データベース・システム　80－2
人　工　知　能　73－2
（1990．11．8）

Belief as a Justified and Consistent Knowledge II

—— Theoretical Foundations of Reasoning with Beliefs ——

Satoshi Matsuda

Computer & Communication Research Center
Tokyo Electric Power Company
1-3, Uchisaiwai-cho, 1 Chome, Chiyoda-ku, Tokyo 100, JAPAN

In the previous paper[6], a formalization of knowledge and belief was presented based on modal logic, where belief was defined as a justified and consistent knowledge. Properties of beliefs and knowledges, which are derived in this system, seem intuitively appropriate. The considerations on theoretical foundation of reasoning with beliefs is made here by using possible world approach. The proof of distribution axiom of belief is also given. Thus our formal system is shown to be an appropriate model where knowledge and belief are represented and reasoned.

知識と信念の論理体系 II

—— 信念による推論の理論的基礎 ——

松田　聖
東京電力（株）　システム研究所

著者が提案した知識と信念に関する様相論理に基づいた体系では[6]、知識を表す様相論理上に、信念はその正当性を示唆する何等かの「根拠」の存在、および既存の知識や信念と「矛盾しない」ことの二つを知っていることと定義され、導かれる知識と信念に関する性質は直観と一致した。本論文では、可能世界を用いることにより、本体系における信念による推論に関する理論的基礎付けを与えるとともに、信念による推論を保証する分配則を証明する。以上により、本体系は知識と信念を表現し推論する妥当な形式モデルであるといえよう。

1. Introduction

In the previous paper[6], according to the view that belief is an intentional object and is introduced to complement the lack of knowledges to explain some phenomina, a formal system based on modal logic has been presented for knowledge and belief. The belief is defined as a justified and consistent knowledge. That is, belief is an information of which consistency with other knowledges and beliefs of the agent and justification suggesting its validity he knows. In that system we can represent and reason knowledges and beliefs of many agents and their nested beliefs including true facts which are not known or believed by agents. The concept of justification has also been introduced, which plays an important role to formalize the concept of belief generally and makes it possible to represent and reason with beliefs including not only default but also other aspects of beliefs, for example, abduction. Several properties of beliefs, and relations between beliefs and knowledges, which are derived in the system, seem intuitively appropriate.

Example of reasoning with knowledges and beliefs was shown, however, the formalization of reasoning with belief in that system was not shown so precisely. We want to reason with beliefs as with knowledges, that is, we want the distribution axiom of belief, i.e., $Bp \wedge B(p \supset q) \supset Bq$, which was given there without proof. In order to prove it, we must make the concept of consistency among knowledges and beliefs more precise. By introducing the possible worlds or concept of submodality of the provability, the concept of consistency can be made precise and some important properties of the system will be derived. The theoretical foundation of reasoning with beliefs and knowledges are given, and the proof of distribution axiom of belief are also given. Thus our formal system is shown to be an appropriate model where knowledge and belief are represented and reasoned.

2. Formal System

The formal system for belief is given as in the previous paper[6].

(1) Formal system

Axioms:

1. $Kp \supset p$ (Knowledge Axiom)
2. $K(p \supset q) \supset (Kp \supset Kq)$ (Distribution Axiom of Knowledge)
3. $Kp \supset KKp$ (Positive Introspection Axiom of Knowledge)

(4. $\neg Kp \supset K \neg Kp$) (Negative Introspection Axiom of Knowledge)

5. $Lp \supset p$ or $p \supset Mp$
6. $L(p \supset q) \supset (Lp \supset Lq)$ or $\neg(Mp \supset Mq) \supset M \neg (p \supset q)$
7. $Lp \supset LLp$ or $MMp \supset Mp$
8. $\neg Lp \supset L \neg Lp$ or $Mp \supset \neg M \neg Mp$
9. $J(p \supset q) \supset (Jp \supset Jq)$ (Distribution Axiom of Justification)
10. $Kp \supset Jp$ (Justification Axiom)

Inference Rules:

1. $p, p \supset q$ infer q (Modus Ponens)
2. p infers Kp (Epistemic Necessitation)
3. p infers Lp (Necessitation for L)
4. "can't infer $\neg p$" infers Mp, where $Mp \equiv \neg L \neg p$ (Possibilitation for M)

(2) Definition of belief

The definitions of knowledge and belief are given as follows.[6]

Def.1) Knowledge of p is defined as Kp, where K is a modal operator defined in the axiom system above.

Def.2) Belief of p, denoted by Bp, is defined as $K(MKp \wedge Jp)$, that is, $Bp \equiv K(MKp \wedge Jp)$, where K, M and J are modal operators defined in axiom system shown above.

