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金融ネットワークのダイナミックス

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概要

この研究の目的は，多くの経済主体が相互作用する金融ネットワークにおける新しい期待形成仮説，「相対的期待形成仮説」を提案することである．相対的期待形成仮説はケインズの美人投票仮説の数学的定式化であると解釈することができる．この期待形成仮説から経済主体の投資態度分布として，ボルツマン分布が導出されることが示される．

キーワード: 相対的期待形成，ケインズの美人投票，ボルツマン分布

Dynamics in a Financial Network

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Abstract

The aim of this study is to propose a new expectation formation hypothesis, the *relative expectation formation hypothesis*, and demonstrate that a standard probabilistic distribution on the trader's opinion, which is called the *Bolztmann-Gibbs distributioni*, is able to be derived from the relative expectation formation hypothesis. The relative expectation formation of interacting agents will be formularized by using *maximum information entropy principle* and is considered as a mathematical formularization of *Keynes' beauty contest.*

Key words : the relative expectation formation, Keynes' beauty contest, and the Boltzmann distribution.

1 Introduction

The Markov switching models of interacting agents have lately attracted considerable attention as an alternative approach to the efficient market approach to finance. A number of studies have already been made on this literature (see Aoki (1994, 1996), Kirman (1993), Lux (1995, 1997, 1998, 1999), Kaizoji (1998, 1999a, 1999b) and the others). Although the interacting agents models are advocated as an alternative approach to the efficient market hypothesis (or rational expectation hypothesis), little attention has been given to the point how probabilistic rules, that an agent switches their opinion, is connected with expectation formations. The aim of this study is to propose a new expectation formation hypothesis, that is, the *relative expectation formation hypothesis*, and demonstrate that a standard probabilistic distribution on the trader's opinion, which is called the *Bolztmann-Gibbs distributioni*, is able to be derived from the relative expectation formation hypothesis. The relative expectation formation of interacting agents will be formularized by using *maximum information entropy principle* and is considered as a mathematical formularization of *Keynes' beauty contest*. In other words, we will reconstruct the master equation approach to speculative activity advocated by Kirman (1993) and Lux (1995) using the information theoretic approach. Therefore the information theoretic approach will make the difference between the master equation approach and the traditional efficient market approach clear from the standpoint of the expectation formation.

Then we consider that the learning algorithm that agents memorize for sequence of the patters of price changes in the model, the so-called *associative memory*. The associative memory may help account for the reason why the similar patterns of the price changes are repeated in financial markets.

2 Information Theoretic Approach to Speculative Activity of Interacting Agents

We think of the financial market that large numbers of traders participate in trading. It is assumed that the total number of traders is $2N$, is constant. Traders are indexed by $j = 1, 2, ..., 2N$. The traders are supposed to determine the mode of holding their wealth by choosing between alternative assets: risky assets (shares or foreign exchange) and a riskless asset (money). Hence, a trader have an investment attitude to the risky asset, the buyer or the seller. x_i denotes the investment attitude of trader i at time t. The investment attitude x_i is defined as follows: if trader i is the buyer of the risky asset at a time, then $x_{it} = +1$. If trader i, in contrast, is the seller of the risky asset at a time, then $x_i = -1$.

2.1 Relative expectations formation hypothesis

Every trader probably has an idea of the price of risky asset which he expects to prevail in the future. Hence, he expects a certain exchange profit through trading. In speculative market the price changes are subject to the law of demand and supply, that the price rises when there is excess demand, and vice versa. Thus a trader will predict the other traders' behavior, and will choose the same attitude as the other behavior as thoroughly as possible he could. Let us assume that trader i tries to minimize the following evaluation function $E_i(x)$,

$$E_i(x) = \frac{1}{2}\sum_{j=1}^{N} \alpha_{ij}(x_i - x_j)^2 - \beta_i(s^* - s(x))x_i. \quad (1)$$

where α_{ij} denotes the connection weight from trader j to trader i, and β_i denotes the strength of the reaction of trader i upon the difference between the fundamental price s^* and the market price $s(x)$, and x denotes the vector of investment attitude $x = (x_1, x_2, ...x_N)$. We assume that the market price depends upon the investment attitude of all traders, x.

