ハミルトン C_k-Bowtie デザイン

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アブストラクト: グラフ理論において、グラフの分解問題は主要な研究テーマである。C_k を k 点を通るサイクルとする。1点を共有する辺素な2個の C_k からなるグラフを C_k-Bowtie という。本研究では、完全多重グラフ λK_n を C_k-Bowtie 全域部分グラフに分解する組合せデザインについて述べる。

キーワード: C_k-bowtie 分解; 完全多重グラフ; グラフ理論

Hamilton C_k-Bowtie Designs

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Abstract: In graph theory, the decomposition problem of graphs is a very important topic. Various types of decompositions of many graphs can be seen in the literature of graph theory. This paper gives a Hamilton C_k-bowtie decomposition of the complete multi-graph λK_n.

Keywords: C_k-bowtie decomposition; Complete multi-graph; Graph theory

1. Introduction

Let K_n denote *the complete graph* of n vertices. *The complete multi-graph* λK_n is the complete graph K_n in which every edge is taken λ times. Let C_k be *the k-cycle* (or *the cycle* on k vertices). *The C_k-bowtie* is a graph of 2 edge-disjoint C_k's with a common vertex and the common vertex is called *the center of the C_k-bowtie*. In particular, a C_k-bowtie satisfying $n = 2(k-1)+1$ is called *the Hamilton C_k-bowtie* because the C_k-bowtie spans λK_n.
When λK_n is decomposed into edge-disjoint sum of Hamilton C_k-bowties, we say that λK_n *has a Hamilton C_k-bowtie decomposition.* This Hamilton C_k-bowtie decomposition of λK_n is called *a Hamilton C_k-bowtie design.*

In this paper, it is shown that the necessary condition for the existence of a Hamilton C_k-bowtie decomposition of λK_n is (i) $n = 2(k-1)+1$ and (ii) $\lambda \equiv 0 \pmod{k}$ for odd k, $\lambda \equiv 0 \pmod{2k}$ for even k. Decomposition algorithms are also given.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[2] and Wallis[15]. Horák and Rosa[3] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or 9 (mod 12). This decomposition is known as *a C_3-bowtie system.* For combinatorial designs, see [1,4,5,15]. Another type of foil-decompositions, see [6–14].

2. Hamilton C_k-bowtie decomposition of λK_n

Notation. We consider the vertex set V of λK_n as $V = \{1, 2, ..., n\}$. We denote a Hamilton C_k-bowtie passing through $v_1 - v_2 - v_3 - ... - v_k - v_1$, $v_1 - v_{k+1} - v_{k+2} - ... - v_{2k-1} - v_1$ by $(v_1, v_2, v_3, ..., v_k) \cup (v_1, v_{k+1}, v_{k+2}, ..., v_{2k-1})$.

Theorem 1. If λK_n has a Hamilton C_k-bowtie decomposition, then (i) $n = 2(k-1)+1$ and (ii) $\lambda \equiv 0 \pmod k$ for odd k, $\lambda \equiv 0 \pmod{2k}$ for even k.
Proof. When $n = 2(k-1)+1$, suppose that λK_n is decomposed into b Hamilton C_k-bowties. Then $b = \lambda n(n-1)/4k = \lambda(2k-1)(k-1)/2k$. Thus, $\lambda \equiv 0 \pmod k$ for odd k, $\lambda \equiv 0 \pmod{2k}$ for even k.

Theorem 2. If λK_n has a Hamilton C_k-bowtie decomposition, then $(s\lambda)K_n$ has a Hamilton C_k-bowtie decomposition for every s.
Proof. Obvious. Repeat s times the Hamilton C_k-bowtie decomposition of λK_n.

Theorem F. Let p be prime and a be integer. Then $a^p \equiv a \pmod p$.

Corollaly F1. Let p be prime and $(a,p) = 1$. Then $a^{p-1} \equiv 1 \pmod p$.

Corollaly F2. Let p be prime and $(a,p) = 1$. Then $sa^{p-1} \equiv s \pmod p$ for $1 \le s \le p-1$.

