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Hamilton Lie 代数のファイバー束を用いた曲面モデル

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概要:　本文は、線形リー代数の１－パラメータ群のファイバー束を用いた曲線モデルを拡張して、ハーミルトンリー代数及び一般的な高次線形常微分方程式の解となる１－パラメータ群のファイバ束を用いた曲面モデルを示している。また、本モデルのユークリッド変換に対する完全不変量を求め、曲面の合成と特徴量抽出方式について、考察している。本モデルは、有限個の完全不変量によって形状が一意に合成或いは復元することができるため、形状合成や認識合成符号化に有利と考えられる。これらの不変量は、画像検索や著作権保護へも応用可能である。また、形状は初等関数で表されるため、合成する際の数値誤差がないという特徴を有する。 .

On Surface Model Based on a Fibre Bundle of 1-Parameter Group of Hamiltonian Lie algebra

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Abstract:　In this paper, the surface model based on 1-parameter groups of linear Lie algebra is extended using 1-parameter groups of Hamilton Lie algebra and high order linear ODE. The complete invariant set of the model under action of Euclidean motion is obtained. Algorithms are also shown on shape synthesis using the proposed model and extraction of invariants from 3D objects. The surface represented by this model is uniquely determined by a finite number of the complete invariants. These invariants can be used in recognition- synthesis encoding of 3D images, image retrieving and copyright protection also. Moreover, the surface can be calculated by elementary functions then shape synthesis is free of numerically integral errors.

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1 Introduction

In [4][1], a fibre bundle model for free surface is presented, furthurmore, a surface model based on 1-parameter groups of linear Lie algebra is proposed. Main features of this model include that it possesses a complete set of invariants with respect to Euclidean motion. Thus, the surface represented by this model is uniquely determined by a finite number (e.g.6) of the complete invariants. These invariants can be used in recognition-synthesis encoding of 3D images, image retrieving and copyright protection also. Moreover, the surface can be calculated by elementary functions then shape synthesis is free of numerically integral errors.

In this paper, this surface model is extended using fibres as 1-parameter groups of Hamilton Lie algebra and high order linear ODE. The complete invariant set of the model under action of Euclidean motion is obtained. Algorithms are also shown on shape synthesis using the proposed model and extraction of invariants from 3D objects.

2 Fibre-bundle model of 1-parameter groups of linear Lie algebra

First we briefly review the fibre bundle surface model shown in [4], which used the fibre curve in the fibre bundle model as 1-paramter groups of linear Lie algebra.

Let the base curve be $\boldsymbol{b} = \{\boldsymbol{b}(v), v \in \mathbb{R}\}$, the fibre curve a 1-parameter Lie group $\boldsymbol{g}_v = \{\boldsymbol{g}_v(u) = e^{uA}\boldsymbol{b}(v), u \in \mathbb{R}\}$. The surface is defined as

$$F := \{\boldsymbol{x}(u,v) = e^{uA}\boldsymbol{b}(v) \quad u, v \in \mathbb{R}\}$$

The points $\boldsymbol{b}(v)$ on the base curve $\boldsymbol{b}$ are initianl points for the integral flow of 1-parameter groups $\boldsymbol{g}_v$. In fact, the base curve needs not to be in a form of parameterized curve $\boldsymbol{b}(v)$.

Thus, the Lie algebra of this fibre bundle is a linear Lie algebra.

$$\mathcal{L}: \quad \frac{\partial \boldsymbol{x}}{\partial u} := \dot{\boldsymbol{x}}_u = Ae^{Au}\boldsymbol{b}_v = A\boldsymbol{x}$$

$$\mathcal{L}: \quad \dot{\boldsymbol{x}}_u = A\boldsymbol{x}$$

Shift the origin by $\{\boldsymbol{c}(v)\}$, one obtains a fibre bundle with Affine Lie algebra.

$$F = \{\boldsymbol{x}(u,v) = e^{Au}(\boldsymbol{b}(v) - \boldsymbol{c}(v)), u, v \in \mathbb{R}\}$$

This is in fact a special case of the fibre bundle

$$F := \{\boldsymbol{x}(u,v) = e^{Au}\boldsymbol{b}(v) + \boldsymbol{d}(v), \quad u, v \in \mathbb{R}\}.$$

