Analysis of Bidirectional Associative Memory using SCSNA and Statistical Neurodynamics

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Abstract *Macroscopic properties of a bidirectional associative memory (BAM) are studied within a framework of S/N analysis called SCSNA. We obtained the relative capacity, which means the relative number of pattern pairs to be memorized and retrieved, as $0.199N$, where N means the units in the system. We also derived dynamical properties by using the statistical neurodynamics and explained the property of BAM from transient process to equilibrium state consistently.*

Keywords: BAM, SCSNA, Statistical neurodynamics

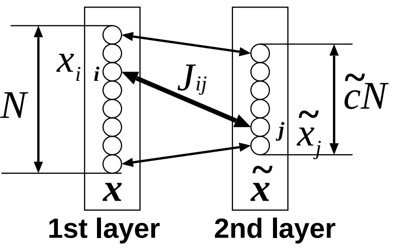

Figure 1: Network structure of BAM

1 Introduction

Bidirectional Associative Memory (BAM) proposed by Kosko [5] is a kind of associative memory neural network. The principle function of associative memory is to memorize multiple patterns and retrieve one of them when its key pattern is given.

Autocorrelation Associative Memory (AAM), sometimes called the Hopfield model [4], is also a kind of associative memory. AAM retrieves a stored pattern when the contaminated or part of it is given as the association key. This is called homogeneous association. In contrast, BAM memorizes pattern pairs and retrieves a stored pattern pair when its pattern is given as the association key. Thus, BAM is used as a heterogeneous pattern association model.

The theoretical analysis of BAM has evolved with a focus on the storage capacity, which means determining how many patterns can be stored in a network. Yanai *et al.* suggested that BAM could be regarded as a kind of AAM, whose connections are systematically removed [9]. They reported the relative storage capacity of BAM, in which a finite amount of retrieval error rate is allowed, to be around $0.22N$.

Recently, Tanaka *et al.* analyzed BAM using a replica method (see [3]), and showed its relative capacity to be $\alpha_c = 0.1998$ [8]. Analysis by replica method is intended for the equilibrium state of BAM, and its dynamical properties are ignored. However, it is important to investigate the transient property of BAM.

In this paper, we analyze the equilibrium state of BAM by Self-Consistent Signal-to-Noise Analysis (SCSNA), which is another equilibrium state analysis method, also known as the cavity method [7]. We show that the relative capacity is $0.199N$, which is identical to the result of Tanaka *et al.* Next, we derive macroscopic dynamical equations from the SCSNA result, and compare them with the computer simulation result. As a result, we show that the dynamic behavior of the macroscopic quantity can be explained by our equations.

We describe BAM formulation in Section 2. In Section 3, we give the results of SCSNA analysis equilibrium. In Section 4, we derive the dynamical equations. We compare the results of various step analyses with the computer simulation result in Section 5. Section 6 concludes our paper.

2 Formulation

BAM can be represented by a two-layered model such as in Figure 1. In the figure, the first layer consists of cN neural units, and the second of $\tilde{c}N$ units. The update rules of the ith unit in the first layer and the jth unit in the second layer are represened as:

$$x_i^{2t} = F(\sum_{j=1}^{\tilde{c}N} J_{ij}\tilde{x}_j^{2t-1}), \quad \tilde{x}_j^{2t-1} = F(\sum_{i=1}^{cN} J_{ij}x_i^{2t-2}), \tag{1}$$

where $F(\cdot)$ is the output function, sometimes represented by a sigmoid function such as $\tanh(\cdot)$ which we used in our simulation. We assumed $F(\cdot)$ as the differentiable function in our analysis.

We adopted a synchronous update rule in each layer, i.e., when $2t = 0$, we set the ith unit in the first layer to x_i^0 for all $1 \le i \le cN$. In the next step $2t-1 = 1$, the whole units in the second layer $\tilde{x}_j^1$ are updated by the right equation in (1). Then, the whole units in the first layer are updated by the left equation in (1). This alternate update is a characteristic of BAM.