Definition. When $sa^{n-1} \equiv s \pmod n$, let $a_i = \underline{sa^{i-1} \bmod n}$ $(i = 1, 2, ..., n)$ for $1 \le s \le n-1$. Find the first i $(i = 2, 3, ..., n)$ such that $a_i = s$. Put the i be L.
Then the sequence $a_1(=s), a_2(=sa), a_3(=sa^2), ..., a_L(=s)$ is called *an L-orbit starting s.*
When there exist $(n-1)$ L-orbits starting $1, 2, ..., n-1$, we say that n *admits L-orbits.*

Note. Let p be prime. It is a widely known result that p admits p-orbits and that a is called *a primitive root w.r.t. mod p.* In particular, the least a denoted g is called *the least primitive root w.r.t mod p.*

Example 1. (p, g) table.
$(p, g) = (2,1), (3,2), (5,2), (7,3), (11,2), (13,2), (17,3), (19,2), (23,5), (29,2), (31,3), (37,2), (41,6), (43,3), (47,5), (53,2), (59,2), (61,2), (67,2), (71,7), (73,5), (79,3), (83,2), (89,3), (97,5)$.

Example 2. p-orbit.
$(p, g) = (5,2)$ p-orbit : $1, 2, 4, 3, 1$.
$(p, g) = (7,3)$ p-orbit : $1, 3, 2, 6, 4, 5, 1$.
$(p, g) = (11,2)$ p-orbit : $1, 2, 4, 8, 5, 10, 9, 7, 3, 6, 1$.
$(p, g) = (13,2)$ p-orbit : $1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1$.
$(p, g) = (17,3)$ p-orbit : $1, 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$.
$(p, g) = (19,2)$ p-orbit : $1, 2, 4, 8, 16, 13, 7, 14, 9, 18, 17, 15, 11, 3, 6, 12, 5, 10, 1$.
$(p, g) = (23,5)$ p-orbit : $1, 5, 2, 10, 4, 20, 8, 17, 16, 11, 9, 22, 18, 21, 13, 19, 3, 15, 6, 7, 12, 14, 1$.

Theorem 3. Let n be prime. When $n = 2(k-1)+1$, $\lambda \equiv 0 \pmod{k}$, and k odd, λK_n has a Hamilton C_k-bowtie decomposition.

Example 3.1. Hamilton C_3-bowtie of $3K_5$.
$(n, g) = (5, 2)$ n-orbit : $1, 2, 4, 3, 1$.
$L_1 : 1, 4, 1$ $L_2 : 2, 3, 2$.
Hamilton C_3-bowtie $= (5, 1, 4) \cup (5, 2, 3)$.
This starter comprises a Hamilton C_3-bowtie decomposition of $3K_5$.

Example 3.2. Hamilton C_7-bowtie of $7K_{13}$.
$(n, g) = (13, 2)$ n-orbit : $1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1$.
$L_1 : 1, 4, 3, 12, 9, 10, 1$ $L_2 : 2, 8, 6, 11, 5, 7, 2$.
Hamilton C_7-bowtie $= (13, 1, 4, 3, 12, 9, 10) \cup (13, 2, 8, 6, 11, 5, 7)$
Hamilton C_7-bowtie $= (13, 4, 3, 12, 9, 10, 1) \cup (13, 8, 6, 11, 5, 7, 2)$
Hamilton C_7-bowtie $= (13, 3, 12, 9, 10, 1, 4) \cup (13, 6, 11, 5, 7, 2, 8)$.
These starters comprise a Hamilton C_7-bowtie decomposition of $7K_{13}$.

