v \in \boldsymbol{P}\}$ converges to f. Such iteration processes promoted by mirror operators can be represented by a feedback loop illustrated in Figure 1. Let $u \in \boldsymbol{P}$ be initially given to terminal A in the left-hand side in the figure. Then its first transformed pattern Tu appears at terminal B in the right-hand side. At the next stage, just after the first round of the loop, UTu returns back to A. After n rounds, $(UT)^n u$ appears at A. Here pay attention to the next stage, at B, where a transformed pattern

$$T(UT)^n u = T(UT \cdots UT)u = (TU \cdots TU)Tu = (TU)^n Tu$$

is obtained. Note $Tu \in \boldsymbol{P}$. From the above discussion, for $n \to \infty$, we have

$(UT)^n u \to f$ at terminal A,

$(TU)^n Tu \to g$ at terminal B.

Where f and g are attractors of UT and TU, respectively (See Figure 1). When f and g are fractal patterns, we call them *dual fractals*. Figure 2 shows an example of dual fractals.

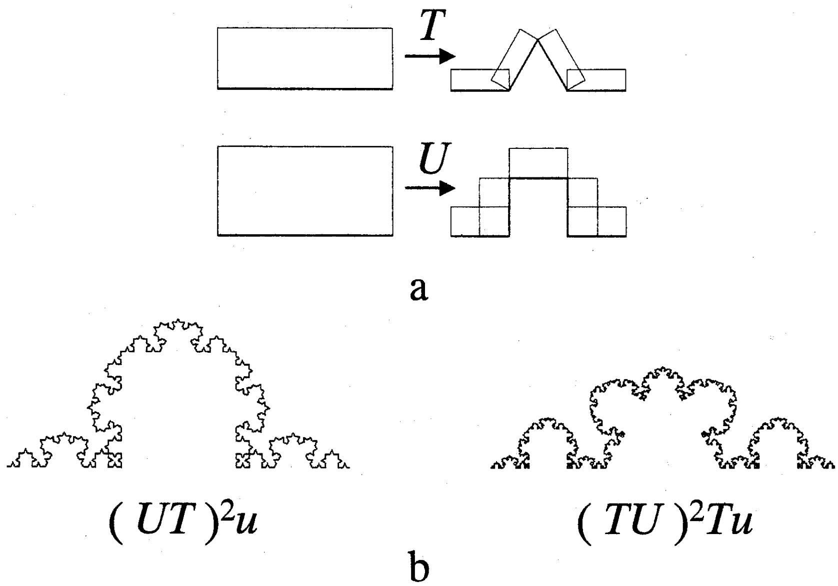


Figure 2: Dual fractals. a: Hutchinson operators. b: $(UT)^2u$ and $(TU)^2Tu$.

Theorem 5 Let $\boldsymbol{P}$ be a complete metric space of patterns. Suppose the following dual-similarity holds between given $f \in \boldsymbol{P}$ and a *reference pattern* $r \in \boldsymbol{P}$:

$r = Tf$ and $f = Ur$

Where T and U are contraction mappings on $\boldsymbol{P}$ with contraction factors s_T and s_U, respectively. Then, for any $p \in \boldsymbol{P}$, we have

$$(1/s_U)\cdot d(UTp, f) \leq d(Tp, r) \leq s_T \cdot d(p, f). \quad (1)$$

(*Fractal Pattern Recognition Theorem*[7])

Proof $d(Tp, r) \leq d(Tp, Tf) \leq s_T \cdot d(p, f)$ and $d(UTp, f) = d(UTp, Ur) \leq s_U \cdot d(Tp, r)$

Q.E.D.

3. Applications

Dual-similarity and dual fractals suggest that we can possibly have new applications to several subjects. Here our basic and tentative discussion will be focused on two subjects; pattern recognition and image coding.

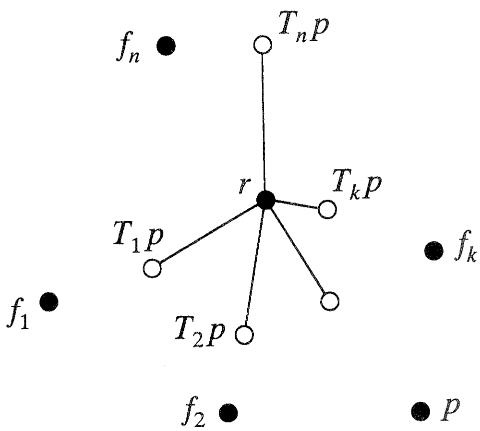


Figure 3: Unknown pattern p and the minimum distance $d(T_k p , r)$ in a metric space.