Def.3) The set of knowledges and beliefs of the agent with the set of proper axioms A, denoted by TH(A), is defined as

$TH(A) = \cup_{N=0}^{\infty} TH^N(A)$,

where

$TH^0(A) = Tk(A)$, $B^0 = \emptyset$, $K^0 = \emptyset$,

$TH^N(A) = Tk(A \cup B^N)$, $B^N = B^{N-1} \cup \{Bp_N\}$,

$K^N = K^{N-1} \cup \{Kp_N\}$ for some proposition p_N such that $\neg Kp_N \notin Th(A \cup K^{N-1})$ and $K(MKp_N \wedge Jp_N) \in Th(A \cup K^{N-1})$, and

$Th(A) = \{p \mid A \vdash p\}$,

$Tk(A) = \{Kp \mid A \vdash Kp\}$.

Note that $A \vdash p$ means that p is infered with set of assumptions A using above modal propositional system except Possibilitation Rule for M. Possibilitation Rule for M is only applied when K^N is introduced from K^{N-1}.

Note also that each proper axiom is a formula formed by using propositional symbols, modal operator K and logical connectives correctively in a usual way.

3. The Possible World Semantics and the Submodality of Provability

(1) Possible world semantics

Firstly let's survey the possible world semantics of modal logic. Following Moore[10], we introduce a predicate T to describe the truth of the proposition in a possible world, that is, T(w,p) means that p is true in possible world $w \in W$. Then we have

$p \equiv T(W_0, p)$,

$Lp \equiv \forall w \in W\ (R_L(W_0, w) \supset T(w,p))$, and

$Mp \equiv \exists w \in W\ (R_L(W_0, w) \wedge T(w,p))$ where W_0 is the actual world.

The following properties of the predicate T are given by him[10].

・Atom

L1: $\forall p_1 (TRUE(p_1) \equiv T(W_0, p_1))$

・Logical connectives

L2: $\forall w_1, p_1, p_2 (T(w_1, AND(p_1, p_2)) \equiv (T(w_1, p_1) \wedge (T(w_1, p_2)))$

L3: $\forall w_1, p_1, p_2 (T(w_1, OR(p_1, p_2)) \equiv (T(w_1, p_1) \vee (T(w_1, p_2)))$

L4: $\forall w_1, p_1, p_2 (T(w_1, IMP(p_1, p_2)) \equiv (T(w_1, p_1) \supset (T(w_1, p_2)))$

L5: $\forall w_1, p_1, p_2 (T(w_1, IFF(p_1, p_2)) \equiv (T(w_1, p_1) \equiv (T(w_1, p_2)))$

L6: $\forall w_1, p_1 (T(w_1, NOT(p_1)) \equiv \neg T(w_1, p_1))$

where AND, OR, IMP, IFF, and NOT are functions in metalanguage and they correspond logical connectives $\wedge, \vee, \supset, \equiv, \neg$ in object language. The above axioms state that logical connectives in object language can be transformed to logical connectives in metalanguage.

・Quantifiers (EXIST/ALL)

L7: $\forall w_1 (T(w_1, EXIST(X,P)) \equiv \exists x_1 (T(w_1, P\,[@(x_1)/X]\,)))$

L8: $\forall w_1 (T(w_1, ALL(X,P)) \equiv \forall x_1 (T(w_1, P\,[@(x_1)/X]\,)))$

where P is a formula in object language, X is a variable of object language, @ is function which maps constant of object language to its rigid designator, that is the value of @(x) is x in all possible worlds, and $P\,[@(x_1)/X]$ is a countpart substituting $@(x_1)$ to all X.

(2) Submodality N of modality L

Let L be the modal operator satisfying the axioms mentioned above, and R_L be the accessibility relation of L on the set of possible worlds W. Now, for any subset $N \subseteq W$, let R_N be subrelation of R_L such that $R_N(w_1, w_2) \equiv w_1 \in N \wedge w_2 \in N \wedge R_L(w_1, w_2)$ for any $w_1, w_2 \in W$. And let N also denote the modal operator corresponing to this accessibility relation R_N. Thus, Np is true iff p is true in any possible world which is accessible from the actual world W_0 through accessibility relation R_N.

Def.4) Let $N \subseteq W$, and R_N be the subrelation of accessibility relation R_L, and N be the modal operator satisfying

$Np \equiv \forall w \in W\ (R_N(W_0, w) \supset T(w,p))$.

Def.5) N is a submodality of L, denoted by $N \subseteq L$, iff $\forall w_1, w_2 \in W\ (R_N(w_1, w_2) \equiv w_1 \in N \wedge w_2 \in N \wedge$

$R_L(w_1, w_2)$), where R_N and R_L are accessibility relation of N and L between possible worlds.

Def.6) Let $N_1 \subseteq L$ and $N_2 \subseteq L$. N_1 is a submodality of N_2, denoted by $N_1 \subseteq N_2$, iff N_1 is a subset of N_2.