Its Lie algebra is an Affine Lie algebra as follows.

$$\mathcal{L}: \quad \dot{\boldsymbol{x}}_u = Ae^{Au}\boldsymbol{b}(v) = A(\boldsymbol{x} - \boldsymbol{d}(v)) = A\boldsymbol{x} - A\boldsymbol{d}(v)$$

$$\mathcal{L}: \quad \dot{\boldsymbol{x}}_u = A\boldsymbol{x} + \boldsymbol{\delta}(v), \qquad \boldsymbol{\delta}(v) = -A\boldsymbol{d}(v).$$

The information to describe the fibre bundle model is the base curve and the six invariants of the linear Lie algebra, i.e. of the matrix A.

A more general model is to use both the base curves and the fibre curves as 1-parameter groups:

$$\boldsymbol{x}(u,v) = e^{uA+vB}\boldsymbol{x}_0 + \boldsymbol{d}$$

The Lie algebra consists of the following two linear algebras.

$$\dot{\boldsymbol{x}}_u = A\boldsymbol{x}, \quad \dot{\boldsymbol{x}}_v = B\boldsymbol{x},$$

In this case, the information to describe the fibre bundle model is a base point $\boldsymbol{x}(0,0) = \boldsymbol{b}(0)$ and twelve invariants of matrices A and B.

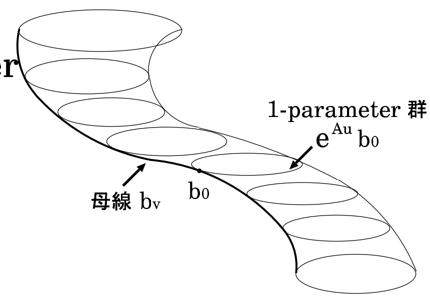

Figure 1: fibre bundle of 1-parameter groups

2.1 Hamilton flow fibres

Consider a spatial curve on a surface M

$$\boldsymbol{x}(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \in M \subset \mathbb{R}^3$$

and a state vector as

$$\boldsymbol{y}(t) := \begin{pmatrix} \dot{\boldsymbol{x}}(t) \\ \boldsymbol{x}(t) \end{pmatrix} \in M \otimes T_{\boldsymbol{x}}M$$

A Hamilton Lie algebra is defined by

$$\begin{pmatrix} \ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix}$$

or

$$\dot{\boldsymbol{y}} = \boldsymbol{H}\boldsymbol{y}$$
$$\boldsymbol{H} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_6(\mathbb{R}) : 6 \times 6 \text{ real matrices}$$

where H is call a representation matrix.

Denote the Laplace transform of $\boldsymbol{y}(t) := \boldsymbol{Y}(s)$.

$$\boldsymbol{Y}(s) = (Y_1(s), ..., Y_6(s)) \in \mathbb{C}^6$$

$$Y_i(s) := \text{ Laplace transform of } y_i(t)$$

Then the Laplace transform of $\dot{\boldsymbol{y}}(t)$ is

$$s\boldsymbol{Y}(s) - \boldsymbol{y}(0)$$

From the Hamilton algebra equation

$$s\boldsymbol{Y}(s) - \boldsymbol{y}(0) = \boldsymbol{H}\boldsymbol{Y}(s)$$

$$(s\boldsymbol{I}_6 - \boldsymbol{H})\boldsymbol{Y}(s) = \boldsymbol{y}(0)$$

$$\boldsymbol{Y}(s) = (s\boldsymbol{I}_6 - \boldsymbol{H})^{-1}\boldsymbol{y}(0)$$

The inverse Laplace transform of each term can be expressed by linear combination of

$$e^{\sigma t}\cos(\omega t + \theta) \quad and \quad e^{\sigma t}\sin(\omega t + \theta).$$

This Hamilton flow obviously defines 1-parameter groups, which can be used in the fibre bundle model for free surface. Since they are expressed by elementary functions, the shape synthesis is free from numerically integral error.