The optimization problem that the market should solve is formalized by

$$\min E(x) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}[\alpha_{ij}(x_i - x_j)^2 - \beta_i(s^* - s(x))x_i]. \quad (2)$$

Assume that trader's decision making is subject to probabilistic rule. Let us introduce a random variable $x^k = (x_1^k, x_2^k, ..., x_N^k)$, $k = 1, 2, ..., K$. Let the state x^k occur with probability

$$P(x^k) = \text{Prob}(x^k) \tag{3}$$

with the requirement $0 < P(x^k) < 1$ and $\sum_{k=1}^{K} P(x^k) = 1$. We define the amount of uncertainty before the occurrence of the state x^k with probability $P(x^k)$ as the logarithmic function:

$$I(x^k) = -\log P(x^k). \tag{4}$$

Formally the optimization problem that the market should solve is formalized by

$$\min\langle E(x)\rangle = \sum_{k=1}^{N} P(x^k)E(x^k) \tag{5}$$

subject to

$$H = -\sum_{k=1}^{N} P(x^k)\log P(x^k), \qquad \sum_{k=-N}^{N} P(x^k) = 1. \tag{6}$$

where

$$E(x^k) = \frac{1}{2}\sum_{i=1}^{N} E_i(x^k) \tag{7}$$

x^k is a state, and H is information entropy. $P(x^k)$ is the relative frequency the occurrence of the state x^k.

We call the traders' optimizing behavior a *relative expectation formation.*

The solution of the above optimization problem is

$$P(x^k) = \frac{1}{Z}\exp(-\mu E(x^k)), \quad k = 1, 2, ..., K \tag{8}$$

$$Z = \sum_{k=1}^{K} \exp(-\mu E(x^k)). \tag{9}$$

$$\langle E(x)\rangle = -\frac{\partial}{\partial\mu}\log Z \tag{10}$$

$$\langle E^2(x)\rangle = \frac{1}{Z}\frac{\partial^2}{\partial\mu^2}Z \tag{11}$$

$$\text{Var}.E = \langle E^2(x)\rangle - \langle E(x)\rangle^2 = \frac{\partial^2}{\partial\mu^2}\log Z \tag{12}$$

$$H_{\max} = \log Z + \mu\langle E(x)\rangle \tag{13}$$

where μ denotes the Lagrangean. The probability distribution $P(x^k)$ is called the *Boltzmann distribution* where $P(x^k)$ is the probability that the network of agents is in the state k with the evaluation function $E(x^k)$, and Z is the partition function.

2.2 Master Equations of $P(x)_t$

Now define a set of transition probabilities $W(x \to x')$ from a state x into another state x'. The master equation is

$$P(x)_t - P(x)_t = \sum_{x \neq x'} W(x' \to x)P(x)_t - \sum_{x \neq x'} W(x \to x')P(x')_t. \tag{14}$$

What is the condition on $W(X \to x')$ so that the system may reach and then remain in the expectational equilibrium ? A necessary condition for maintaining equilibrium is that the average number of transitions from x to x' and from x' to x be equal:

$$W(x' \to x)P(x)_t = W(x \to x')P(x')_t, \tag{15}$$

or, by dividing by $W(x \to x')$ and assuming the Boltzmann distribution (8)

$$\frac{W(x \to x')}{W(x' \to x)} = \frac{P(x)_t}{P(x')_t} = \frac{\mu\exp(-E(x))}{\mu\exp(-E(x'))} = \exp(-\mu\Delta E), \tag{16}$$

where $\Delta E = E(x) - E(x') =$. If $\alpha_{ij} = \alpha_{ji}$ and $\alpha_{ii} = 0$, then the resulting transition probability

$$W(x_i \to -x_i) = \frac{\exp(\mu\Delta E_i(x))}{\mu\exp(\Delta E_i(x)) + \exp(-\mu\Delta E_i(x))}, \tag{17}$$

where $\Delta E_i(x) = \sum_{i=1}^{N} \alpha_{ij}x_i + \beta_i(s^* - s(x))$.