Note that submodality $N \subseteq L$ is not reflexive since $R_N(w,w)$ does not hold for $w \notin N$. So, $Np \supset p$ does not hold.

Lemma1) The accessibility relation R_N is transitive, i.e., $R_N(w_1,w_2) \wedge R_N(w_2,w_3) \supset R_N(w_1,w_3)$.

pr) $R_N(w_1,w_2) \wedge R_N(w_2,w_3)$

$$\equiv (w_1 \in N \wedge w_2 \in N \wedge R_L(w_1,w_2)) \wedge (w_2 \in N \wedge w_3 \in N \wedge R_L(w_2,w_3))$$

$$\supset w_1 \in N \wedge w_3 \in N \wedge R_L(w_1,w_3) \quad \text{(since } R_L \text{ is transitive)}$$

$$\equiv R_N(w_1,w_3). \qquad \text{Q.E.D.}$$

Using the concept of submodality of L, Lp and Mp can be represented as follows.

$$Lp \equiv \forall w \in W\ (R_L(W_0,w) \supset T(w,p))$$

$$\equiv \forall N \subseteq L\ \forall w \in W\ (R_N(W_0,w) \supset T(w,p))$$

For the proposition p satisfying Mp, that is, $\exists w \in W(R_L(W_0,w) \wedge T(w,p))$, by letting $N^p \subseteq W$ such that $\forall w \in W\ (w \in N^p \equiv R_L(W_0,w) \wedge T(w,p))$, and R_{N^p} be the accessibility relation such that $\forall w \in W\ (R_{N^p}(W_0,w) \equiv w \in N^p)$, that is, $\forall w \in W\ R_{N^p}(W_0,w) \equiv R_L(W_0,w) \wedge T(w,p)$, and N^p be the modal operator corresponing to this accessibility relation R_{N^p}, then we have

$$Mp \equiv \exists w \in W\ (R_L(W_0,w) \wedge T(w,p))$$

$$\equiv \forall w \in W\ (R_{N^p}(W_0,w) \supset T(w,p)).$$

That is,

$$Mp \equiv \exists N \leq L\ \forall w \in W\ (R_N(W_0,w) \supset T(w,p)).$$

Thus, we obtain

$$Lp \equiv \forall N \subseteq L\ Np, \text{ and}$$

$$Mp \equiv \exists N \subseteq L\ Np.$$

The truth of Np, Lp and Mp in any possible world are extended as follows,

$$T(w,Np) \equiv \forall w_1 \in W\ (R_N(w,w_1) \supset T(w_1,p))$$

$$T(w,Lp) \equiv \forall N \subseteq L\ \forall w_1 \in W\ (R_N(w,w_1) \supset T(w_1,p))$$

$$T(w,Mp) \equiv \exists N \subseteq L\ \forall w_1 \in W\ (R_N(w,w_1) \supset T(w_1,p)).$$

(3) Properties of submodality N

Some properties of the submodality N are examined here.

Lemma2) The submodality $N \subseteq L$ satisfies distribution axiom, i.e., $N(p \supset q) \supset (Np \supset Nq)$.

pr) $N(p \supset q) \equiv \forall w \in W\ (R_N(W_0,w) \supset T(w, IMP(p,q))$

$$\equiv \forall w \in W\ (R_N(W_0,w) \supset (T(w,p) \supset T(w,q))$$

$$\supset \forall w \in W\ (R_N(W_0,w) \supset T(w,p) \supset \forall w \in W\ (R_N(W_0,w) \supset T(w,q))$$

$$\equiv Np \supset Nq \qquad \text{Q.E.D.}$$

Cor.1) The submodality $N \subseteq L$ satisfies normal system K.

Cor.2) If there is a possible world which is accessible from the actual world W_0 through the accessibility relation R_N, then submodality $N \subseteq L$ satisfies KD.

pr) $\neg N(F) \equiv \neg \forall w \in W\ (R_N(W_0,w) \supset T(w,F))$

$$\equiv \exists w \in W\ \neg(R_N(W_0,w) \supset T(w,F))$$

$$\equiv \exists w \in W\ (R_N(W_0,w) \wedge \neg T(w,F))$$

$$\equiv \exists w \in W\ (R_N(W_0,w) \wedge T(w,NOT(F)))$$

$$\equiv \exists w \in W\ (R_N(W_0,w) \wedge T(w,T))$$

$$\equiv \exists w \in W\ R_N(W_0,w)$$

And, since N satisfies distribution axiom, N satisfies KD. Q.E.D.

Cor.3) If Mp is true in all the possible worlds and the submodality $N \subseteq L$ satisfies $Mp \equiv Np$ in all the possible worlds, then N satisfies KD.

pr) Let N be the submodalitu such that $Mp \equiv Np$ for all possible worlds.