Now this can be extended to an affine Hamilton Lie algebra in a 6-dim state space $\mathbb{R}^6$

$$\dot{\boldsymbol{y}} = \boldsymbol{H}\boldsymbol{y} + \boldsymbol{\beta}, \quad H \in M_6(\mathbb{R})$$

$$\begin{pmatrix} \ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{x} \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}$$

Then

$$(s\boldsymbol{I}_6 - \boldsymbol{H})\boldsymbol{Y}(s) = \boldsymbol{\beta} + \boldsymbol{y}(0)$$

$$\boldsymbol{Y}(s) = (s\boldsymbol{I}_6 - \boldsymbol{H})^{-1}(\boldsymbol{\beta} + \boldsymbol{y}(0))$$

3 Invariants of the Hamilton Lie algebra

One major advantage to using 1-parameter groups of linear Lie algebra is that the flow therefore the surface can be uniquely determined by a complete set of invariants with respect to Euclidean motion.

As in the case of linear Lie algebras, we wish to determine the complete set of invariants of the orbit traced out by the representation matrix $\boldsymbol{H}$ in $\mathfrak{gl}_6(\mathbb{R})$ under action of $SO_3(\mathbb{R})$. As shown in [6], [8], the Hamilton Lie algebra also possesses such desirable properties.

When $SO_3(\mathbb{R})$ acts on the Hamiltonian flow M, $M \otimes TM$ is subjected to action of $SO_3(\mathbb{R}) \otimes SO_3(\mathbb{R})$ by

$$\mathcal{R} = \begin{pmatrix} R & O \\ O & R \end{pmatrix}, \quad R \in SO_3(\mathbb{R}).$$

Thus, the representation matrix $\boldsymbol{H}$ is transformed to

$$\boldsymbol{H}^{\mathcal{R}} = \mathcal{R}^T \boldsymbol{H} \mathcal{R} = \begin{pmatrix} R^T A R & R^T B R \\ R^T C R & R^T D R \end{pmatrix}$$

We can see that this action is an extension of the adjoint action on the linear Lie algebras. Furthermore, the orbit of $\boldsymbol{H}$ is uniquely determined by four orbits of A, B, C under adjoint action of $SO_3(\mathbb{R})$ and the relative phases between these orbits.

Theorem 1. *[6][8]*

$$^{\forall}\boldsymbol{H} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$
$$A, B, C, D \in \mathfrak{gl}_6(\mathbb{R}),$$

the orbit of $\boldsymbol{H}$ *insides* $\mathfrak{gl}_6(\mathbb{R})$ *under action of* $SO_3(\mathbb{R}) \otimes SO_3(\mathbb{R})$ *is uniquely determined by the orbits of* A, B, C, D *under the adjoint action of* $SO_3(\mathbb{R})$ *in* $\mathfrak{gl}_3(\mathbb{R})$. *The complete set of invariants for the orbit of* $\boldsymbol{H}$ *is*

$$\begin{Bmatrix} \mu_{A1}, & \mu_{A2}, & \mu_{A3}, & \theta_A, & \phi_A, & \psi_A \\ \mu_{B1}, & \mu_{B2}, & \mu_{B3}, & \theta_B, & \phi_B, & \psi_B \\ \mu_{C1}, & \mu_{C2}, & \mu_{C3}, & \theta_C, & \phi_C, & \psi_C \\ \mu_{D1}, & \mu_{D2}, & \mu_{D3}, & \theta_D, & \phi_D, & \psi_D \end{Bmatrix}$$

Here, denotes the singular decomposition of $E = U^T W V$, $W = diag\{\mu_{E1}, \mu_{E2}, \mu_{E3}\}$, with signs of the singular-values adjusted such that $U, V \in SO_3(\mathbb{R})$, $\mu_{iE}, i = 1, ..., 3$ are such singular values of matrix E and θ_E, ϕ_E, ψ_E are Euler's angles of $R^0 = VU^T \in SO_3(\mathbb{R})$.

4 Compactness of the Hamilton flow

It is interesting to produce bounded surface or if possible closed ones by the above models.

Stability theory provides certain sufficient conditions for with bounded flow, in particular period flow. i.e. the conditions on Eigenvalue $\lambda(H) = e^{\sigma + i\phi} \in \mathbb{C}$ of H. However, e.g. the $|\lambda| = 1$ condition simply means that H is a unitary matrix: $H^*H = I$, in fact since H is a real matrix this means H is a orthogonal matrix $H^T H = I$.

