Independent Spanning Trees of Product Graphs and Their Construction

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SUMMARY A graph G is called an n-channel graph at vertex r if there are n independent spanning trees rooted at r. A graph G is called an n-channel graph if G is an n-channel graph at every vertex. Independent spanning trees of a graph play an important role in fault-tolerant broadcasting in the graph. In this paper we show that if G_1 is an n_1-channel graph and G_2 is an n_2-channel graph, then $G_1 \times G_2$ is an (n_1+n_2)-channel graph. We prove this fact by constructing n_1+n_2 independent spanning trees of $G_1 \times G_2$ from n_1 independent spanning trees of G_1 and n_2 independent spanning trees of G_2.

1 Introduction

For a pair of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the product of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph with the vertex set $V_1 \times V_2 = \{(x,y) \mid x \in V_1, y \in V_2\}$ and the edge set such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = v_1$ and $u_2v_2 \in E_2$, or $u_2 = v_2$ and $u_1v_1 \in E_1$. The definition of the product of two graphs can be generalized to the product of n graphs in the natural way. $G_1 \times G_2 \times G_3$ is $(G_1 \times G_2) \times G_3$ or $G_1 \times (G_2 \times G_3)$. Note that $(G_1 \times G_2) \times G_3$ and $G_1 \times (G_2 \times G_3)$ are isomorphic. The product of n graphs $G_1 \times G_2 \times \cdots \times G_n$ is $(G_1 \times \cdots \times G_k) \times (G_{k+1} \times \cdots \times G_n)$ for some k $(1 \le k \le n-1)$, where each G_i $(1 \le i \le n)$ is called a component of $G_1 \times G_2 \times \cdots \times G_n$.

Some of popular interconnection networks are product graphs. For example, the n-dimensional hypercube Q_n is $Q_{n-1} \times K_2 = Q_{n-2} \times K_2 \times K_2 = \cdots = K_2 \times K_2 \times \cdots \times K_2$, where K_2 is the complete graph of order 2. The $(m_1 \times \cdots \times m_n)$-mesh is $L_{m_1} \times \cdots \times L_{m_n}$, and the $(m_1 \times \cdots \times m_n)$-torus is $R_{m_1} \times \cdots \times R_{m_n}$, where L_i and R_i are a linearly linked graph of order i and a ring of order i, respectively. Denote the vertex connectivity of a graph G by $\kappa(G)$. Youssef [7] showed that for a pair of graphs G_1 and G_2, $\kappa(G_1 \times G_2) = \kappa(G_1) + \kappa(G_2)$.

A set of paths connecting a pair of vertices in a graph are said to be internally disjoint if and only if any pair of paths of the set have no common vertices and no common edges except for their extreme vertices. Two spanning trees of a graph $G = (V, E)$ are said to be independent if they are rooted at the same vertex, say r, and for each vertex v in V, the two paths from r to v, one path in each tree, are internally disjoint. A set of spanning trees of G are said to be independent if they are pairwise independent. A graph G is called an n-channel graph at vertex r, if there are n independent spanning trees rooted at r of G. If G is an n-channel graph at every vertex, G is called an n-channel graph.

Itai and Rodeh [5] gave a linear time algorithm for finding two independent spanning trees in a biconnected graph. Cheriyan and Maheshwari [4] showed how to find three independent spanning trees of $G = (V, E)$ in $O(|V||E|)$ time. Zehavi and Itai [8] also showed that for any 3-connected graph G and any vertex r there are three independent spanning trees rooted at r. They conjectured that any κ-vertex connected graph has κ independent spanning trees rooted at an arbitrary vertex r [6][8]. This conjecture is still open for any $\kappa > 3$.

It has been shown that broadcasting along independent spanning trees are efficient and reliable [2][3][5]. In fact, if G is an n-channel graph and the source vertex is not faulty, then there exists a broadcasting scheme that tolerates up to $n-1$ faults of crash type and up to $\lfloor (n-1)/2 \rfloor$ faults of Byzantine type even in the worst case. All transmissions by such a broadcasting scheme contribute to the majority voting to obtain the correct message, and its communication complexity is optimal to tolerate up to $\lfloor (n-1)/2 \rfloor$ faults of Byzantine type [2][3].