Then,

$T(w, \neg N(F))$

$$\equiv \neg \forall w \in W\ (R_N(w,w_1) \supset T(w_1,F))$$

$$\equiv \exists w \in W\ \neg(R_N(w,w_1) \supset T(w_1,F))$$

$$\equiv \exists w \in W\ (R_N(w,w_1) \wedge \neg T(w_1,F))$$

$$\equiv \exists w \in W\ (R_N(w,w_1) \wedge T(w_1,NOT(F)))$$

$$\equiv \exists w \in W\ (R_N(w,w_1) \wedge T(w_1,T))$$

$$\equiv \exists w \in W\ R_N(w,w_1)$$

Since $T(w,Mp) \equiv \exists w \in W\ (R_L(w,w_1) \wedge T(w_1,p))$ and $\forall w \in W\ R_N(w,w_1) \equiv R_L(w,w_1) \wedge T(w_1,p)$, $T(w,Mp) \equiv \exists w \in W\ R_N(w,w_1)$. And, since $T(w,Mp) \equiv T$, $T(w,\neg N(F)) \equiv T$. Thus, N satisfies KD. Q.E.D.

Cor.4) Followings hold for any submodality $N \subseteq L$;
$p \supset q$ infers $Np \supset Nq$, and
$N(p \wedge q) \equiv Np \wedge Nq$.

Lemma3) $Mp \equiv Np$ for some $N \subseteq L$.

pr) $Mp \equiv \exists N \subseteq L\ \forall w \in W\ (R_N(W_0,w) \supset T(w,p))$
$\equiv \forall w \in W\ (R_N(W_0,w) \supset T(w,p))$ for some $N \subseteq L$
$\equiv Np$ for some N Q.E.D.

Lemma4) $p \supset Np$ for some $N \subseteq L$.

pr) $p \supset Mp$
$\equiv Np$ for some $N \subseteq L$. Q.E.D.

Lemma5) $Lp \supset Np$ for any $N \subseteq L$.

pr) $Lp \equiv \forall w \in W\ (R_L(W_0,w) \supset T(w,p))$
$\supset \forall w \in W\ (R_N(W_0,w) \supset T(w,p))$ for any $N \subseteq L$
$\equiv Np$ for any $N \subseteq L$. Q.E.D.

Lemma6) $N_1(p \supset q) \wedge N_2 p \supset N_3 q$ where $R_{N3}(W_0,w) \equiv R_{N1}(W_0,w) \wedge R_{N2}(W_0,w)$ for all $w \in W$.

pr) $N_1(p \supset q) \wedge N_2 p$
$\equiv \forall w \in W\ (R_{N1}(W_0,w) \supset T(w,p \supset q)) \wedge \forall w \in W\ (R_{N2}(W_0,w) \supset T(w,p))$
$\supset \forall w \in W\ (R_{N1}(W_0,w) \wedge R_{N2}(W_0,w) \supset T(w,p \supset q) \wedge T(w,p))$
$\equiv \forall w \in W\ (R_{N3}(W_0,w) \supset T(w,p \supset q) \wedge T(w,p))$
$\equiv \forall w \in W\ (R_{N3}(W_0,w) \supset T(w,(p \supset q) \wedge p))$
$\equiv \forall w \in W\ (R_{N3}(W_0,w) \supset T(w,(\neg p \vee q) \wedge T(w,p))$
$\equiv \forall w \in W\ (R_{N3}(W_0,w) \supset (T(w,\neg p) \vee T(w,q)) \wedge T(w,p))$
$\equiv \forall w \in W\ (R_{N3}(W_0,w) \supset T(w,q)) \wedge T(w,p)$
$\supset \forall w \in W\ (R_{N3}(W_0,w) \supset T(w,q))$
$\equiv N_3 q$ Q.E.D.

This is a generalization of Lemma 2.

Lemma7) $N_1 p \supset N_2 p$ if $N_2 \subseteq N_1$.

pr) $N_1 p \equiv \forall w \in W\ (R_{N1}(W_0,w) \supset T(w,p))$
$\equiv \forall w \in W\ ((R_{N2}(W_0,w) \supset R_{N1}(W_0,w)) \wedge (R_{N1}(W_0,w) \supset T(w,p)))$
since $\forall w \in W\ (R_{N2}(W_0,w) \supset R_{N1}(W_0,w))$
$\supset \forall w \in W\ (R_{N2}(W_0,w) \supset T(w,p))$
$\equiv N_2 p$. Q.E.D.