Since the flow are 1D submanifold on the level set of the Hamiltonian. One can use condition for the level set to be compact to ensure the compactness of flow.

Theorem 2. *[6][8] If the representation matrix $\boldsymbol{H}$*

$$^{\forall}\boldsymbol{H} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

$$B = B^T,\ C = C^T\ A^T = -A,$$

then the hamiltonian H is a quadratic form defined by symmetric matrix $\boldsymbol{H}'$

$$\boldsymbol{H}' = \frac{1}{2}\begin{pmatrix} -C & -A \\ A & B \end{pmatrix}.$$

When $\boldsymbol{H}'$ is positive or negative definite, the Hamilton flows therefore the level sets of H are compact.

5 Extension to high order linear differential equations

The above results can be extended to using fibres as 1-parameter groups defined by an differential equation

$$\sum_{i=0}^{n} A_i \boldsymbol{x}^{(i)}(t) + \boldsymbol{b} = 0$$

here $x^{(i)}$ denotes the i-th derivative.

$$\boldsymbol{x}^{(i)} \iff s^i X(s) - \sum_{r=0}^{i-1} s^{i-r-1}\boldsymbol{x}^{(r)}(0)$$

$$(\sum_{i=0}^{n} A_i s^i)\boldsymbol{X}(s) - \sum_{i=0}^{n-1}\sum_{r=0}^{i-1} s^{i-r-1}\boldsymbol{x}^{(r)}(0) + \boldsymbol{b} = 0$$

$$\boldsymbol{X}(s) = \left(\sum_{i=0}^{n} A_i s^i\right)^{-1} \left(\sum_{i=0}^{n-1}\sum_{r=0}^{i-1} s^{i-r-1}\boldsymbol{x}^{(r)}(0) - \boldsymbol{b}\right)$$

This is a again rational function, i.e. the integral can be expressed by elementary functions.

When A_n is invertible, one can use a simper notation. Consider a new state variable in $\mathbb{R}^{3n}$

$$\boldsymbol{y} := \begin{pmatrix} \boldsymbol{x}^{(n-1)} \\ \vdots \\ \boldsymbol{x}^{(1)} \\ \boldsymbol{x} \end{pmatrix} \in \mathbb{R}^{3n}.$$

Assume it is defined by an affine Lie algebra in $\mathbb{R}^{3n}$

$$\boldsymbol{y}^{(1)} = \mathcal{A}\boldsymbol{y} + \boldsymbol{\beta} \qquad \mathcal{A} \in M_{3n}(\mathbb{R}) \tag{1}$$

$$\begin{pmatrix} \boldsymbol{x}^{(n)} \\ \vdots \\ \boldsymbol{x}^{(1)} \end{pmatrix} = \begin{pmatrix} H_{11} & \cdots & H_{1n} \\ \vdots & \cdots & \vdots \\ H_{n1} & \cdots & H_{nn} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^{(n-1)} \\ \vdots \\ \boldsymbol{x} \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_n \end{pmatrix}$$

Then the 1-parameter group defined by

$$\sum_{i=0}^{n} A_i \boldsymbol{x}^{(i)} + \boldsymbol{b} = 0$$

when $A_n = I$ or invertible, can be expressed as $\boldsymbol{y}^{(1)} = \mathcal{A}\boldsymbol{y} + \boldsymbol{\beta}$,

$$\mathcal{A} = \begin{pmatrix} -A_{n-1} & -A_{n-2} & \cdots & -A_0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

Then

$$(s\boldsymbol{I}_{3n} - \mathcal{A})Y(s) = \boldsymbol{\beta} + \boldsymbol{y}(0)$$

$$Y(s) = (s\boldsymbol{I}_{3n} - \mathcal{A})^{-1}(\boldsymbol{\beta} + \boldsymbol{y}(0))$$

6 Fibres as flows of time-invariant diffeo-intral equations

Now we consider to fibres as flows defined by time-invariant diffeo- intral equations such as

$$\sum_{i=0}^{n} A_i \frac{d^i}{dt^i}\boldsymbol{x} + \sum_{j=1}^{m} A_j \left(\int_0^t\right)^j \boldsymbol{x} + \boldsymbol{b} = 0$$