3.1 Pattern Recognition

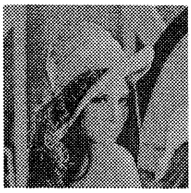
A type of definition of pattern recognition is written as follows: Let $f_1, f_2, \cdots, f_n$ be standard patterns which are the representatives of n classes of patterns, respectively. Suppose there exist infinitely many patterns belonging to i-th class (i=1, 2, ⋯, n) around f_i in a metric space of patterns. Then, given a unknown pattern p, we have a special problem, termed *pattern recognition*, to identify p; i.e. to judge what class p belongs to.

The simplest way to identify a unknown pattern p is to detect the minimum distance among $\{d(p, f_i) \mid i=1, 2, \cdots, n\}$. We judge p belongs to k-th class when the minimum distance is obtained for $i=k$.

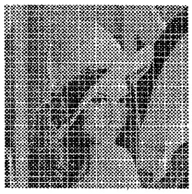
Instead of such direct comparison between p and standard patterns $\{f_i\}$, another procedure based on dual-similarity can be introduced as follows: First, we build n dual-similarity relations between an appropriate reference pattern r and standard patterns $\{f_i\}$. Symbolically we denote

$$r = T_i f_i \quad \text{and} \quad f_i = U_i r \quad (i=1, 2, \cdots, n).$$

Given a unknown pattern p, we identify it with a pattern in k-th class when the minimum distance among $\{d(T_i\, p, r)\}$ is obtained where $i=k$ (See Figure 3). As mentioned, by Theorem 5, the necessary condition is satisfied for identifying p based on $\{d(T_i p, r)\}$. The fractal pattern recognition theorem is, however, incomplete in the sense that the sufficient condition has not been proved. One of the most important tasks in the fractal pattern recognition might be concerned with how to find the optimal reference pattern which acts well for properly classifying unknown patterns.

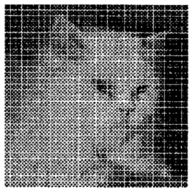


Lenna (256×256)



Range-cells of Lenna

Pepe (256×256)



Range-cells of Pepe

Figure 4: Two sample images and their range-cells.

3.2 Image Coding

The fractal image coding technique has been developed based on self-similarity assumed within a given image[2,5]. It should be noted that most images handled in this technique have never been *fractals*. Dual-similarity can provide an individual image coding technique as well as self-similarity[7]. Here an "image" is treated as a shading pattern denoted by a density function defined over the domain I^2.

First, we discuss how to compose a pair of Hutchinson operators which approximates two contraction mappings founding dual-similarity between two images. Suppose we are given two images f and g defined as density functions over I^2. According to Definition 9, we assume there exist two contraction mappings T and U such that

$$g=Tf \quad \text{and} \quad f=Ug. \tag{2}$$

The simplest way to compose Hutchinson operators approximating T and U starts with partitioning each of the images into sub-images: As shown in Figure 4, g is partitioned into very small-sized sub-images on disjoint sub-squares. Call each sub-square a *range-cell*. Then, for each range-cell of g, we search over the domain of f an optimal sub-square on which sub-image is well fit for that of the range-cell by applying an contractive affine transformation. We call such a sub-square of f a *domain-cell*. Since reduction is the core of contraction mapping, every domain-cell is to be larger than the range-cell.

Suppose we have n range-cells and their corresponding sub-images $g^{(1)}, g^{(2)}, \cdots, g^{(n)}$ which cover the entire image g. Each of $\{g^{(i)}\}$ is approximated by fitting an appropriate sub-image on a domain-cell of f, which is obtained by an affine transformation. We assume this can be written by such a contraction mapping as

$$g^{(i)}=T^{(i)}f \qquad (i=1, 2, \cdots, n).$$

Since $g=g^{(1)} \cup g^{(2)} \cup \cdots \cup g^{(n)}$, we obtain a Hutchinson operator as follows:

$$Tf=T^{(1)}f \cup T^{(2)}f \cup \cdots \cup T^{(n)}f,$$

or, as denoted in (11), we write shortly

$$T=T^{(1)} \cup T^{(2)} \cup \cdots \cup T^{(n)}. \qquad (3)$$

In the same manner, we have the following Hutchinson operator U such that $f=Ug$, which is composed corresponding to m range-cells of f (See Figure 4).

$$U=U^{(1)} \cup U^{(2)} \cup \cdots\cdots \cup U^{(m)} \qquad (4)$$

Two Hutchinson operators T and U in (3) and (4) can practically be employed as contraction mappings actualizing dual-similarity between f and g as presented in (2).