Lemma8) $N_1 N_2 p \supset N_3 p$ where $R_{N3}(W_0,w_2) \equiv R_{N1}(W_0,w_1) \wedge R_{N2}(w_1,w_2)$ for all w_1.

pr) $N_1 N_2 p \equiv \forall w_1 \in W\ (R_{N1}(W_0,w_1) \supset T(w_1,N_2 p))$
$\equiv \forall w_1 \in W\ (R_{N1}(W_0,w_1) \supset T(w_1, \forall w_2 \in W\ (R_{N2}(w_1,w_2) \supset T(w_2,p))$
$\equiv \forall w_1 \in W\ (R_{N1}(W_0,w_1) \supset \forall w_2 \in W\ T(w_1,(R_{N2}(w_1,w_2) \supset T(w_2,p))$
$\equiv \forall w_1,w_2 \in W\ (R_{N1}(W_0,w_1) \supset T(w_1,(R_{N2}(w_1,w_2) \supset T(w_2,p))$
$\equiv \forall w_1,w_2 \in W\ (R_{N1}(W_0,w_1) \supset (R_{N2}(w_1,w_2) \supset T(w_2,p)))$
$\equiv \forall w_1,w_2 \in W\ (R_{N1}(W_0,w_1) \wedge R_{N2}(w_1,w_2) \supset T(w_2,p))$
$\supset \forall w \in W\ (R_{N3}(W_0,w) \supset T(w,p))$
where $R_{N3}(W_0,w_2) \equiv R_{N1}(W_0,w_1) \wedge R_{N2}(w_1,w_2)$ for all w_1 and w_2.
Note that $R_{N3} \subset R_L$.
$\equiv N_3 p$. Q.E.D.

By denoting N_3 by $N_{1 \cdot 2}$, Lemma 8 can be represented by $N_1 N_2 p \supset N_{1 \cdot 2} p$. Note that if $N_1 \subseteq L$ and $N_2 \subseteq L$, then $N_{1 \cdot 2} \subseteq L$ since R_L is transitive.

Lemma9) $NNp \supset Np$.

Lemma10) $N_1 p \wedge N_2 q \supset N_3(p \wedge q)$ where $R_{N3}(W_0,w) \equiv R_{N1}(W_0,w) \wedge R_{N2}(W_0,w)$ for all $w \in W$.

pr) $N_1 p \wedge N_2 q \equiv \forall w \in W\ (R_{N1}(W_0,w) \supset T(w,p)) \wedge \forall w \in W\ (R_{N2}(W_0,w) \supset T(w,q))$
$\equiv \forall w \in W\ (R_{N1}(W_0,w) \supset T(w,p)) \wedge (R_{N2}(W_0,w) \supset T(w,q))$
$\supset \forall w \in W\ (R_{N1}(W_0,w) \wedge R_{N2}(W_0,w) \supset T(w,p) \wedge T(w,q))$
$\equiv \forall w \in W\ (R_{N3}(W_0,w) \supset T(w,p) \wedge T(w,q))$
$\equiv \forall w \in W\ (R_{N3}(W_0,w) \supset T(w,AND(p,q))$
$\equiv N_3(p \wedge q)$. Q.E.D.

By denoting N_3 by $N_{1\&2}$, the Lemma 10 can be represented by $N_1 p \wedge N_2 q \supset N_{1\&2}(p \wedge q)$.

4. Reasoning with Beliefs

(1) Coexistence of knowledges and beliefs

Before proving the distribution axiom of belief, it's better to examine the distribution axiom of belief over conjunction.

Th.1) Distribution axiom of knowledge and belief over conjunction, i.e., $Kp \wedge Bq \supset B(p \wedge q)$ holds.

pr) $Kp \wedge Bq \equiv Kp \wedge K(N_1 Kq \wedge Jq)$ for some submodality $N_1 \subseteq L$

$\supset K(N_2 Kp \wedge Jp) \wedge K(N_1 Kq \wedge Jq)$ for some submodality $N_2 \subseteq L$

$\equiv KN_2 Kp \wedge KN_1 Kq \wedge KJ(p \wedge q)$

$\supset KN_{1\&2} K(p \wedge q) \wedge KJ(p \wedge q)$ by Lemma10

$\equiv B(p \wedge q)$

In the case of $N_{1\&2} = \varnothing$, it means that Bq causes contradiction. However, there can be possible worlds where Kq and all the knowledges including Kp coexist because the model of all the knowledges have a possible world where Kq exists. Thus, $N_{1\&2} \neq \varnothing$, and $Kp \wedge Bq \supset B(p \wedge q)$ holds. Q.E.D.

Fig.1 illustrates Th.1 in a manner of possible worlds.