Then

$$\left(\sum_{i=0}^{n} A_i s^i\right) \boldsymbol{X}(s) - \sum_{i=0}^{n-1}\sum_{r=0}^{i-1} s^{i-r-1}\boldsymbol{x}^{(r)}(0) + \left(\sum_{j=1}^{m} A_j s^{-j}\right) \boldsymbol{X}(s) + \boldsymbol{b} = 0$$

$$\boldsymbol{X}(s) = \left(\sum_{i=-m}^{n} A_i s^i\right)^{-1} \left(\sum_{i=0}^{n-1}\sum_{r=0}^{i-1} s^{i-r-1}\boldsymbol{x}^{(r)}(0) - \boldsymbol{b}\right)$$

In fact, all integral parts can be reduced to differentials. so the equation is almost the same as pure differential equations. Difference between differentials and integrals is that the differential equations needs or reserves initial values informations.

7 Invariants

The invariants of 1-parameter grous defined by

$$\boldsymbol{y}^{(1)} = \mathcal{A}\boldsymbol{y} \tag{2}$$

can also be obtained as follows.

Action of $\forall R \in SO_3(\mathbb{R})$ induced an action $R^{\otimes n}$ on the state space $\mathbb{R}^{3n}$.

$$\forall R \in SO_3(\mathbb{R}), \boldsymbol{y} \longmapsto \mathcal{R}\boldsymbol{y}, \quad \mathcal{R} := \text{diag}(R, ..., R)$$

Thus the representation matrix is transformed to

$$\forall R \in SO_3(\mathbb{R}), \boldsymbol{H} \longmapsto \boldsymbol{H}' = \mathcal{R}\boldsymbol{H}\mathcal{R}^T$$

Since

$$\boldsymbol{H} = (H_{ij}), \boldsymbol{H}' = \left(RH_{ij}R^T\right)$$

The complete set of invariants of the flow is the singular values and Euler's angles.

$$\lambda_{kH_{ij}}, k = 1, ..., 3, \theta_{H_{ij}}, \phi_{H_{ij}}, \psi_{H_{ij}}$$

7.0.1 Linear differential equations

The flow defined by linear differential equations with $\det A_n \neq 0$ can be regarded as a special case of above discussion.

When A_n is non-invertible, the flow defined by

$$A_n \boldsymbol{x}^{(n)} + \sum_{i=0}^{n-1} A_i \boldsymbol{x}^{(i)}(t) = 0$$

can be treated as follows. $\forall R \in SO(3)$

$$RA_nR^{-1}R\boldsymbol{x}^{(n)} + \sum_{i=0}^{n-1} RA_iR^{-1}R\boldsymbol{x}^{(i)}(t) = 0$$

$$RA_nR^{-1}\boldsymbol{z}^{(n)} + \sum_{i=0}^{n-1} RA_iR^{-1}\boldsymbol{z}^{(i)}(t) = 0, \ \boldsymbol{z} = R^T\boldsymbol{x}$$

i.e. the equation is under conjugate action of $A_i \longmapsto RA_iR^{-1}$, whose invariants can be obtained as

$$\lambda_{kA_j}, k = 1, ..., 3, \theta_{A_j}, \phi_{A_j}, \psi_{A_j}.$$

8 Estimation of representation matrx and invariants

The representation matrix and invariants can be extracted in the similar way in for the 1-parameter grouprs of linear Lie algebras[4] [3][5]. Least mean square fitting will be used to extract the representation matrix. The algebraic surface fitting are also useful in robust extraction.

9 Simulation

Fig.2-7 shown several shapes generated using the Hamilton Lie algebra fibres. The representation matrix is obtained from the complete set of invariants. For simplicity, only S^1 is used as the base curve.