Example 3.3. Hamilton C_9-bowtie of $9K_{17}$.
$(n, g) = (17, 3)$ n-orbit : $1, 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6, 1$.
$L_1 : 1, 9, 13, 15, 16, 8, 4, 2, 1$ $L_2 : 3, 10, 5, 11, 14, 7, 12, 6, 3$.
Hamilton C_9-bowtie $= (17, 1, 9, 13, 15, 16, 8, 4, 2) \cup (17, 3, 10, 5, 11, 14, 7, 12, 6)$
Hamilton C_9-bowtie $= (17, 9, 13, 15, 16, 8, 4, 2, 1) \cup (17, 10, 5, 11, 14, 7, 12, 6, 3)$
Hamilton C_9-bowtie $= (17, 13, 15, 16, 8, 4, 2, 1, 9) \cup (17, 5, 11, 14, 7, 12, 6, 3, 10)$
Hamilton C_9-bowtie $= (17, 15, 16, 8, 4, 2, 1, 9, 13) \cup (17, 11, 14, 7, 12, 6, 3, 10, 5)$.
These starters comprise a Hamilton C_9-bowtie decomposition of $9K_{17}$.

Theorem 4. Let n be prime. When $n = 2(k-1)+1$, $\lambda \equiv 0 \pmod{2k}$, and k even, λK_n has a Hamilton C_k-bowtie decomposition.

Example 4.1. Hamilton C_4-bowtie of $8K_7$.
$(n, g) = (7, 3)$ n-orbit : $1, 3, 2, 6, 4, 5, 1$.
$L : 1, 3, 2, 6, 4, 5, 1$.
Hamilton C_4-bowtie $= (7, 1, 3, 2) \cup (7, 6, 4, 5)$
Hamilton C_4-bowtie $= (7, 3, 2, 6) \cup (7, 4, 5, 1)$
Hamilton C_4-bowtie $= (7, 2, 6, 4) \cup (7, 5, 1, 3)$.
These starters comprise a Hamilton C_4-bowtie decomposition of $8K_7$.

Example 4.2. Hamilton C_6-bowtie of $12K_{11}$.
$(n, g) = (11, 2)$ n-orbit : $1, 2, 4, 8, 5, 10, 9, 7, 3, 6, 1$.
$L : 1, 2, 4, 8, 5, 10, 9, 7, 3, 6, 1$.
Hamilton C_6-bowtie $= (11, 1, 2, 4, 8, 5) \cup (11, 10, 9, 7, 3, 6)$
Hamilton C_6-bowtie $= (11, 2, 4, 8, 5, 10) \cup (11, 9, 7, 3, 6, 1)$
Hamilton C_6-bowtie $= (11, 4, 8, 5, 10, 9) \cup (11, 7, 3, 6, 1, 2)$
Hamilton C_6-bowtie $= (11, 8, 5, 10, 9, 7) \cup (11, 3, 6, 1, 2, 4)$
Hamilton C_6-bowtie $= (11, 5, 10, 9, 7, 3) \cup (11, 6, 1, 2, 4, 8)$.
These starters comprise a Hamilton C_6-bowtie decomposition of $12K_{11}$.

Example 4.3. Hamilton C_{10}-bowtie of $20K_{19}$.
$(n, g) = (19, 2)$ n-orbit : $1, 2, 4, 8, 16, 13, 7, 14, 9, 18, 17, 15, 11, 3, 6, 12, 5, 10, 1$.
$L : 1, 2, 4, 8, 16, 13, 7, 14, 9, 18, 17, 15, 11, 3, 6, 12, 5, 10, 1$.

Hamilton C_{10}-bowtie = $(19, 1, 2, 4, 8, 16, 13, 7, 14, 9) \cup (19, 18, 17, 15, 11, 3, 6, 12, 5, 10)$
Hamilton C_{10}-bowtie = $(19, 2, 4, 8, 16, 13, 7, 14, 9, 18) \cup (19, 17, 15, 11, 3, 6, 12, 5, 10, 1)$
Hamilton C_{10}-bowtie = $(19, 4, 8, 16, 13, 7, 14, 9, 18, 17) \cup (19, 15, 11, 3, 6, 12, 5, 10, 1, 2)$
Hamilton C_{10}-bowtie = $(19, 8, 16, 13, 7, 14, 9, 18, 17, 15) \cup (19, 11, 3, 6, 12, 5, 10, 1, 2, 4)$
Hamilton C_{10}-bowtie = $(19, 16, 13, 7, 14, 9, 18, 17, 15, 11) \cup (19, 3, 6, 12, 5, 10, 1, 2, 4, 8)$
Hamilton C_{10}-bowtie = $(19, 13, 7, 14, 9, 18, 17, 15, 11, 3) \cup (19, 6, 12, 5, 10, 1, 2, 4, 8, 16)$
Hamilton C_{10}-bowtie = $(19, 7, 14, 9, 18, 17, 15, 11, 3, 6) \cup (19, 12, 5, 10, 1, 2, 4, 8, 16, 13)$
Hamilton C_{10}-bowtie = $(19, 14, 9, 18, 17, 15, 11, 3, 6, 12) \cup (19, 5, 10, 1, 2, 4, 8, 16, 13, 7)$
Hamilton C_{10}-bowtie = $(19, 9, 18, 17, 15, 11, 3, 6, 12, 5) \cup (19, 10, 1, 2, 4, 8, 16, 13, 7, 14)$.
These starters comprise a Hamilton C_{10}-bowtie decomposition of $20K_{19}$.