Th.2) The distribution axiom of beliefs over conjunction, i.e., $Bp \wedge Bq \equiv B(p \wedge q)$ holds.

pr) Any belief Bp can be represented as $K(NKp \wedge Jp)$ for the same submodality $N \subseteq L$. So we obtain

$Bp \wedge Bq \equiv K(NKp \wedge Jp) \wedge K(NKq \wedge Jq)$

$\equiv K(NKp \wedge NKq) \wedge K(Jp \wedge Jq)$

$\equiv KN(Kp \wedge Kq) \wedge K(Jp \wedge Jq)$

$\equiv KNK(p \wedge q) \wedge KJ(p \wedge q)$ by Cor.4

$\equiv B(p \wedge q)$. Q.E.D.

(2) Reasoning with knowledges and beliefs

We want to reason with beliefs as well as knowledges. It's time to prove the distribution axiom of belief by distribution axiom of belief over conjunction.

Th.3) Distribution axiom of knowledge and belief, i.e., $Kp \wedge B(p \supset q) \supset Bq$, holds.

pr) $Kp \wedge B(p \supset q) \supset B(p \wedge (p \supset q))$ by Lemma2

$\supset Bq$ by Th.12 in (6) Q.E.D.

Similarly we obtain the following theorem.

Th.4) Distribution axiom of knowledge and belief, i.e., $Bp \wedge K(p \supset q) \supset Bq$ holds.

Th.5) Distribution axiom of beliefs, i.e., $Bp \wedge B(p \supset q) \supset Bq$ holds.

pr) $Bp \wedge B(p \supset q) \equiv B(p \wedge (p \supset q))$ by Th.2

$\supset Bq$ by Th.12 in (6) Q.E.D.

Thus any new belief Bq, derived from current beliefs and knowledges by deduction, can be also represented as $K(NKq \wedge Jq)$ for the same submodality $N \subseteq L$. So, by the mathematical induction, we can prove that all the beliefs derived from two of the current beliefs and knowledges by deduction are represented as $K(NKq \wedge Jq)$ for the same submodality $N \subseteq L$ for which new belief $Bp_0 \equiv K(NKp_0 \wedge Jp_0)$ is introduced by Possibilitation Rule.

(3) Belief as a justified and consistent knowledge

Thus, once a belief is introduced, one can reason with this belief as well as previous knowledges and also can reason with beliefs derived from above reasonings.

Another way to get new belief is to get it as a consistent and justified knwledge, i.e., Bp as $K(NBp \wedge Jp)$ for some submodality $N \subseteq L$. In possible worlds semantics, new belief is infered in the current possible worlds in the same way as in the actual world. New belief, say Bq, introduced as a justified and consistent knowledge, can be written as $K(NBq \wedge Jp)$ for some submodality $N \subseteq L$. Thus, we obtain next theorem.

Th.6) All the beliefs can be represented as $K(NBq \wedge Jp)$ for some submodality $N \subseteq L$.

The set of beliefs $TH^N(A)$ is the intersection of all the sets of propositions which are true in a possible world which contains Bp_N, that is, $TH^N(A)$ is the smallest set of proposi-

