割当て問題の拡張について
—最悪コスト最小化とコストベクトル化割当て問題

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概要

割当て問題の拡張したモデルを提示し, その解法について考察した. 本論文では, 最悪のコストを最小化する問題と, コストをベクトル化した問題の二種類の拡張について考えた. 最悪コスト最小化問題については, 簡単な解法を示し, この問題が多項式時間で解けることを示した. また, コストをベクトル化した問題では, パラメトリックな解析を提案した. そこで, パラメトリックな割当て問題の最適解がこの問題の準最適解を与えることを示した.

Variants of the Assignment Problem

— Worst Cost Minimization and Vector Cost Assignment —

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Abstract

Extended models of the well known assignment problem are presented, and new algorithms are proposed and analyzed. The extensions are made in two directions; worst cost minimization and vector cost assignment. For worst cost minimization, a simple algorithm is proposed and the time complexity is shown to be polynomial. For vector cost assignment, parametric analysis is proposed. We show that the optimal solution of the parametric assignment problem is a quasi optimal solution for the vector cost assignment.

1 Introduction

Suppose you are an employer and you have n employees for n jobs. Each employee can do only one job, and each job is to be done by only one employee, so you must assign a job to each employee. Since employees have different skills, you should find the "optimal" assignment for them. Let the *cost* for the ith employee to do the jth job be denoted by c_{ij}. Assuming that it is the total cost you want to minimize, then we can state the assignment problem as,

P0: Minimize

$$z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n),$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n),$$

$$x_{ij} \geq 0 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n). \qquad \square$$

The variable x_{ij} takes value 1 if the jth job is assigned to the ith employee, and 0 otherwise. P0 is a linear programming problem, and it is well known that the optimal solution of P0 is integer valued, i.e., 0 or 1.

It is also known that the assignment problem is a special case of the minimum cost flow problem in a network [3]. It is clear that we can solve the assignment problem with $O(n^4)$ time complexity. And, as the progress is made in minimum cost flow algorithms and computational geometry, faster assignment algorithms have been proposed, e.g., Tomizawa [8], Tokuyama and Nakano [7], etc.

The objective function of problem P0 is to minimize total cost. It may happen that under such an optimal solution some employees are assigned very inadequate jobs. To avoid such cases, let us consider minimization by another objective function:

$$w = \max c_{ij} x_{ij},$$

where the maximum is taken over all $i = 1, \ldots, n$ and $j = 1, \ldots, n$. This new approach can be stated as the following problem:

P1: Minimize $\quad w$

subject to

$$\begin{aligned}
& c_{ij} x_{ij} \le w \qquad (i = 1, \ldots, n; j = 1, \ldots, n), \\
& \sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n), \\
& \sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n), \\
& x_{ij} = 0 \text{ or } 1 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n). \quad \square
\end{aligned}$$

Sometimes costs are given in vector form, i.e., the cost for the ith employee to do the jth job is evaluated as c_{ij} from one point of view and as c'_{ij} from another point of view. There are two total costs, $\sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}$ and $\sum_{j=1}^{n} \sum_{i=1}^{n} c'_{ij} x_{ij}$, so the problem is:

P2: Minimize $\quad z$

subject to

$$\begin{aligned}
& \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij} \le z, \\
& \sum_{j=1}^{n} \sum_{i=1}^{n} c'_{ij} x_{ij} \le z, \\
& \sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n), \\
& \sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n), \\
& x_{ij} = 0 \text{ or } 1 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n). \quad \square
\end{aligned}$$

Problem P2 is an integer programming problem, because the unimodularity of the constraints does not hold.

Sometimes we would like to minimize the largest element of $\max c_{ij}$ and $\max c'_{ij}$ instead of $\sum_j \sum_i c_{ij} x_{ij}$ and $\sum_j \sum_i c'_{ij} x_{ij}$. Thus we have

P3: Minimize $\quad w$

subject to

$$\begin{aligned}
& \max_{i,j=1,\ldots,n} c_{ij} x_{ij} \le w, \\
& \max_{i,j=1,\ldots,n} c'_{ij} x_{ij} \le w, \\
& \sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n), \\
& \sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n), \\
& x_{ij} = 0 \text{ or } 1 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n). \quad \square
\end{aligned}$$

Our goal is to propose algorithms for problems P1, P2 and P3. In **2** we restate problem P1 and show that it can be solved by repeatedly finding, in the fashion of a binary search, the maximum matching of bipartite graphs. The time complexity of this algorithm is of a polynomial order. In **3** we show that we can solve P3 in almost the same way as P1 and that the time complexity is the same. In **4** we consider problem P2, which is a compound problem of the ordinary assignment problem. Rather than solving for P2 exactly, which is NP-hard, we find a solution that is better than the two known subproblem solutions. The algorithm is also shown to be of polynomial time complexity.

