ボロノイ図あてはめ法の改良と考察

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細胞の断面などの現実に見られる模様を、ボロノイ図であるとみなしてモデル化することがある。その際、与えられた模様が、どの程度ボロノイ図に近いかを定量的に知りたい。それによって、その模様の性質を論じることができるからである。ここでは、ボロノイ図らしさの定量化の、知られているいくつかの方法の内の１つに焦点をしぼる。この方法は、ある別の方法と比べ扱いが易しい。しかし、幾何学的な意味が不明瞭であり、さらに重大な欠点がある。そこで、この方法に変更を加えて、この方法に対する幾何学的意味を与え、欠点を解消するための方法を論じる。

Improvement on a Voronoi Fitting Method and Some Considerations

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One of the applications of the Voronoi diagram is to model actual patterns. It is important to ask how close a given pattern is to the Voronoi diagram. Actually, if we can measure the closeness quantitatively, we can analize the models in more detail. In this paper, we focus on one of existing methods. This method is comparatively easy to deal with. But it has a defect. This paper modifies the method and gives an intuitive interpretation of the method. It also argues about the solution to the defect.

1 Introduction

The Voronoi diagram [6] is applied to various fields, especially to engineering. One of such applications is to model actual patterns with the Voronoi Diagram. In fact, there are many patterns regarded as close to the Voronoi diagram [7] [8] [9] [10] [11] [12] [13] [14] [15]. In this context, it is important to ask how close a given pattern is to the Voronoi diagram. Actually, if we can measure the closeness quantitatively, we can analize the models in more detail; for example, we can discuss whether the pattern has inhomogeneity in directions or in strength of each cell. Examples of patterns considered in practice are a cross-section of a cluster of cells [7] [8] [9], territories of animals or plants [10] [11] [12], cracks of a rock [4] and an administrative district such as an electoral district [3]. It is easy to execute this kind of simulation because it is not necessary to deal with a complicated topological structure of a pattern.

In Section 2, we define the Voronoi diagram. In Section 3, Voronoi fitting problem is formulated. In Section 4, some existing methods for the problem are reviewed. In the subsequent sections, we focus on one of existing methods, named P-B method. This method is comparatively easy to deal with. But it has a defect. This paper proposes modified P-B method and an intuitive interpretation of the method in Section 5. It also argues about the solution to the defect in the same section. Lastly, some conclusions and discussions are given in Section 6.

2 Voronoi Diagram

In this section, we define the Voronoi diagram [6] in the 2-dimensional plane. Given a set of n distinct points $\{P_i(x_i, y_i) \mid i = 1, 2, \ldots, n\}$ in the Euclidean plane, we associate all locations in the plane with the closest member of the point set. The result is a tessellation of the plane into a set of regions associated with members of the point set. This is called the planar ordinary Voronoi diagram generated by the point set. Figure 1 is an example. The members of the point set are called generators. The region associated with a member P_i is denoted by $\mathcal{V}(P_i)$ and is called the Voronoi polygon associated with P_i.

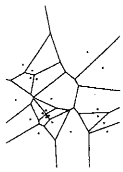

Figure 1: Example of Voronoi diagram

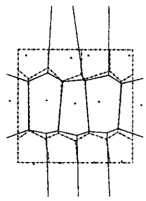

Figure 2: Example of Voronoi fitting problem

3 Voronoi Fitting Problem

In this section, we define the Voronoi fitting problem. At first, we define a tessellation, which will be the input to the Voronoi fitting problem. Let $\mathcal{R}$ be a bounded planar region such as a square. A set of regions $\{\mathcal{R}_i \mid i = 1, 2, \ldots, n\}$, where $\mathcal{R} = \bigcup_{i=1,\ldots,n} \mathcal{R}_i$ and $|\mathcal{R}_i \bigcap \mathcal{R}_j| = 0$ $(\forall\ i,\ \forall\ j \neq i)$, is called a tessellation, provided that the notation $|\ |$ means the area of a region.

Then, the Voronoi fitting problem is the following. Given a tessellation $\{\mathcal{R}_i\}$, arrange unknown n generators $\{P_i\}$ so that the Voronoi diagram generated by the generators is close to the given tessellation. In figure 2, chained lines represent the given tessellation, solid circles represent generators calculated by some method and solid lines represent the Voronoi diagram generated by the generators.

Although the meaning of "close" is ambiguous in the above definition, it is strictly defined for each method in the following sections.

4 Existing Methods

In this section, some existing methods of solving Voronoi fitting problem are reviewed.

4.1 Voronoi Recognition Problem

Some methods of solving a slightly different problem, called Voronoi recognition problem, have been proposed [1] [2] [5]. Because these methods give only the decision if a given tessellation is a Voronoi diagram precisely or not, that is to say, give no measure of closeness to the Voronoi diagram, they don't seem to be useful in practice. But they gave this paper some useful hints.

Some paper has utilized the necessary and sufficient condition that a given tessellation is a Voronoi diagram [5]. The necessary and sufficient condition is that we can select generators satisfying the following conditions.