J_{ij} denotes the connection weight, and we assumed correlation-based learning as follows: $J_{ij} = \frac{1}{N}\sum_{\mu=1}^{\alpha N} \xi_i^\mu \tilde{\xi}_j^\mu$, where $\boldsymbol{\xi}^\mu, \tilde{\boldsymbol{\xi}}^\mu (\mu = 1, \cdots, \alpha N)$ are pattern pairs for association and μ denotes the pattern pair index. We assumed αN pattern pairs were stored in the network. The quantity $\alpha (0 \sim 1)$ controls the amount of pattern pairs to be stored. Therefore α is commonly used as the capacity index. Moreover, each pattern pair $(\boldsymbol{\xi}^\mu, \tilde{\boldsymbol{\xi}}^\mu)$ is generated by $\mathrm{Prob}[\xi_i^\mu] = \mathrm{Prob}[\tilde{\xi}_j^\mu] = 1/2$.

3 Equilibrium state analysis

Self-Consistent Signal-to-Noise Analysis (SCSNA) was developed by Shiino & Fukai [7]. Since SCSNA deals with the equilibrium state of a system, the time index variable t becomes negligible.

Following the prescriptions of SCSNA, we introduced overlaps between the equilibrium states $(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ and the μth pattern pair $(\boldsymbol{\xi}^\mu, \tilde{\boldsymbol{\xi}}^\mu)$ as follows:

$$m_\mu = \frac{1}{cN}\sum_{i=1}^{cN} x_i \xi_i^\mu, \quad \tilde{m}_\mu = \frac{1}{\tilde{c}N}\sum_{j=1}^{\tilde{c}N} \tilde{x}_j \tilde{\xi}_j^\mu. \quad (2)$$

Following the method of S/N analysis, we need to decouple the input into signal and noise. Assuming the first pattern pair is retrieved, the overlap of pair m_1 and $\tilde{m}_1$, which indicates how well the first pattern pair is retrieved, is the signal pattern. Thus, we can derive equilibrium equation as:

$$x_i = F(\tilde{c}\tilde{m}_1 \xi_i^1 + z_i), \quad \tilde{x}_j = F(c m_1 \tilde{\xi}_j^1 + \tilde{z}_j), \quad (3)$$

where $z_i, \tilde{z}_j$ are crosstalk noises, which indicate the effects from other $(\mu = 2, \cdots, \alpha N)$ pattern pairs. These crosstalk noises can be described as:

$$z_i = \frac{\sum_{\mu=2}^{\alpha N}\sum_{j=1}^{\tilde{c}N} \xi_i^\mu \tilde{\xi}_j^\mu \tilde{x}_j}{N}, \quad \tilde{z}_j = \frac{\sum_{\mu=2}^{\alpha N}\sum_{i=1}^{cN} \tilde{\xi}_j^\mu \xi_i^\mu x_i}{N}. \quad (4)$$

SCSNA [7] evaluates the effective self-depend term, which comes from the νth pattern pair, in the crosstalk noises. We took these effects into consideration, and derived self-consistent equations called order parameter equations. The followings are the order parameter equations of BAM.

$$\begin{aligned}
Y_i &= F(\tilde{c}\tilde{m}^1 \xi_i^1 + \frac{\alpha \tilde{c}\tilde{U}}{1 - c\tilde{c}U\tilde{U}} Y_i + \sqrt{\alpha r} z), \\
\tilde{Y}_j &= F(c m^1 \tilde{\xi}_j^1 + \frac{\alpha c U}{1 - c\tilde{c}U\tilde{U}} \tilde{Y}_j + \sqrt{\alpha \tilde{r}} z), \\
m^1 &= \int Dz \langle \xi_i^1 Y_i \rangle_i, \quad \tilde{m}^1 = \int Dz \langle \tilde{\xi}_j^1 \tilde{Y}_j \rangle_j, \\
q &= \int Dz \langle Y_i^2 \rangle_i, \quad \tilde{q} = \int Dz \langle \tilde{Y}_j^2 \rangle_j, \\
U &= \frac{1}{\sqrt{\alpha r}} \int Dz z \langle Y_i \rangle_i, \quad \tilde{U} = \frac{1}{\sqrt{\alpha \tilde{r}}} \int Dz z \langle \tilde{Y}_j \rangle_j, \\
r &= \frac{\tilde{c}}{(1 - c\tilde{c}U\tilde{U})^2} (\tilde{q} + c\tilde{c}\tilde{U}^2 q), \\
\tilde{r} &= \frac{c}{(1 - c\tilde{c}U\tilde{U})^2} (q + c\tilde{c}U^2 \tilde{q}). \quad (5)
\end{aligned}$$