2 Problem P1: Minimizing the Maximum Cost

In order to state the algorithm for P1, let us first consider the following problem.

Problem Q(ξ): Given ξ, determine if there is a feasible solution for

$$\begin{aligned}
& c_{ij} x_{ij} \le \xi \qquad (i = 1, \ldots, n; j = 1, \ldots, n), \\
& \sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n),
\end{aligned}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n),$$
$$x_{ij} \geq 0 \quad (i = 1, \ldots, n; j = 1, \ldots, n). \qquad \square$$

Problem Q(ξ) is easily solved by the following algorithm:

Step 1 Create an n by n bipartite graph, where the ith left vertex and the jth right vertex are connected by an edge if $c_{ij} \leq \xi$.
Step 2 Solve the maximum matching problem in the above bipartite graph. $\square$

If the maximum matching obtained in Step 2 includes n edges, Q(ξ) is feasible; otherwise, it is infeasible. Since an $O(n^{5/2})$ algorithm for the maximum matching problem is known [4], Q(ξ) can be solved in time complexity $O(n^{5/2})$.

We can solve the problem P1 by solving the Q(ξ)'s for various ξ's with binary searches. The algorithm for P1 is :

Arrange the c_{ij}'s in ascending order;
Let $l = 1$ and $r = n^2$;
while $(l < r - 1)$
  begin
    Let $m = \lceil (l + r)/2 \rceil$
      and ξ be the mth smallest
      value of the c_{ij}'s;
    if Q(ξ) is infeasible
      then let $l = m$ **else** $r = m$
  end

After the algorithm completes, the rth smallest value for c_{ij} gives the value of the objective function of P1, and the corresponding maximum matching is the optimal solution of P1. Apparently, this algorithm is of time complexity $O(n^{5/2} \log n)$.

3 Problem P3: Minimizing the Maximum Cost under Vector Cost Assignment

Problem P3 is solved in almost the same way as P1. Instead of Q(ξ), we consider the problem R(ξ), where the constraints are the same as in Q(ξ) except that

$$c'_{ij} x_{ij} \leq \xi \qquad (i = 1, \ldots, n; j = 1, \ldots, n),$$

is added. Then R(ξ) is solved in almost the same way as Q(ξ) except that in the bipartite graph the ith left vertex and the jth right vertex are connected if $c_{ij} \leq \xi$ and $c'_{ij} \leq \xi$.

The algorithm for P3 is almost the same as for P1 except that Q(ξ) is replaced by R(ξ), so the time complexity is $O(n^{5/2} \log n)$.

4 Problem P2 and the Parametric Assignment Problem

Since problem P2 is difficult to solve, we consider a modified version of P2, i.e., a parametric assignment problem:

Q2: Minimize

$$z = t \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij} + (1 - t) \sum_{j=1}^{n} \sum_{i=1}^{n} c'_{ij} x_{ij}(t)$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, \ldots, n),$$
$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, \ldots, n),$$
$$x_{ij} \geq 0 \quad (i = 1, \ldots, n;\ j = 1, \ldots, n) \qquad \square$$

for a given t $(0 \leq t \leq 1)$.

If t is fixed, problem Q2 is an ordinary assignment problem.

Let us denote the optimal solution of Q2 by $\hat{x}_{ij}(t)$ and the value of the objective function by $F(t)$. Note that $\hat{x}_{ij}(1)$ and $\hat{x}_{ij}(0)$ are the optimal solutions of the ordinary assignment problem with costs c_{ij} and c'_{ij}. Noting that there are only a finite number of distinct $\hat{x}_{ij}(t)$'s (at most $n!$ in total), we have

Lemma 1 $F(t)$ is piecewise linear.

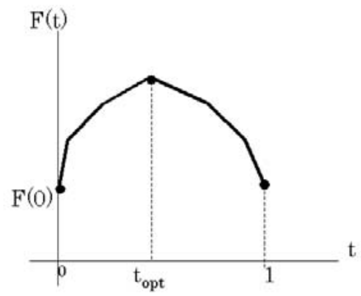

Figure 1: F(t)

Also, we can show
Lemma 2 $F(t)$ is concave, i.e.,

$$F(\lambda t_1 + (1 - \lambda) t_2) \geq \lambda F(t_1) + (1 - \lambda) F(t_2)$$

for any λ $(0 \leq \lambda \leq 1)$, where $0 \leq t_1 \leq 1$ and $0 \leq t_2 \leq 1$.