In this paper we focus attention on the construction of independent spanning trees of a given product graph. This problem was discussed in [1], but they only showed a weaker result than the main result in this paper. We show that if G_1 is an n_1-channel graph and G_2 is an n_2-channel graph, then $G_1 \times G_2$ is an (n_1+n_2)-channel graph. We construct n_1+n_2 independent spanning trees of $G_1 \times G_2$ from n_1 independent spanning trees of G_1 and n_2 independent spanning trees of G_2. This construction is not straightforward. From our construction we can say that if for each component graph G_i $(1 \le i \le n)$, the vertex connectivity of G_i coincides with the number of independent spanning trees rooted at the same vertex of G_i, then the vertex connectivity and the number of independent spanning trees rooted at the same vertex of $G_1 \times \cdots \times G_n$ coincide.

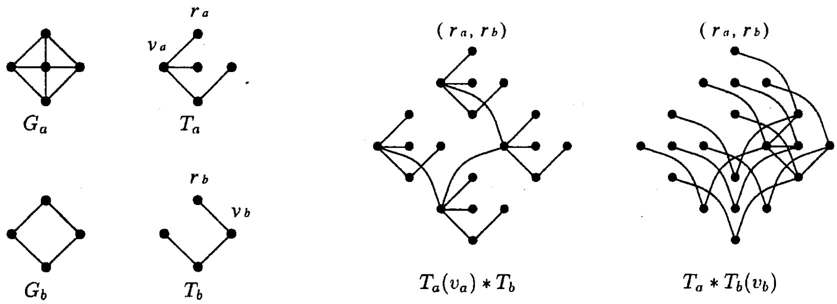

Figure 1: Examples of $T_a(v_a) * T_b$ and $T_a * T_b(v_b)$

2 Spanning Trees of Product Graphs

We first define an operation "*" on spanning trees. The set of vertices and the set of edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The cardinality of a set α is denoted by $|\alpha|$. Let G_a and G_b be two graphs, r_a be a vertex of G_a, and r_b be a vertex of G_b. Let T_a be a spanning tree rooted at r_a of G_a, and let T_b be a spanning tree rooted at r_b of G_b. Assume that the number of sons of r_a in T_a is k_a, and let the set of sons of r_a in T_a be $C_a = \{s_a^1, \cdots, s_a^{k_a}\}$. Let v_a be a vertex in C_a. For each i $(1 \le i \le k_a)$, let S_a^i be the subtree rooted at s_a^i of T_a. We now construct a spanning tree rooted at (r_a, r_b) of $G_a \times G_b$, denoted by $T_a(v_a) * T_b$, from T_a and T_b as follows:

(1) For each u in C_a, connect (r_a, r_b) with (u, r_b).

(2) For each $y_1y_2 \in E(T_b)$, if $u \in C_a$ then connect (u, y_1) with (u, y_2).

(3) For each i $(1 \le i \le k)$, if $x_1x_2 \in E(S_a^i)$ and $y \in V(G_b)$ then connect (x_1, y) with (x_2, y).

(4) For each $y \in V(G_b) - \{r_b\}$, connect (r_a, y) with (v_a, y).

Assume that the number of sons of r_b in T_b is k_b, and let the set of sons of r_b in T_b be $C_b = \{s_b^1, \cdots, s_b^{k_b}\}$. Let v_b be a vertex in C_b. For each i $(1 \le i \le k_b)$, let S_b^i be the subtree rooted at s_b^i of T_b. Symmetrically we can construct $T_a * T_b(v_b)$.

Examples of $T_a(v_a) * T_b$ and $T_a * T_b(v_b)$ are shown in Fig. 1.

To specify a path in $G_a \times G_b$ we use the following notations. If x_1x_2 is an edge of a subgraph T of G_a, then path with length 1 from (x_1, y) to (x_2, y) is denoted by $(x_1, y) \xrightarrow{T} (x_2, y)$. The reflexive and transitive closure of $\xrightarrow{T}$ is denoted by $\xRightarrow{T}$. Alternatively, $A \xRightarrow{T} B$ means that B follows from A by application of $\xrightarrow{T}$ zero or more times. Similarly, if y_1y_2 is an edge of a subgraph T' of G_b, then path with length 1 from (x, y_1) to (x, y_2) is denoted by $(x, y_1) \xrightarrow{T'} (x, y_2)$. The reflexive and transitive closure of $\xrightarrow{T'}$ is denoted by $\xRightarrow{T'}$.

We have the following four lemmas. The proofs of these lemmas are omitted here due to the page limit.