For f and g in (2), we employ two example images Lenna and Pepe, respectively, as is shown in Figure 4. They are 256×256 pixel images in which each pixel can be one of 256 levels of gray from black to white. Range-cells of an image are defined as 8×8 pixel disjoint sub-squares covering the 256×256 pixel domain. Then every image is

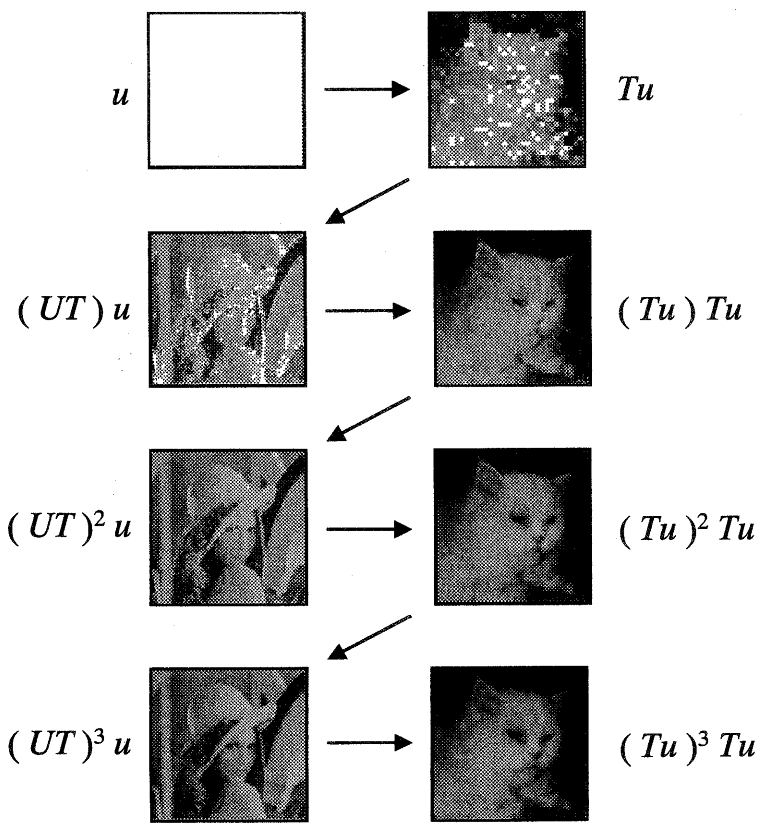


Figure 5: Two sequences of images generated from the feedback loop.

partitioned into 32×32=1024 sub-images. A domain-cell is defined as a 16×16 pixel sub-square in the 256×256 pixel domain of each image. Hence either of two Hutchinson operators is to be composed of 1024 contraction mapping operators from $m=n=1024$ in (3) and (4).

Employing these Hutchinson operators T and U, we can build a feedback loop same as Figure 1. Figure 5 presents both sequences of images appeared at terminals A and B of the feedback loop, respectively. Where this iteration process begins with the initial image $f(x)=1$ (everywhere "white" over the domain) given at terminal A. When iteration reaches the limit state, a unique attractor of a mirror operator TU appears at terminal A which is an approximate image of Lenna. At terminal B, another attractor of UT also appears as an approximate image of Pepe.

Our image coding, termed *dual-similarity coding* (*DSC*), is nothing but determining all parameters (codes) of U such that $f=Ug$. Then any receiver can approximately reconstruct f by U sent from a sender, provided that the receiver also keeps g as a key for decoding U. In this case, decoding is carried out very quickly by $f=Ug$; i.e. no iteration process is required at all for decoding U. This is quite different from the existing fractal image coding procedures based on self-similarity. Another way of decoding is to reconstruct f and g as attractors of mirror operators UT and TU, provided both U and T are sent from a sender.

4. Conclusion

In this paper, dual-similarity between two patterns has been defined and discussed in relation to fractal geometry. It would be very suggestive for new fractal applications that we can handle inter-pattern relations by introducing dual-similarity relations between two patterns. Namely possible applications to pattern recognition and image coding have also been discussed. As a matter of course, our next step will be concerned with studies on definite practical problems such as character recognition, face recognition or image compression. As previously mentioned, no matter what type of application is taken up, it should be very important for our success either how to compose Hutchinson operators or how to select the reference pattern.

References

[1]B.B. Mandelbrot, *Fractals: Form, Chance, and Dimension*, W.H.Freeman and Co., San Francisco, 1977.

[2]M.F. Barnsley, *Fractals Everywhere*, Academic Press, San Diego, 1988.

[3]H-O. Peitgen, H. Juergens and D. Saupe, *Fractals for the Classroom, Part One Introduction to*

Fractals and Chaos, Springer-Verlag, New York, 1991.

[4]Y. Fisher (ed), *Fractal Image Compression*, Springer-Verlag, New York, 1995.

[5]J. Hutchinson, Fractals and self-similarity, *Indiana University Journal of Mathematics*, Vol.30, 713-747, 1981.

[6]D.G. Luenberger, *Optimization By Vector Space Method*, John Wiley & Sons, Inc., New York, 1969.

[7]K. Ozawa, Dual fractals, *Technical report of Information Science Center* (Osaka Electro-Comm. Univ.), ISC2000, 2000 (to appear).