Let the maximum value of $F(t)$ $(0 \leq t \leq 1)$ be $F(t_{\rm opt})$ $(0 \leq t_{\rm opt} \leq 1)$. Then, we have

Theorem 1

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(t_{\rm opt}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(0),$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\hat{x}_{ij}(t_{\rm opt}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\hat{x}_{ij}(1).$$

Proof

Since $\hat{x}_{ij}(t_{\rm opt})$ is the optimal solution for the assignment problem with cost $t_{\rm opt}c_{ij} + (1 - t_{\rm opt})c'_{ij}$,

$$F(t_{\rm opt}) \ \leq \ \sum_{j=1}^{n}\sum_{i=1}^{n}\{t_{\rm opt}c_{ij} + (1 - t_{\rm opt})c'_{ij}\}\hat{x}_{ij}(1)$$

However,

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(1) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(t_{\rm opt}).$$

Therefore

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\hat{x}_{ij}(t_{\rm opt}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c'_{ij}\hat{x}_{ij}(1).$$

Similarly, we have

$$\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(t_{\rm opt}) \leq \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\hat{x}_{ij}(0).$$

Q.E.D.

Theorem 1 shows that $\hat{x}_{ij}(t_{\rm opt})$ is a "better" solution than $\hat{x}_{ij}(0)$ or $\hat{x}_{ij}(1)$ for P2.

Finally, we show how to obtain $t_{\rm opt}$.
Since, the function $F(t)$ is a piecewise linear and concave function, it is clear that

$$\frac{d}{dt}F(t) > 0 \qquad \text{when} \qquad t < t_{\rm opt}$$

and

$$\frac{d}{dt}F(t) < 0 \qquad \text{when} \qquad t > t_{\rm opt}.$$

Note that

$$\frac{d}{dt}F(t) = \sum_{j=1}^{n}\sum_{i=1}^{n}(c_{ij} - c'_{ij})\hat{x}_{ij}(t),$$

where $\hat{x}_{ij}(t)$ is the optimal solution of Q2. The following algorithm based on a binary search gives such a $t_{\rm opt}$, where ϵ is the so called "machine epsilon," i.e., the smallest value that a computer is able to handle.

Algorithm

$l = 0 \ \ r = 1;$
while $(r - l > \epsilon)$
  begin
    $t = (l + r)/2$; solve Q2 from t;
    if $\frac{d}{dt}F(t) > 0$ **then** $l = t$ **else** $r = t$
  end

Since the "while" loop repeats L times, where L is the number of bits in the computer "word," the time complexity is $O(n^3L)$.

5 Conclusions

We have extended the classical assignment problem in two directions, i.e., worst cost minimization and vector cost assignment. For the former, we proposed a simple and efficient algorithm. For the latter, we considered a parametric assignment problem whose optimal solution is better than the known ones. Detailed analysis and the development of more efficient algorithms of the parametric assignment problem are left for further research.

References

[1] Ahuya, R. K., A. V. Goldberg, J. B. Orlin, and R. E. Tarjan, Finding minimum-cost flows by double scaling, *Mathematical Programming, Ser. A*, **53**, 243-266 (1992).

[2] Dantzig, G. B., *Linear Programming and Extensions*, Princeton Univ. Press, 1963.

[3] Ford, L. R. Jr. and D. R. Fulkerson, *Flows in Networks*, Princeton Univ. Press, 1962.

[4] Hopcroft, J. E. and R. M. Karp, An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs, *SIAM J. Computing*, **4**, 225-231 (1973).

[5] Kamura, Y., M. Nakamori, and Y. Shinano, Combining imperfect components (II) — The case of multidimensional error, *Proc. PDPTA 2002*, **1**, 228-232 (2002).

[6] Tardos, É., A strongly polynomial minimum cost circulation algorithms, *Combinatorica*, **5**, 247-255 (1985).

[7] Tokuyama, T. and J. Nakano, Geometric algorithms for a minimum cost assignment problem, *Proc. 7th Annual Symp. Computational Geometry*, 262-271 (1991).

[8] Tomizawa, N., On some techniques useful for solution of transportation network problems, *Networks*, **1**, 173-194 (1971).