We have described the order parameter equations (5) in the manner of Shiino & Fukai [7]. Note that the operator $\langle\cdot\rangle$ means the expectations for the stochastic variable ξ_i^1 or $\tilde{\xi}_j^1$, which yield to the distribution for generating stored patterns. The stored pattern pairs can be considered to be the set of stochastic variables which come from independent and identical distribution (i.i.d.). Thus, this substitution is reasonable and proper.

In equations (5), Y_i and $\tilde{Y}_j$ mean x_i and $\tilde{x}_j$ respectively. The inner summation of function $F(\cdot)$ consists of three parts. The first term comes from the signal term, the second is the self-dependent term in the crosstalk noise, and the third term comes from the other crosstalk noise components. We assumed each noise term to be an independent Gaussian noise. Thus, we evaluated the means and the variances of noise terms which are given by equation (4). Both means are equal to 0, and the variances are equal to αr and $\alpha \tilde{r}$ respectively. We deal with this noise term by averaging, which can be described by the integral of standard normal distribution as follows: $\int Dz = \frac{1}{\sqrt{2\pi}} \int dz \exp(-\frac{z^2}{2})$.

We solved the order parameter equations (5) numerically and obtained the critical capacity $\alpha_c = 0.199$. This result agrees with the Tanaka *et al.*'s result ($\alpha_c = 0.1998$) which they obtained by using replica method[8]. In the simulations, we conducted 10 trials and indicated medians with quartile deviations. SCSNA analysis also quantitatively explained simulations very well.

4 BAM's dynamics

As we have seen, SCSNA can the describe equilibrium state of BAM. In this section, we consider the dynamics which describes transient state of BAM. In the limit of the dynamics, it should be settled into the equilibrium state. The dynamics equations can be derived from the same concept of SCSNA. First, we used one-step analysis method

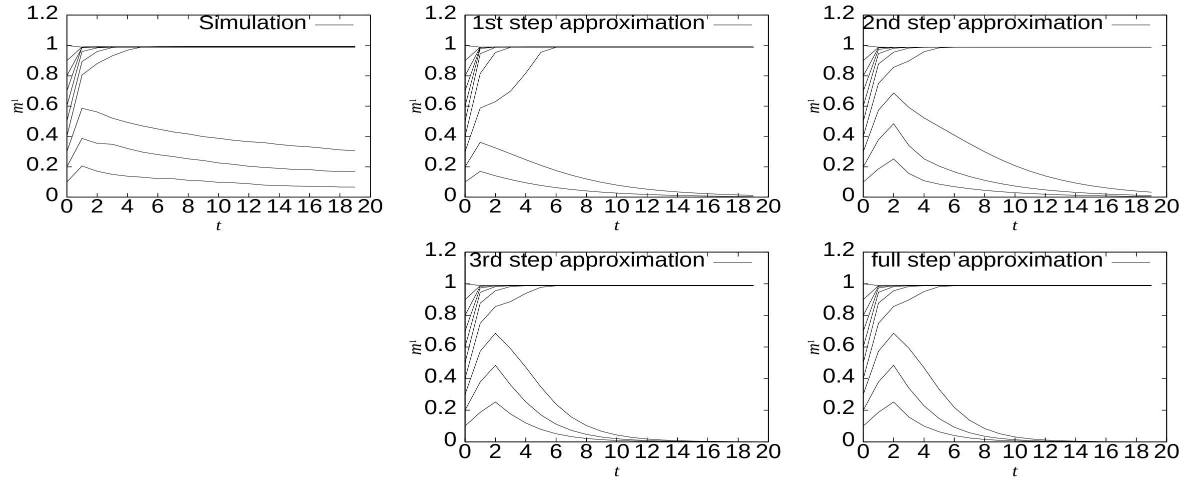

Figure 2: Time dependence of evolution of the overlap m_1

proposed by Amari & Maginu [2]. Then, we extended the one-step analysis to statistical neurodynamics.