Main Conjecture. λK_n has a Hamilton C_k-bowtie decomposition if and only if (i) $n = 2(k-1)+1$ and (ii) $\lambda \equiv 0 \pmod{k}$ for odd k, $\lambda \equiv 0 \pmod{2k}$ for even k.

References

[1] C. J. Colbourn, CRC Handbook of Combinatorial Designs, CRC Press, 1996.
[2] C. J. Colbourn and A. Rosa, Triple Systems, Clarendom Press, Oxford, 1999.
[3] P. Horák and A. Rosa, Decomposing Steiner triple systems into small configurations, *Ars Combinatoria*, Vol. 26, pp. 91–105, 1988.
[4] C. C. Lindner, Design Theory, CRC Press, 1997.
[5] K. Ushio, G-designs and related designs, *Discrete Math.*, Vol. 116, pp. 299–311, 1993.
[6] K. Ushio, Bowtie-decomposition and trefoil-decomposition of the complete tripartite graph and the symmetric complete tripartite digraph, *J. School Sci. Eng. Kinki Univ.*, Vol. 36, pp. 161–164, 2000.
[7] K. Ushio, Balanced bowtie and trefoil decomposition of symmetric complete tripartite digraphs, *Information and Communication Studies of The Faculty of Information and Communication Bunkyo University*, Vol. 25, pp. 19–24, 2000.
[8] K. Ushio and H. Fujimoto, Balanced bowtie and trefoil decomposition of complete tripartite multigraphs, *IEICE Trans. Fundamentals*, Vol. E84-A, No. 3, pp. 839–844, March 2001.
[9] K. Ushio and H. Fujimoto, Balanced foil decomposition of complete graphs, *IEICE Trans. Fundamentals*, Vol. E84-A, No. 12, pp. 3132–3137, December 2001.
[10] K. Ushio and H. Fujimoto, Balanced bowtie decomposition of complete multigraphs, *IEICE Trans. Fundamentals*, Vol. E86-A, No. 9, pp. 2360–2365, September 2003.
[11] K. Ushio and H. Fujimoto, Balanced bowtie decomposition of symmetric complete multi-digraphs, *IEICE Trans. Fundamentals*, Vol. E87-A, No. 10, pp. 2769–2773, October 2004.
[12] K. Ushio and H. Fujimoto, Balanced quatrefoil decomposition of complete multigraphs, *IEICE Trans. Information and Systems, Special Section on Foundations of Computer Science*, Vol. E88-D, No. 1, pp. 19–22, January 2005.
[13] K. Ushio and H. Fujimoto, Balanced C_4-bowtie decomposition of complete multigraphs, *IEICE Trans. Fundamentals, Special Section on Discrete Mathematics and Its Applications*, Vol. E88-A, No. 5, pp. 1148–1154, May 2005.
[14] K. Ushio and H. Fujimoto, Balanced C_4-trefoil decomposition of complete multigraphs, *IEICE Trans. Fundamentals, Special Section on Discrete Mathematics and Its Applications*, Vol. E89-A, No. 5, pp. 1173–1180, May 2006.
[15] W. D. Wallis, Combinatorial Designs, Marcel Dekker, New York and Basel, 1988.