Lemma 1 *Let T_a and T_b be a spanning tree rooted at r_a of G_a and a spanning tree rooted at r_b of G_b, respectively, and let v_a and v_b be a son of r_a in T_a and a son of r_b in T_b, respectively. Then each of $T_a(v_a) * T_b$ and $T_a * T_b(v_b)$ is a spanning tree rooted at (r_a, r_b) of $G_a \times G_b$.*

Lemma 2 *Let T_1 and T_2 be independent spanning trees rooted at r_a of G_a, and let v_1 be a son of r_a in T_1 and v_2 be a son of r_a in T_2. Let T_b be a spanning tree rooted at r_b of G_b. Then $T_1(v_1) * T_b$ and $T_2(v_2) * T_b$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.*

Lemma 3 *Let T_a be a spanning tree rooted at r_a of G_a. Let T_1 and T_2 be independent spanning trees rooted at r_b of G_b, and let v_1 be a son of r_b in T_1 and v_2 be a son of r_b in T_2. Then $T_a * T_1(v_1)$ and $T_a * T_2(v_2)$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.*

Lemma 4 *Let $T_{a,1}$ and $T_{a,2}$ be independent spanning trees of G_a, and let v_a be a son of r_a in $T_{a,1}$. Let $T_{b,1}$ and $T_{b,2}$ be independent spanning trees of G_b, and let v_b be a son of r_b in $T_{b,1}$. Then $T_{a,1}(v_a) * T_{b,2}$ and $T_{a,2} * T_{b,1}(v_b)$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.*

Suppose that G_a is an n_a-channel graph and G_b is an n_b-channel graph. Let $T_{a,1}, \cdots, T_{a,n_a}$ be n_a independent spanning trees rooted at r_a of G_a, and let $T_{b,1}, \cdots, T_{b,n_b}$

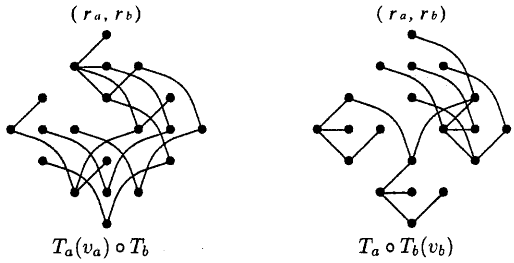

Figure 2: Examples of $T_a(v_a) \circ T_b$ and $T_a \circ T_b(v_b)$

be n_b independent spanning trees rooted r_b of G_b. For each i ($1 \le i \le n_a$), let $v_{a,i}$ be a son of r_a in $T_{a,i}$, and for each j ($1 \le j \le n_b$), let $v_{b,j}$ be a son of r_b in $T_{b,j}$. We now consider the following $n_a + n_b$ spanning trees:

$$T_{a,1}(v_{a,1}) * T_{b,1},\ \cdots,\ T_{a,n_a}(v_{a,n_a}) * T_{b,1},$$

$$T_{a,1} * T_{b,1}(v_{b,1}),\ \cdots,\ T_{a,1} * T_{b,n_b}(v_{b,n_b})$$

From Lemma 2, Lemma 3 and Lemma 4, the $n_a + n_b - 2$ trees listed above excepting $T_{a,1}(v_{a,1}) * T_{b,1}$ and $T_{a,1} * T_{b,1}(v_{b,1})$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.

The goal of this paper is to construct $n_a + n_b$ independent spanning trees rooted at the same vertex of the product graph of an n_a-channel graph and an n_b-channel graph. For this purpose we introduce another operation, denoted by "$\circ$", on spanning trees. Let T_a be a spanning tree rooted at r_a of G_a, and let T_b be a spanning tree rooted at r_b of G_b. Let v_a be a son of r_a in T_a. The vertex set of $T_a(v_a) \circ T_b$ is $V(G_a \times G_b)$, and its edge set consists of the following edges:

(1) For each $x_1x_2 \in E(T_a)$, connect (x_1, r_b) with (x_2, r_b).

(2) For each $y_1y_2 \in E(T_b)$, if $x \in V(T_a) - \{r_a\}$ then connect (x, y_1) with (x, y_2).

(3) For each $y \in V(T_b) - \{r_b\}$, connect (r_a, y) with (v_a, y).

Let v_b be a son of r_b in T_b. Symmetrically we can define $T_a \circ T_b(v_b)$.