Amari & Maginu proposed an analysis method called the one-step theory [2] which is derived from the S/N analysis, and applied it to AAM. Here, we applied the one-step analysis to BAM and obtained:

$$\begin{aligned}
Y_i^{2t} &= F(\tilde{c}\tilde{m}^1_{2t-1}\xi^1_i + \sqrt{\alpha r_{2t-1}}z),\\
\tilde{Y}_j^{2t+1} &= F(cm^1_{2t}\tilde{\xi}^1_j + \sqrt{\alpha\tilde{r}_{2t}}z),\\
m^1_{2t} &= \int Dz\langle\xi^1_i Y_i^{2t}\rangle_i, \quad \tilde{m}^1_{2t+1} = \int Dz\langle\tilde{\xi}^1_j\tilde{Y}_j^{2t+1}\rangle_j,\\
q_{2t} &= \int Dz\langle(Y_i^{2t})^2\rangle_i, \quad \tilde{q}_{2t+1} = \int Dz\langle(\tilde{Y}_j^{2t+1})^2\rangle_j,\\
U_{2t} &= \frac{1}{\sqrt{\alpha r_{2t-1}}}\int Dzz\langle Y_i^{2t}\rangle_i,\\
\tilde{U}_{2t+1} &= \frac{1}{\sqrt{\alpha\tilde{r}_{2t}}}\int Dzz\langle\tilde{Y}_j^{2t+1}\rangle_j,\\
r_{2t+1} &= \tilde{c}(\tilde{q}_{2t+1} + c\tilde{c}\tilde{U}^2_{2t+1}q_{2t}),\\
\tilde{r}_{2t} &= c(q_{2t} + c\tilde{c}U^2_{2t}\tilde{q}_{2t-1}).
\end{aligned} \tag{6}$$

The important point is that these recurrence formulae can be described by the one-step before state. These equations are identical form to those of the sequence association model, which is a variety of AAM, analyzed by Amari [1]. We derived critical capacity as the limit of the dynamics (6) and obtained $\alpha_c = 0.27$, which was also suggested by Amari. However, this critical capacity seems to be overestimated. We consider that the reason for this overestimation is the assumption that noise distribution parameters r_{2t+1}, and $\tilde{r}_{2t}$ have no correlation with the previous state in each update. Therefore, we need to evaluate these noise correlations exactly in accordance with the concept of SCSNA.

Thus, we introduced the statistical neurodynamics proposed by Okada [6]. Just like one-step analysis derives from S/N analysis, statistical neurodynamics analysis is also derived from SCSNA. By using statistical neurodynamics, Okada succeeded in describing the transient process of AAM by recurrence formulae of macroscopic parameters [6]. This analysis evaluated the noise correlation more exact rather than one-step analysis.

To apply statistical neurodynamics to BAM, we needed only to evaluate the order-parameters r_{2t+1} and $\tilde{r}_{2t}$. There was no need to re-evaluate the other parameters $Y_i^{2t}, \tilde{Y}_j^{2t+1}, m^1_{2t}, \tilde{m}^1_{2t+1}, q_{2t}, \tilde{q}_{2t+1}, U_{2t}, \tilde{U}_{2t+1}$.

Thus, we expanded crosstalk noises by considering the effect of the n-step before state:

$$\begin{aligned}
r_{2t+1} &= \tilde{c}\tilde{q}_{2t+1} + c(\tilde{c}\tilde{U}_{2t+1})^2 q_{2t}\\
&\quad + 2\tilde{c}\sum_{\eta=1}^{n}(c\tilde{c})^\eta\tilde{q}_{2t+1,2(t-\eta)+1}\prod_{\tau=t-\eta+1}^{t}\tilde{U}_{2\tau+1}U_{2\tau}\\
&\quad + 2c(\tilde{c}\tilde{U}_{2t+1})^2\sum_{\eta=1}^{n-1}(c\tilde{c})^\eta q_{2t,2(t-\eta)}\prod_{\tau=t-\eta+1}^{t}U_{2\tau}\tilde{U}_{2\tau-1}\\
&\quad + (c\tilde{c}\tilde{U}_{2t+1}U_{2t})^2 r_{2t-1}
\end{aligned} \tag{7}$$