There are many possible choices of the transition probabilities which could have been made in Equation (17), but the choice of the sigmoidal function like Equation (17) is motivated by statistical mechanics (Glauber (1963), Amari (1972), Little (1974), and Hinton and Sejnowski (1983)).

2.3 The volume of investment

The trading volume should depends upon the attitudes of all traders. The aggregate excess demand at time t is given by

$$\text{Excess Demand} = g(x). \quad (18)$$

where $g(\cdot)$ denotes the excess demand function.

2.4 The price adjustment process

There is a market-maker in the market, and he/she compares the buying and selling orders by traders, and executes trading. If the aggregate demand for the risky asset at time t exceeds the aggregate supply of the risky asset at time t, then the market-maker raises the price at time t, and vice versa. Hence, an adjustment process of the price can be described as follows,

$$s_t - s_{t-1} = \lambda g(x_t). \quad (19)$$

where the parameter, λ represent the flexibility of the market price change. The necessary condition for the equilibrium prices is to satisfy the following

$$g(x) = 0. \quad (20)$$

The mean value of the price is

$$\langle \Delta s \rangle = \sum_{k=1}^{N} P(x^k) g(x^k) \quad (21)$$

where $P(x^k)$ is the stationary distribution (8).

3 Mean-Field Approximation

It was convincingly demonstrated that the two-step probabilistic Markov process in the agents' network can be replaced by a deterministic equations in the so-called mean field theory approximation. The mean field theory approximation is a well known technique in physics, particularly for spin-systems (Glauber, (1963)). extensive studies of the applicability of this approximation and refinements thereof has been made for spin-glass system (Mezard, Parisi, and Virasoro (1987), and which are closely related to bidirectional neural network model (Amit, et. al. (1985)). Here we briefly list the key equations. The summation over all possible agent configurations $x = (x_1, ..., x_N)$ is computationally explosive with problem size. Let us replace this discrete sum with the mean field variable

$$\langle x \rangle = \langle \frac{1}{N} \sum_{i=1}^{N} x_i \rangle. \quad (22)$$

If $\alpha_{ij} = \alpha$ and $\beta_i = \beta, (i = 1, ..., N)$, then the evaluation function (2), that trader try to minimizes, is approximated by

$$E(x) \approx - \sum_{i=1}^{N} \sum_{j=1}^{N} [\alpha N \langle x \rangle x_i - \beta (s^* - \langle s \rangle) x_i)] \quad (23)$$

The first term of the right side of Equation (23) is interpreted as a mathematical formularization of *Keynes beauty contest*, because the first term represents that a trader tries to takes the same investment attitude as prediction of the average opinion. The second term of the right side represents the arbitrage trading because traders would buy (sell short) an underpriced (overpriced) asset, driving its price back to the fundamental value.

The mean field $\langle x \rangle$ is given by

$$\langle x \rangle = \tanh(\mu \alpha N \langle x \rangle + \mu \beta (s^* - s(\langle x \rangle)) \quad (24)$$

which represent steady state solutions. A straightforward iteration of Equation (24) gives

$$\langle x \rangle_{t+1} = \tanh(\mu \alpha N \langle x \rangle_t + \mu \beta (s^* - \langle s \rangle_t)). \quad (25)$$

Next, let us approximate the adjustment process of the exchange rate using $\langle x \rangle$.

$$\langle \Delta s \rangle_t = \lambda g(N \langle x \rangle). \quad (26)$$

We assume that $g(\cdot)$ is linear function with respect to $\langle x \rangle$. Then we get

$$\langle \Delta s \rangle_t = \lambda N \langle x \rangle \quad (27)$$

and

$$\langle x \rangle_{t+1} = \tanh(\mu \alpha N \langle x \rangle_t + \mu \beta (s^* - \langle s \rangle_t)). \quad (28)$$

where $\langle \Delta s \rangle_t = \langle s \rangle_t - \langle s \rangle_{t-1}$.