Examples of $T_a(v_a) \circ T_b$ and $T_a \circ T_b(v_b)$ are shown in Fig. 2, where T_a, T_b, v_a, and v_b are given in Fig. 1.

We have the following four lemmas. The proofs of these lemmas are omitted here.

Lemma 5 *Let T_a and T_b be a spanning tree rooted at r_a of G_a and a spanning tree rooted at r_b of G_b, respectively, and let v_a and v_b be a son of r_a in T_a and a son of r_b in T_b, respectively. Then each of $T_a(v_a) \circ T_b$ and $T_a \circ T_b(v_b)$ is a spanning tree rooted at (r_a, r_b) of $G_a \times G_b$.*

Lemma 6 *Let T_a and T_b be a spanning tree rooted at r_a of G_a and a spanning tree rooted at r_b of G_b, respectively. Let v_a and v_b be a son of r_a in T_a and a son of r_b in T_b, respectively. Then $T_a(v_a) \circ T_b$ and $T_a \circ T_b(v_b)$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.*

Lemma 7 *Let T_1 and T_2 be independent spanning trees rooted at r_a of G_a, and let v_1 and v_2 be a son of r_a in T_1 and a son of r_a in T_2, respectively. Let T_b be a spanning tree rooted at r_b of G_b. Then $T_1(v_1) \circ T_b$ and $T_2(v_2) * T_b$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.*

Lemma 8 *Let T_a be a spanning tree rooted at r_a of G_a. Let T_1 and T_2 be independent spanning trees rooted at r_b of G_b, and let v_1 and v_2 be a son of r_b in T_1 and a son of r_b in T_2, respectively. Then $T_a \circ T_1(v_1)$ and $T_a * T_2(v_2)$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.*

Consider the following $n_a + n_b$ spanning trees rooted at (r_a, r_b) of $G_a \times G_b$:

$$T_{a,1}(v_{a,1}) \circ T_{b,1},$$

$$T_{a,2}(v_{a,2}) * T_{b,1},\ \cdots,\ T_{a,n_a}(v_{a,n_a}) * T_{b,1},$$

$$T_{a,1} \circ T_{b,1}(v_{b,1}),$$

$$T_{a,1} * T_{b,2}(v_{b,2}),\ \cdots,\ T_{a,1} * T_{b,n_b}(v_{b,n_b})$$

These spanning trees are obtained from the $n_a + n_b$ spanning trees listed earlier in this section by replacing $T_{a,1}(v_{a,1}) * T_{b,1}$ and $T_{a,1} * T_{b,1}(v_{b,1})$ with $T_{a,1}(v_{a,1}) \circ T_{b,1}$ and $T_{a,1} \circ T_{b,1}(v_{b,1})$, respectively. The set of these spanning trees are still not independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$. To construct $n_a + n_b$ independent spanning trees, we need further modifications.

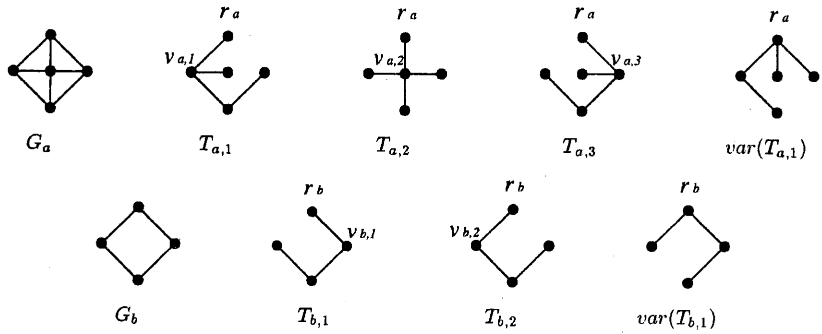

Figure 3: Examples of $var(T_{a,1})$ and $var(T_{b,1})$

3 Construction of Independent Spanning Trees

Let $T_{a,1}, \cdots, T_{a,n_a}$ be n_a independent spanning trees rooted at r_a of G_a, and let T_b be a spanning tree rooted at r_b of G_b. For each i $(1 \leq i \leq n_a)$, let k_a^i be the number of sons of r_a in $T_{a,i}$, and let $C_{a,i} = \{s_{a,i}^1, \cdots, s_{a,i}^{k_a^i}\}$ be the set of sons of r_a in $T_{a,i}$. Let $C_a = C_{a,1} \cup C_{a,2} \cup \cdots \cup C_{a,n_a}$. For each $v \in V(T_{a,1}) - \{r_a\}$, let $p_{a,1}(v)$ be the parent of v in $T_{a,1}$.