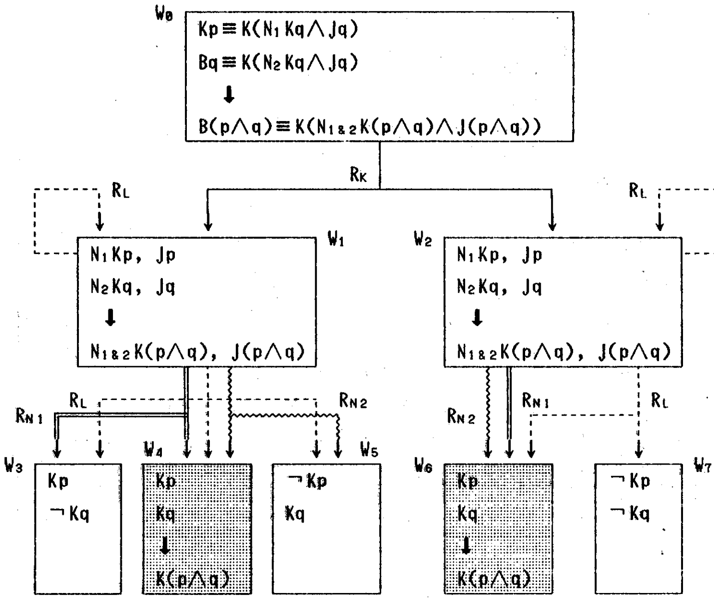


Fig.1 Distribution axiom of knowledge and belief over conjunction

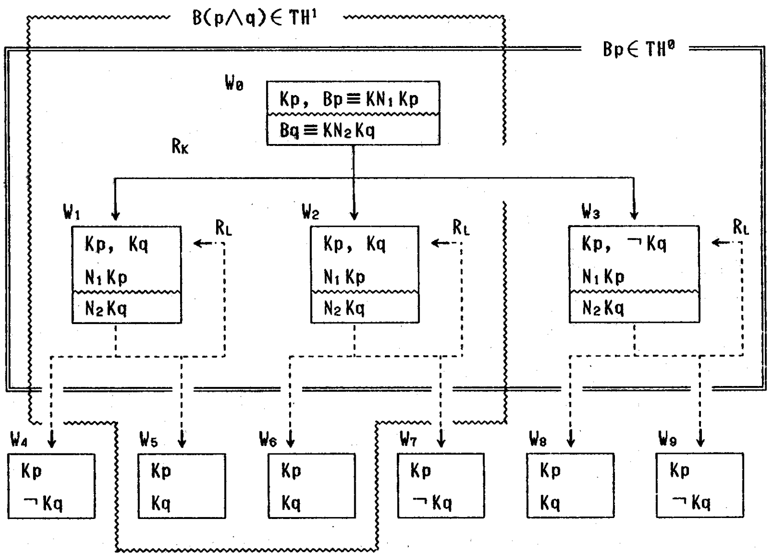


Fig.2 Possible world view of the hierarchy of belief set TH^0 and TH^1

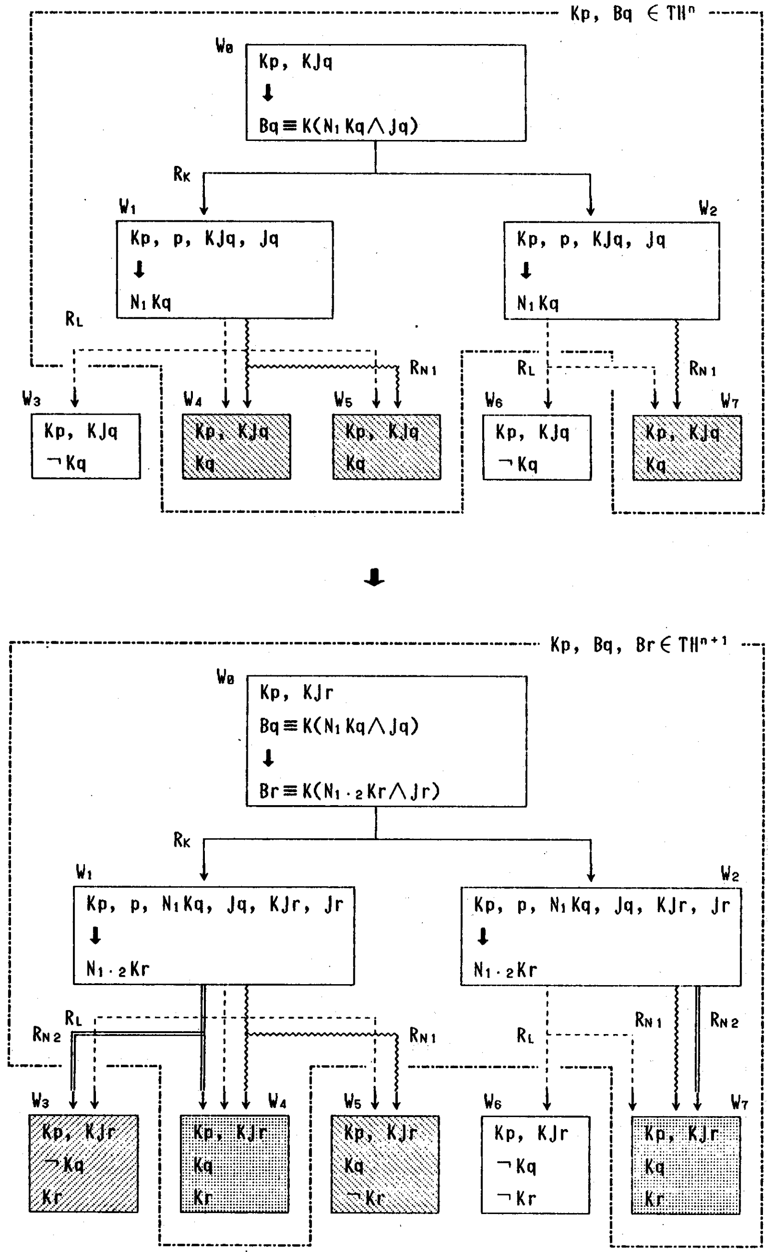


Fig.3 Possible world view of the hierarchy of belief set TH^n and TH^{n+1}

tions which are true in a possible world which contains Bp_N and the smallest closure of the propositions including all the axioms and Bp_N under the formal system except rule of possibilitation (Fig.2,3).

5. Consistency of the System

In the previous paper[6], we examined properties of belief derived in our system, and gave $B\neg p \supset \neg Bp$ as a theorem without proof. Although it is intuitively appropriate, it may not hold in a formal system based on nonmonotonic or modal logic since $M\neg p \supset \neg Mp$ does not hold in modal logic. Thus it is a characteristic of our formalization. Now, we can give its proof.