- Every generator is in the associated region of the given tessellation.
- Every edge in the given tessellation is the perpendicular bisector of the line connecting the generators associated with the regions on both sides of the edge.

This necessary and sufficient condition is utilized in Section 5 in this paper again.

4.2 Minimizing Area of Discrepancy

An iterative method of minimizing the area of discrepancy of a given tessellation and the Voronoi diagram generated by unknown generators has been proposed and the method has been applied to the analysis for administrative districts [3]. This criterion seems to be the most acceptable. Although the solution is guaranteed to be only a minimal one, it is useful enough in practice.

5 P-B Method

The fundamental idea of the method, called P-B method in this paper, has been proposed [4]. It utilizes only the second condition in Subsection 4.1. That is the defect. But it has some merits. First of all, it's easy to deal with in the sense that it's not necessary to consider the topological structure of a given tessellation. Furthermore, it's not necessary to use an iterative method like the method in Section 4.2. This paper proposes modified

P-B method in Subsection 5.1, gives a new intuitive interpretation of it in Subsection 5.3 and argues about the solution to the defect in Subsection 5.4.

5.1 Modified P-B Method, Perpendicularity and Bisectability

At first, we define non-perpendicularity and non-bisectability (See Figure 3). Let V_{ij} and V_{ji} be the end points of the line separating $\mathcal{R}_i$ and $\mathcal{R}_j$ in the given tessellation so that the left side of the half-line $V_{ij}V_{ji}$ is $\mathcal{R}_i$. P_j' is the reflection of P_i in the line $V_{ij}V_{ji}$. For a given tessellation and unknown generators, non-per-pendicularity p and non-bisectability b are defined as

$$|\mathcal{R}|^2 p \equiv \sum_{(i,j)\in\mathcal{A}} \left(\overrightarrow{V_{ij}V_{ji}} \cdot \overrightarrow{P_iP_j}\right)^2 , \qquad (1)$$

$$|\mathcal{R}|^2 b \equiv \sum_{(i,j)\in\mathcal{A}} (2\triangle V_{ij}V_{ji}P_i - 2\triangle V_{ji}V_{ij}P_j)^2 \quad (2)$$

respectively, where $\mathcal{A}$ is the set of couples of indices of regions adjacent to each other in the given tessellation, and m is the number of members in the set. The notation $\triangle ABC$ means the signed area of a triangle, which is positive if the path ABC turns left. Both sides of Equations (1) and (2) have the dimension of square of area. The coefficient $|\mathcal{R}|^2$ was decided so that similar magnification of the given tessellation has no effect on p and b. Then, we can formulate the problem as an optimization problem whose objective function is

$$f \equiv (1-k)p + kb \qquad (3)$$

and will be minimized, where k is a fixed real number satisfying $0 \leq k \leq 1$. We call k importance of bisectability. It is expected that we can get more detail information, which is non-perpendicularity and non-bisectability in addition to closeness to the Voronoi diagram, by means of applying this method with various values of k to an actual tessellation. This problem is solved comparatively easily because minimization of the objective function is reduced to the method of least squares. Then we can use the minimum value of the objective function f in Equation (10) as the measure of closeness to the Voronoi diagram.

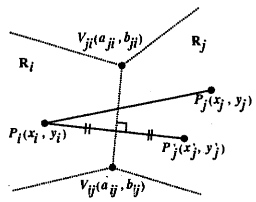

Figure 3: P-B method

5.2 Existing P-B Method, τ Value

There are 2 differences between the method above and the existing one. One of the differences is the objective functions. The other is the adopted measures of closeness to the Voronoi diagram.

The existing P-B method adopts other optimization problems, although it uses p and b [4]. Furthermore, it does not adopt f in Equation (3) but another measure, named τ Value [4], as a measure of closeness to the Voronoi diagram.

5.3 New Intuitive Interpretation of P-B Method

The second condition in Subsection 4.1 means that P_j' coincides with P_j (See Figure 3 again). A slightly different method considered in this subsection lets the objective function be the appropriately weighted mean value of the square of the length P_jP_j' over all vertices in the given tessellation.

This objective function is easier to visualize than that in Subsection 5.1. However, in fact, the objective function here is completely equivalent to that in Subsection 5.1 where the importance of bisectability k is 0.5. After all, this interpretation has the same meaning as visualization of the measure of closeness derived from P-B method in Subsection 5.1.

5.4 Method Utilizing Convex Quadratic Programming

In fact, P-B methods above do not consider the first condition in Subsection 4.1. So, generators calculated by P-B method can be outside the associated regions in a given tessellation [5].

In order to avoid this kind of undesired solutions, this paper proposes to adopt the idea of optimization problem with constraints. Concretely, we minimize the objective function in Equation (3) with the constraints representing the first condition in Subsection 4.1. Then, we obtain a convex quadratic programming problem. As is well known, convex quadratic programming problems can be solved effectively.

6 Conclusions and Discussions

Some conclusions and discussions are given as follows.

- We have modified P-B method slightly. Then, it is expected that we can get more detail information by means of applying the method.
- We got an intuitive interpretation of the objective function of P-B method.
- This paper has proposed to adopt convex quadratic programming in order to avoid the defect of P-B method.

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