The variation of $T_{a,1}$, denoted by $var(T_{a,1})$, is a graph with vertex set $V(T_{a,1})$ and the edge set $E(T_{a,1}) \cup \{r_a x \mid x \in C_a - C_{a,1}\} - \{p_{a,1}(x)x \mid x \in C_a - C_{a,1}\}$. Note that $p_{a,1}(x)x$ is an edge connecting $p_{a,1}(x)$ and x, and that $r_a x$ is an edge connecting r_a and x in G_a. Apparently $var(T_{a,1})$ is also a spanning tree rooted at r_a of G_a.

We now modify $T_{a,1}(v_{a,1}) \circ T_b$, where $v_{a,1}$ is one of the sons of r_a in $T_{a,1}$. For each $x \in C_{a,i}$ $(2 \leq i \leq n_a)$ and each $y_1y_2 \in E(T_b)$, edge $(x, y_1)(x, y_2)$ of $T_{a,1}(v_{a,1}) \circ T_b$ is also used in $T_{a,i}(v_{a,i}) * T_b$. For each $x \in C_a - C_{a,1}$ and each $y_1y_2 \in E(T_b)$, we remove edge $(x, y_1)(x, y_2)$ from $T_{a,1}(v_{a,1}) \circ T_b$, and for each $x \in C_a - C_{a,1}$ and each $y \in V(G_b) - \{r_b\}$, we add edge $(p_{a,1}(x), y)(x, y)$. This modification is called the transformation. The graph obtained from $T_{a,1}(v_{a,1}) \circ T_{b,1}$ by the transformation is denoted by $tr(T_{a,1}(v_{a,1}) \circ T_b)$. More formally $tr(T_{a,1}(v_{a,1}) \circ T_b)$ is a graph with the vertex set $V(G_a \times G_b)$ and the edge set $E(T_{a,1}(v_{a,1}) \circ T_b) \cup \{(p_{a,1}(x), y)(x, y) \mid x \in C_a - C_{a,1},\ y \in V(G_b) - \{r_b\}\} - \{(x, y_1)(x, y_2) \mid x \in C_a - C_{a,1},\ y_1y_2 \in E(T_b)\}$. For each $x \in C_a - C_{a,1}$ and each $y \in V(G_b) - \{r_b\}$, $(p_{a,1}(x), y)(x, y)$ is an edge of $G_a \times G_b$ since $p_{a,1}(x)x$ is an edge of $T_{a,1}$. From the construction of $tr(T_{a,1}(v_{a,1}) \circ T_b)$, it is also a spanning tree rooted at (r_a, r_b) of $G_a \times G_b$.

Let $T_{b,1}, \cdots, T_{b,n_b}$ be n_b independent spanning trees rooted at r_b of G_b, and let T_a be a spanning tree rooted at r_a of G_a. For each i $(1 \leq i \leq n_b)$, let k_b^i be the number of sons of r_b in $T_{b,i}$, and let $C_{b,i} = \{s_{b,i}^1, \cdots s_{b,i}^{k_b^i}\}$ be the set of sons of r_b in $T_{b,i}$. Let $C_b = C_{b,1} \cup C_{b,2} \cup \cdots \cup C_{b,n_b}$. For each $v \in V(T_{b,1}) - \{r_b\}$, let $p_{b,1}(v)$ be the parent of v in $T_{b,1}$. Symmetrically we define $var(T_{b,1})$ and $tr(T_a \circ T_{b,1}(v_{b,1}))$, where $v_{b,1}$ is one of the sons of r_b in $T_{b,1}$. The variation of $T_{b,1}$, denoted by $var(T_{b,1})$, is a graph with the vertex set $V(T_{b,1})$ and the edge set $E(T_{b,1}) \cup \{r_b y \mid y \in C_b - C_{b,1}\} - \{p_{b,1}(y) \mid y \in C_b - C_{b,1}\}$. It is also a spanning tree rooted at r_b of G_b. The transformation of $T_a \circ T_{b,1}(v_{b,1})$, denoted by $tr(T_a \circ T_{b,1}(v_{b,1}))$, is a graph with the vertex set $V(G_a \times G_b)$ and the edge set $E(T_a \circ T_{b,1}(v_{b,1})) \cup \{(x, p_{b,1}(y))(x, y) \mid x \in V(G_a) - \{r_a\},\ y \in C_b - C_{b,1}\} - \{(x_1, y)(x_2, y) \mid y \in C_b - C_{b,1},\ x_1x_2 \in E(T_a)\}$. The transformation of $T_a \circ T_{b,1}(v_{b,1})$ is also a spanning tree rooted at (r_a, r_b) of $G_a \times G_b$. Examples of $var(Ta, 1)$ and $var(T_{b,1})$ are shown in Fig. 3. Examples of $T_{a,1}(v_{a,1}) \circ var(T_{b,1})$ and $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ are shown in Fig. 4, where $T_{a,1}, T_{b,1}$, and $v_{a,1}$ are given in Fig. 3.