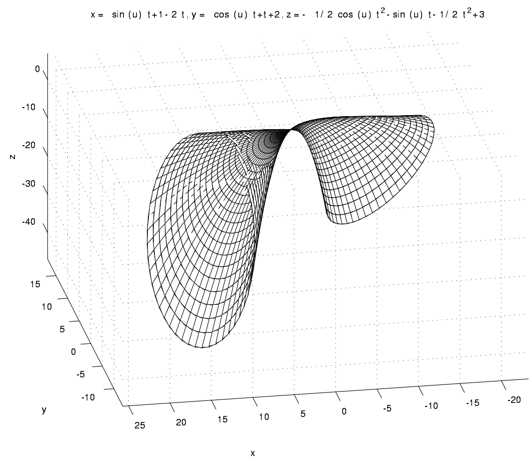

Figure 2: Shape 1

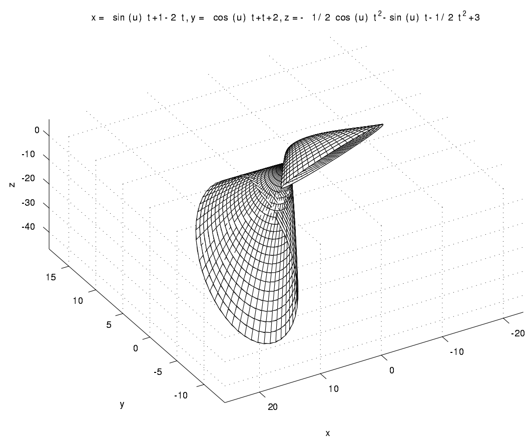

Figure 3: Shape 2

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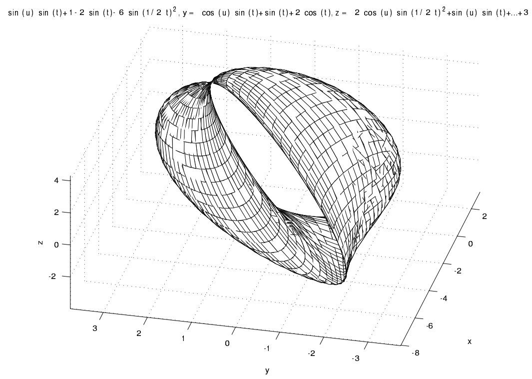

Figure 4: Shape 3

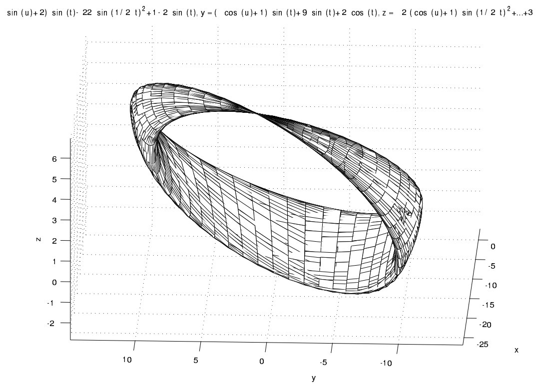

Figure 5: Shape 4

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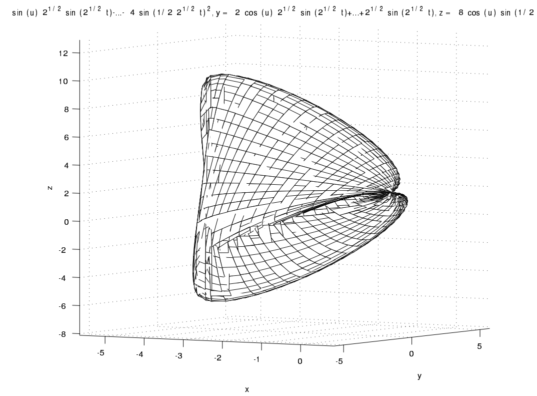

Figure 6: Shape 5

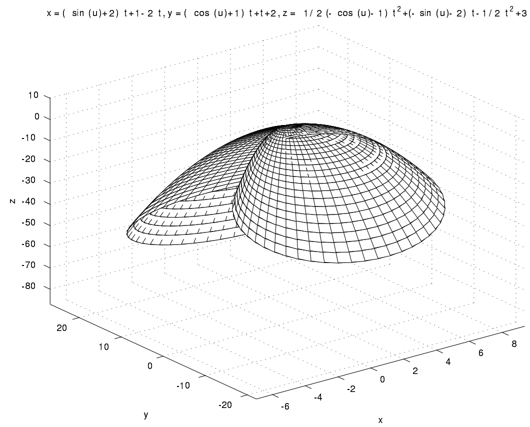

Figure 7: Shape 6

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