Th.7) Consistency with respect to Bp and $B\neg p$, i.e., $\neg(Bp \wedge B\neg p)$.

pr) By Th.6 we can assume that any belief Bp can be written as $K(NKp \wedge Jp)$ for the same submodality N. So we have

$$\begin{aligned} &\neg(B\neg p \wedge Bp) \\ &\equiv \neg(KNK\neg p \wedge KJ\neg p \wedge KNKp \wedge KJp) \\ &\equiv \neg K(NK\neg p \wedge J\neg p \wedge NKp \wedge Jp) \\ &\equiv \neg K(NK(\neg p \wedge p) \wedge J(\neg p \wedge p)) \\ &\equiv \neg K(NK(F) \wedge J(F)) \\ &\equiv \neg K(N(F) \wedge J(F)) \\ &\equiv \neg K(F \wedge J(F)) \\ &\equiv \neg K(F) \\ &\equiv \neg F \\ &\equiv T \end{aligned}$$

Q.E.D.

In our logic, the law of exclusive middle, $Bp \vee B\neg p$ or $Bp \vee B\neg p$, does not hold, of course. However, as shown above, the weak consequence, the law of contradiction, $\neg(Bp \wedge B\neg p)$, or equivalently, $\neg Bp \vee \neg B\neg p$ holds.

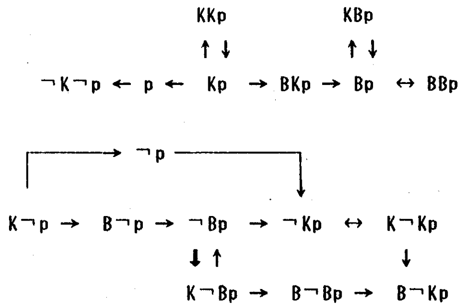
Cor.5) Belief of negation implies negation of belief, i.e., $B\neg p \supset \neg Bp$.

pr) $(B\neg p \supset \neg Bp) \equiv \neg B\neg p \vee \neg Bp$

$$\begin{aligned} &\equiv \neg(B\neg p \wedge Bp) \\ &\equiv T \quad \text{(by Th.2)} \end{aligned}$$

Q.E.D.

Thus, we can summarize the relation between knowledges and beliefs as follows[6]:



Remark) → means the implication which holds in S4 and S5. ➡ means the implication which holds only in S5. Note that proof of $BBp \supset Bp$ is shown in neither of the previous or current paper, however, it is easily shown.

6. Conclusion

In the previous paper[6], a formal system based on modal logic was presented for knowledge and belief. The belief was defined as a justified and consistent knowledge. That is, belief is an information ofwhich consistency with other knowledges and beliefs of the agent and justification suggesting its validity he knows. In that system we can represent and reason knowledges and beliefs of many agents and their nested beliefs including true facts which are not known or believed by agents. Several properties of beliefs, and relations between beliefs and knowledges, which were derived in the system, seem intuitively appropriate.

In this paper, by introducing the concept of submodality of modality L which represents provability, the concept of consistency among knowledges and beliefs can be made precise and some important properties of the system, including the distribution axiom of beliefs, i.e., $Bp \wedge B(p \supset q) \supset Bq$, have been proved. Thus, we can reason with beliefs as well as knowledges in our system. Thus, the theoretical foundations of reasoning with beliefs have been presented.

References:

(1)B.Chellas:Modal Logic:An Introduction, Cambridge University Press (1980)

(2)M.Genesereth and N.Nilsson:Logical Foundations of Artificial Intelligence, Morgan Kaufmann Pub.(1987)

(3)G.Hughes and M.Cresswell:An Introduction to Modal Logic (1968)

(4)D.Israel,K.Konolige,M.Pollack,M.Stickel,and S.Matsuda:Theoretical Foundations of User Interfaces, Internal Report(1990)

(5)W.Lukaszewicz:Two results on default logic, IJCAI-85 pp.459-461(1985)

(6)S.Matsuda:Belief as a justified and consistent knowledge, Information Processing Society of Japan, 90-AI-72-4 (1990)

(7)D.McDermott:Nonmonotonic logic II: nonmonotonic modal theories, JACM 29,1 pp.33-57 (1982)

(8)D.McDermott and Doyle:Non-monotonic logic I, Artif.Intell.13,1,2 pp.41-72 (1980)

(9)R.Moore:Semantical considerations on nonmonotonic logic, Artif.Intell. 25,1 pp.75-94 (1985)

(10)R.Moore:A formal theory of knowledge and action, in "Formal Theories of the Commonsense World", pp.319-358, Ablex Pub. (1985)

(11)R.Reiter:On reasoning by default, Proc.2nd Symp.on TINLAP (1978)