3.1 Speculative Dynamics

There is the formal equivalence between the stochastic models of speculative activity and the spin-glass models in statistical mechanics. The striking similarity between the behavior of spin-glass models and the above interacting agents models will lead to the discovery of important properties of the interacting agents model. The most important difference between the two is that the interacting agents models has a self-feedback from the market exchange rate. The investment attitude of traders changes the market exchange rate and a change of the exchange rate has great influence on the investment attitude. The interacting agents model is a self-organizing system.

The local stability conditions for the equilibrium $(\langle x \rangle_t = 0, \langle s \rangle_t = s^*)$ are as follows

$$-1 + \frac{\beta \lambda N}{2} < \alpha < 1, \quad \beta > 0$$

The figure illustrates that the dynamics of the mean values are complex. Let us illustrate the dynamics of the model via numerical simulation. We will show the dynamics in two case.

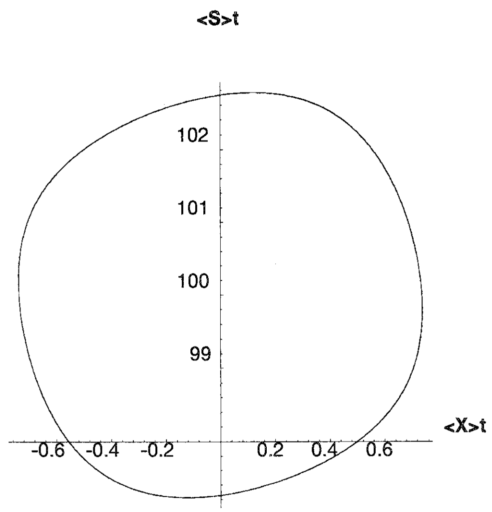

Figure 1: The qusi-periodic attractor for $(\alpha, \beta) = (1.2, 0.1)$

Figure 1 shows an attractor in the $(\langle x \rangle_t, \langle s \rangle_t)$ plane with $(\alpha, \beta) = (1.2, 0.1)$. In the figure, the orbits converges to attracting invariant 'circle' created in the Hopf bifurcation. If the coefficient α is above 1 in the case with $\beta = 0.1$, then a Hopf bifurcation occurs at the fundamental equilibrium, and the orbits converges to attracting invariant 'square' created in the Hopf bifurcation.

Figure 2 shows an attractor in the $(\langle x \rangle_t, \langle s \rangle_t)$ plane with $(\alpha, \beta) = (1.9, 6.0)$. The figure shows the occurence of strange attractor[1].

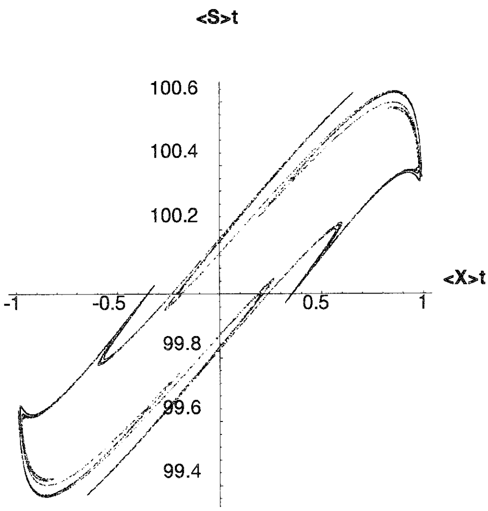

Figure 2: The chaotic attractor for $(\alpha, \beta) = (1.9, 6.0)$.