$$\begin{aligned}
\tilde{r}_{2t} &= cq_{2t} + \tilde{c}(cU^2_{2t})^2\tilde{q}_{2t-1}\\
&\quad + 2c\sum_{\eta=1}^{n}(\tilde{c}c)^\eta q_{2t,2(t-\eta)}\prod_{\tau=t-\eta+1}^{t}U_{2\tau}\tilde{U}_{2\tau-1}\\
&\quad + 2\tilde{c}(cU_{2t})^2\sum_{\eta=1}^{n-1}(\tilde{c}c)^\eta\tilde{q}_{2t-1,2(t-\eta)-1}\prod_{\tau=t-\eta+1}^{t}\tilde{U}_{2\tau-1}U_{2\tau-2}\\
&\quad + (\tilde{c}cU_{2t}\tilde{U}_{2t-1})^2\tilde{r}_{2t-2}
\end{aligned} \tag{8}$$

The first two terms in equations (7) and (8) also appeared in the one-step analysis equations (6). Thus, the residual terms are important in this analysis. The quantities $\tilde{q}_{2t+1,2(t-n)+1}$ and $q_{2t,2(t-n)}$ are the auto-correlations between the current state (whose suffix is described as $2t+1$ or

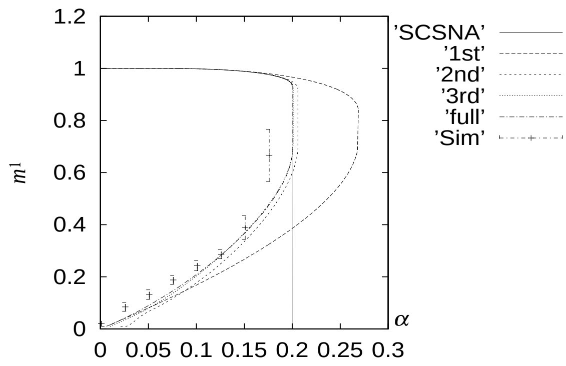

Figure 3: Capacity comparing the statistical neurodynamics with SCSNA

$2t$) and n-step before state (described as $2(t-n)+1$ or $2(t-n)$).

5 Result

We compared the derived macroscopic parameter dynamics with the computer simulation. Figure 2 shows the time developing of the overlap m_1, which means how well the pattern $\boldsymbol{\xi}^1$ is retrieved in the first layer. Each abscissa axis is a time step and the ordinate axis is the overlap m^1_{2t}. Converging the overlap m^1_{2t} to 1.0 means successful retrieval. In these figures, each line shows overlap evolution with several initial overlap states.

The figure in the first column of the top row shows a computer simulation result. We chose a capacity index as $\alpha = 0.15$ and the units number $N = 10,000$. In the simulation, retrieval was successful when we set the initial overlap to be larger than 0.4.

The figure in the second column of the top row shows the result of the one-step analysis and third column shows the result of two-step analysis. As shown in the one-step analysis, retrieval succeeded when the initial overlap state was 0.3. This did not agree with the simulation result. On the other hand, the two-step analysis result exhibited the same behavior to as the simulation.

The first column of the bottom row shows the three-step analysis result and the second column shows the full-step analysis, which means expanding noise correlation tracing to the initial state. As far as we can see from these figures, three-step analysis can approximate the simulation and it is very similar to the full-step analysis.

Figure 3 shows the equilibrium state and the basin derived from statistical neurodynamics. The abscissa axis is the capacity index α and the ordinate axis is the overlap m_1. The solid line shows the SCSNA result. The dashed lines are derived from statistical neurodynamics. In each curve derived from neurodynamics, the upper part shows the equilibrium overlap m^1_∞ and the lower part shows the basin, which means the retrievable limit of initial overlap m^1_0. We also show the basin derived from the simulation result in Figure 3. The two-step and above analyses show agreement with these simulation results.

6 Conclusion

In this research, we derived the macroscopic parameter of BAM in the equilibrium state with SCSNA and showed that the result agreed with the Tanaka *et al.*'s result. Moreover, we confirmed that the result also agreed with a computer simulation. Then, we analyzed the transient process of BAM by using statistical neurodynamics and confirmed that this analysis could explain the process from transient to equilibrium state. As a result, when we consider the three-step analysis of the previous correlation, the computer simulation result can be explained quantitatively.

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