Assume that G_a is an n_a-channel graph and G_b is an n_b-channel graph. We now consider the following $n_a + n_b$ graphs:

$$tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1})),$$

$$T_{a,2}(v_{a,2}) * var(T_{b,1}), \cdots, T_{a,n_a}(v_{a,n_a}) * var(T_{b,1}),$$

$$tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1})),$$

$$var(T_{a,1}) * T_{b,2}(v_{b,2}), \cdots, var(T_{a,1}) * T_{b,n_b}(v_{b,n_b}),$$

where for each i $(1 \leq i \leq n_a)$, $v_{a,i}$ is one of the sons of r_a in $T_{a,i}$, and for each j $(1 \leq j \leq n_b)$, $v_{b,j}$ is one of the sons of r_b in $T_{b,j}$. From Lemma 1, Lemma 5 and the discussion above, these graphs are $n_a + n_b$ spanning trees rooted at (r_a, r_b) of $G_a \times G_b$. We show that these $n_a + n_b$ spanning trees are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.

Lemma 9 *For each i $(2 \leq i \leq n_a)$ and each j $(2 \leq j \leq n_b)$, $T_{a,i}(v_{a,i}) * var(T_{b,1})$ and $var(T_{a,1}) * T_{b,j}(v_{b,j})$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.*

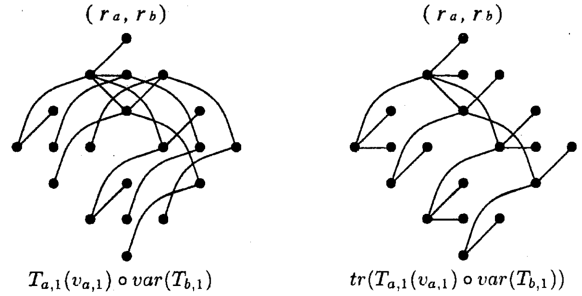

Figure 4: Examples of $T_{a,1}(v_{a,1}) \circ var(T_{b,1})$ and $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$

Proof: From Lemma 4, $T_{a,i}(v_{a,i}) * T_{b,1}$ and $T_{a,1} * T_{b,j}(v_{b,j})$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$. Consider the paths from (r_a, r_b) to (x, y) in $T_{a,i}(v_{a,i}) * T_{b,1}$ and in $T_{a,i}(v_{a,i}) * var(T_{b,1})$. If the former path does not contain any node (s, t) such that $s \in C_{a,i}$ and $t \in C_b - C_{b,1}$, then these two paths are identical. Suppose that the former path contains a node (s, t) such that $s \in C_{a,i}$ and $t \in C_b - C_{b,1}$. Assume that x is in $S^h_{a,i}$ (i.e., the subtree rooted at $s^h_{a,i}$ of $T_{a,i}$). Then the former path is $(r_a, r_b) \xrightarrow{T_{a,i}} (s^h_{a,i}, r_b) \overset{T_{b,1}}{\Longrightarrow} (s^h_{a,i}, t) \overset{T_{b,1}}{\Longrightarrow} (s^h_{a,i}, y) \overset{S^h_{a,i}}{\Longrightarrow} (x, y)$. On the other hand, the latter path is $(r_a, r_b) \xrightarrow{T_{a,i}} (s^h_{a,i}, r_b) \xrightarrow{var(T_{b,1})} (s^h_{a,i}, t) \overset{var(T_{b,1})}{\Longrightarrow} (s^h_{a,i}, y) \overset{S^h_{a,i}}{\Longrightarrow} (x, y)$. Since $(s^h_{a,i}, t) \overset{var(T_{b,1})}{\Longrightarrow} (s^h_{a,i}, y)$ and $(s^h_{a,i}, t) \overset{T_{b,1}}{\Longrightarrow} (s^h_{a,i}, y)$ are identical paths, the latter path is obtained from the former path by cutting the subpath $(s^h_{a,i}, r_b) \overset{T_{b,1}}{\Longrightarrow} (s^h_{a,i}, t)$ short to edge $(s^h_{a,i}, r_b)(s^h_{a,i}, t)$. Note that edge $(s^h_{a,i}, r_b)(s^h_{a,i}, t)$ is not used in $var(T_{a,1}) * T_{b,j}(v_{b,j})$. For the paths from (r_a, r_b) to (x, y) in $T_{a,1} * T_{b,j}(v_{b,j})$ and in $var(T_{a,1}) * T_{b,j}(v_{b,j})$, we can apply the same argument. Hence, for each $(x, y) \in V(G_a \times G_b)$, the paths from (r_a, r_b) to (x, y) in $T_{a,1}(v_{a,1}) * var(T_{b,1})$ and in $var(T_{a,1}) * T_{b,1}(v_{b,1})$ are internally disjoint. Therefore, $T_{a,i}(v_{a,i}) * var(T_{b,1})$ and $var(T_{a,1}) * T_{b,j}(v_{b,j})$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$. □