4 Associative Memory

In the proceeding section we show that the variety of dynamics appear corresponding to the different values of the parameters. These results will lead us further into a consideration of learning. In this section we will consider how the agents decide the parameters α and β. It seems reasonable to suppose that the past patterns of the price changes are stored in memory through a learning process by agents, and the stored patters is recalled later

[1] For further details of dynamical properties in the model see Kaizoji (1999b).

when a key pattern appears in the market. This kind of memory is called *associative memory.*

4.1 The gradient-descent-based learning rule

Here we use a gradient-descent-based learning rule as the learning algorithm for storage of the patterns[2]. Inserting (28) into (27), we get

$$F(\langle \Delta s\rangle_t) = \lambda N \tanh(\bar{\lambda}\langle \Delta s\rangle_t + \bar{\beta}(s^* - \langle s\rangle_t)), \quad \bar{\alpha} = \frac{\mu\alpha}{\lambda}, \quad \bar{\beta} = \mu\beta. \tag{29}$$

The gradient-descent algorithm is a stable and robust procedure for minimizing the following one-step-prediction error function

$$E(\bar{\alpha}_l, \bar{\beta}_l) = \frac{1}{2}\sum_{t=1}^{n}[\langle \Delta s\rangle_t - F(\langle \Delta s\rangle_{t-1})]^2. \tag{30}$$

More specifically, the gradient-descent algorithm changes the parameter vector $(\bar{\alpha}_l, \bar{\beta}_l)$ to satisfy the following condition :

$$\Delta E(\bar{\alpha}_l, \bar{\beta}_l) = \frac{\partial E(\bar{\alpha}_l, \bar{\beta}_l)}{\partial \bar{\alpha}_l}\Delta\bar{\alpha}_l + \frac{\partial E(\bar{\alpha}_l, \bar{\beta}_l)}{\partial \bar{\beta}_l}\Delta\bar{\beta}_l < 0, \tag{31}$$

where $\Delta E(\bar{\alpha}_l, \bar{\beta}_l) = E(\bar{\alpha}_l, \bar{\beta}_l) - E(\bar{\alpha}_{l-1}, \bar{\beta}_{l-1})$, $\Delta\bar{\alpha}_l = \bar{\alpha}_l - \bar{\alpha}_{l-1}$, and $\Delta\bar{\beta}_l = \bar{\beta}_l - \bar{\beta}_{l-1}$. To accomplish this the gradient-descent algorithm adjusts each parameters $\bar{\alpha}_l$ and $\bar{\beta}_l$ by amounts $\Delta\bar{\alpha}_l$ and $\Delta\bar{\beta}_l$ proportional to the negative of the gradient of $E(\bar{\alpha}_l, \bar{\beta}_l)$ at the current location:

$$\bar{\alpha}_{l+1} = \bar{\alpha}_l - \eta\frac{\partial E(\bar{\alpha}_l, \bar{\beta}_l)}{\partial \bar{\alpha}_l}, \quad \bar{\beta}_{l+1} = \bar{\beta}_l - \eta\frac{\partial E(\bar{\alpha}_l, \bar{\beta}_l)}{\partial \bar{\beta}_l} \tag{32}$$

where η is a learning rate.

When a stored pattern of price changes as a initial state is observed in the stock market, the financial network responds by producing the sequence of the stored patterns. Hence, the recall is through association of the observed pattern of price changes with the information memorized into the financial network. In financial market the similar patterns have been often observed in different time. Why does history repeat itself in financial markets ? The associative memory may help account for the reason.

[2]The gradient-descent method is often used for training multilayer feedforward networks. It is called the ***back-propagation learning algorithm*** which is one of the most important historical developments in neural networks (Rumelhart et al. (1986)). Another example of a supervised learning algorithm is the Boltzmann machin (Acley, et. al. (1985) and Peterson and Anderson (1987)).

5 Conclusion

This work presents a markov switching model of interacting agents. I would now like to go on to develop the model by extending my investigations to the following directions, (1) the models with heterogeneous agents, (2) the models in the financial market that many securities are listed.

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