For clear description of the proofs of the following three lemmas we introduce functions $A_{a,1}$ and $A_{b,1}$. Remember that $p_{a,1}(v)$ and $p_{b,1}(v)$ denote the parents of v in $T_{a,1}$ and in $T_{b,1}$, respectively. For each $v \in V(T_{a,1}) - \{r_a\}$, let

$$A_{a,1}(v) = \begin{cases} A_{a,1}(p_{a,1}(v)) & \text{if } v \in C_a - C_{a,1} \\ v & \text{if } v \notin C_a - C_{a,1}. \end{cases}$$

Symmetrically, for each $v \in V(T_{b,1}) - \{r_b\}$, let

$$A_{b,1}(v) = \begin{cases} A_{b,1}(p_{b,1}(v)) & \text{if } v \in C_b - C_{b,1} \\ v & \text{if } v \notin C_b - C_{b,1}. \end{cases}$$

Lemma 10 *A pair of $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.*

Proof: From Lemma 5 and a similar argument to the proof of Lemma 7, $T_{a,1}(v_{a,1}) \circ var(T_{b,1})$ and $var(T_{a,1}) \circ T_{b,1}(v_{b,1})$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.

Consider the paths from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and in $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$. If $x \notin C_a - C_{a,1}$ and $y \notin C_b - C_{b,1}$, then the paths from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and in $T_{a,1}(v_{a,1}) \circ var(T_{b,1})$ are identical, and the paths from (r_a, r_b) to (x, y) in $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$ and in $var(T_{a,1}) \circ T_{b,1}(v_{b,1})$ are identical. Hence, in this case the paths from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and in $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$ are internally disjoint. If $x \notin C_a - C_{a,1}$ and $y \in C_b - C_{b,1}$, then the path from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ is $(r_a, r_b) \overset{T_{a,1}}{\Longrightarrow} (x, r_b) \xrightarrow{var(T_{b,1})} (x, y)$, and the path from (r_a, r_b) to (x, y) in $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$ is $(r_a, r_b) \overset{T_{b,1}}{\Longrightarrow} (r_a, A_{b,1}(y)) \overset{var(T_{a,1})}{\Longrightarrow} (x, A_{b,1}(y)) \overset{T_{b,1}}{\Longrightarrow} (x, y)$. These paths are internally disjoint. If $x \in C_a - C_{a,1}$ and $y \in C_b - C_{b,1}$, then the path from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ is $(r_a, r_b) \overset{T_{a,1}}{\Longrightarrow} (A_{a,1}(x), r_b) \xrightarrow{var(T_{b,1})} (A_{a,1}(x), y) \overset{T_{a,1}}{\Longrightarrow} (x, y)$, and the path from (r_a, r_b) to (x, y) in $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$ is $(r_a, r_b) \overset{T_{b,1}}{\Longrightarrow} (r_a, A_{b,1}(y)) \xrightarrow{var(T_{a,1})} (x, A_{b,1}(y)) \overset{T_{b,1}}{\Longrightarrow} (x, y)$. These paths are also internally disjoint. Therefore, $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$. □

Lemma 11 *For each* i $(2 \le i \le n_a)$*,* $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ *and* $T_{a,i}(v_{a,i}) * var(T_{b,1})$ *are independent spanning trees rooted at* (r_a, r_b) *of* $G_a \times G_b$*.*

Proof: Since $var(T_{b,1})$ is a spanning tree rooted at r_b of G_b, from Lemma 7, $T_{a,1}(v_{a,1}) \circ var(T_{b,1})$ and $T_{a,i} * var(T_{b,1})$ are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$.

Consider the paths from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and in $T_{a,i}(v_{a,i}) * var(T_{b,1})$. If $x \notin C_a - C_{a,1}$, then the paths from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and in $T_{a,1}(v_{a,1}) \circ var(T_{b,1})$ are identical. Hence, in this case the paths from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and in $T_{a,i}(v_{a,i}) * var(T_{b,1})$ are internally disjoint. Suppose that $x \in C_a - C_{a,1}$ and x is in subtree $S^h_{a,i}$ of $T_{a,i}$. Then the former path from (r_a, r_b) to (x, y) is $(r_a, r_b) \overset{T_{a,1}}{\Longrightarrow} (A_{a,1}(x), r_b) \overset{var(T_{b,1})}{\Longrightarrow} (A_{a,1}(x), y) \overset{T_{a,1}}{\Longrightarrow} (x, y)$, and the latter path from (r_a, r_b) to (x, y) is $(r_a, r_b) \overset{T_{a,i}}{\longrightarrow} (s^h_{a,i}, r_b) \overset{var(T_{b,1})}{\Longrightarrow} (s^h_{a,i}, y) \overset{S^h_{a,i}}{\Longrightarrow} (x, y)$. Since $T_{a,1}$ and $T_{a,i}$ are independent spanning trees rooted at r_a of G_a, path $(A_{a,1}(x), y) \overset{T_{a,1}}{\Longrightarrow} (x, y)$ in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and path $(s^h_{a,i}, y) \overset{S^h_{a,i}}{\Longrightarrow} (x, y)$ have no common vertices excepting (x, y). Hence, in this case the paths from (r_a, r_b) to (x, y) in $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ and in $T_{a,i}(v_{a,i}) * var(T_{b,1})$ are internally disjoint. Therefore, a pair of these trees are independent spanning trees rooted at (r_a, r_b) of $G_a \times G_b$. □

Symmetrically we have following three lemmas.

Lemma 12 *For each* j $(2 \le j \le n_b)$*,* $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$ *and* $var(T_{a,1}) * T_{b,j}(v_{b,j})$ *are independent spanning trees rooted at* (r_a, r_b) *of* $G_a \times G_b$*.*

Lemma 13 *For each* i $(1 \le i \le n_a)$*,* $tr(var(T_{a,1}) \circ T_{b,1}(v_{b,1}))$ *and* $T_{a,i}(v_{a,i}) * var(T_{b,1})$ *are independent spanning trees rooted at* (r_a, r_b) *of* $G_a \times G_b$*.*

Lemma 14 *For each* j $(1 \le j \le n_b)$*,* $tr(T_{a,1}(v_{a,1}) \circ var(T_{b,1}))$ *and* $var(T_{a,1}) * T_{b,j}(v_{b,j})$ *are independent spanning trees rooted at* (r_a, r_b) *of* $G_a \times G_b$*.*

Theorem 1 *If* G_a *is an* n_a*-channel graph and* G_b *is an* n_b*-channel graph, then* $G_a \times G_b$ *is an* $(n_a + n_b)$*-channel graph.*

Proof: It is immediate from Lemmas $2 - 4$ and Lemmas $9 - 14$. □

4 Concluding Remarks

The following two problems arise from our approach if we consider general graphs.

(1) How can we construct independent spanning trees rooted at the same vertex of an arbitrary graph? This is a very hard problem. It is open whether every n-connected graph is an n-channel graph.

(2) How can we design efficient broadcasting protocols, in particular for one-port broadcasting, based on message transmissions through independent spanning trees. Since such a broadcasting scheme consists of sub-broadcasts, each through one of the independent spanning trees, there are few hints how each vertex should use a strategy to achieve short broadcasting time.

These problems would be worthy for further investigation.

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