# 劣モジュラ多面体の辞書引き順最適基を求めるプライマルー デュアル アルゴリズム と その ポセット グリードイド との関係

岩村覚三
城西大学数学教室

劣モジュラ多面体とそのデュアル優モジュラ多面体と与えられた重みベクトルに対し辞書引き順最適基が唯一つ存在する。デュアル優多面体上で動く この基を求めるデュアル アルゴリズムを提出する。このデュアル アルゴリズムはMorton, G., von Randow, R. そして Ringwald, K.［1985］のアルゴリズム，分配束としては チェイン ポセット グリードイドではあるが，と完全に一致する。最後に劣モジュラ多面体の辞書引き順最適基を求めることは本質的には分配束がポセット グリード イドであるような単純な劣モジュラ多面体のそれを求めることと同値であることを示す。この事実は劣モジュラ システム 理論において グリードイドが重要であることを示している。

# PRIMAL DUAL ALGORITHM FOR THE LEXICOGRAPHICALLY OPTIMAL BASE OF A SUBMODULAR POLYHEDRON AND ITS RELATION TO A POSET GREEDOID

Kakuzo Iwamura
Department of Mathematics, Josai University
Sakado, Saitama 350-02
Japan

We show that for a submodular polyhedron and its dual supermodular polyhedron there exists a unique lexicographically optimal base with respect to a weight vector and they coincide. We also present a dual algorithm to get the lexicographically optimal base of a submodular polyhedron which works on its dual supermodular polyhedron. This dual algorithm completely agrees to the algorithm of Morton, G. and von Randow, R. and Ringwald, K. [1985], where their underlying distributive lattice is a chain poset greedoid. Finally we show that finding the lexicographically optimal base of a submodular system is essentially equivalent to finding the lexicographically optimal base of a simple submodular system, where its underlying distributive lattice is a poset greedoid. This fact indicates the importance of greedoids in a further development of submodular system theory.

## 1. Introduction

In the preceeding paper, we proved the existence and the uniqueness of a lexicographically optimal base of a submodular system with respect to a weight vector (Iwamura, K. [1987]). There we presented a greedy procedure to get it, which is quite different from Fujishige's algorithm [1980,1987] and explains the algorithm of the first problem of Morton, G. and von Randow, R. and Ringwald, K. [1985]. There, we noticed that the greedy procedure proceeds inversely to the algorithm of Morton, G and von Randow, R. and Ringwald, K. [1985] and asked ourselves why?

Here, we present another algorithm to get a lexicographically optimal base of a submodular system with respect to a weight vector. When the distributive lattice of a submodular system is simple, it is, in fact, a poset greedoid. It is well known that there exist two algorithms to find an optimal base of a matroid and/or a shelling structure (Korte, B. and Lovaśz, L. [1984c], Iwamura, K. [1985]) for a linear objective function. Hence our result can be considered as another example for which there exist more than one greedy algorithm.

## 2. Definition

Let $E$ be a finite set and denote by $2^E$ the set of all the subsets of $E$. Let a collection $\mathcal{F}$ of subsets of $E$ be a *distributive lattice* with set union and intersection as the lattice operations, i.e., for any $X, Y \in \mathcal{F}$ we have $X \cup Y, X \cap Y \in \mathcal{F}$. A function $f$ from $\mathcal{F}$ to the set $R$ of reals is called a *submodular function* (Fujishige, S. [1984]) on $\mathcal{F}$ if for each pair of $X, Y \in \mathcal{F}$

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y).$$

A triple $(E, \mathcal{F}, f)$ of a finite set $E$ and a distributive lattice $\mathcal{F} \subseteq 2^E$ and a submodular function $f : \mathcal{F} \to R$ is called a *submodular system*. We assume that $\emptyset, E \in \mathcal{F}$ and $f(\emptyset) = 0$. It is well known that for a distributive lattice $\mathcal{F} \subseteq 2^E$ with $\emptyset, E \in \mathcal{F}$ there uniquely exist a partition $\Pi = \{A_1, \ldots, A_k\}$ of $E$ and a partial order $\leq$ on $\Pi$ satisfying $\mathcal{F} \ni X$ iff there exists an ideal $I$ of poset $(\Pi, \leq)$ such that $X = \cup\{A_i \mid A_i \in I\}$ (Birkhoff, G. [1937], Fujishige, S. and Tomizawa, N. [1983]). Note that the correspondence $X \leftrightarrow I$ is a bijection. For a submodular system $(E, \mathcal{F}, f)$, by indentifying each $X \in \mathcal{F}$ with $I \subseteq \Pi$, we obtain a distributive lattice $\mathcal{F}' \subseteq 2^{E'}$ with $E' = \Pi$ and a submodular function $f' : \mathcal{F}' \to R$. That is to say, $\mathcal{F}' := \{I \subseteq \Pi \mid \cup\{A_i \mid A_i \in I\} \in \mathcal{F}\} = \{I \subseteq \Pi \mid I \text{ is an ideal of } (\Pi, \leq)\}$, $f'(I) := f(\cup\{A_i \mid A_i \in I\})$ for $I \in \mathcal{F}'$. We see that $(E', \mathcal{F}')$ is a poset greedoid (Korte, B. and Lovaśz, L. [1983, 1984a]) and hence $(E', \mathcal{F}', f')$ is still a (*simple*) submodular system. $(E', \mathcal{F}', f')$ is called a *simplification* of $(E, \mathcal{F}, f)$.

For a submodular sytem $(E, \mathcal{F}, f)$, define a *submodular polyhedron* $P(f)$ and a *submodular base polyhedron* $B(f)$ by

$$P(f) = \{x \in R^E \mid x(X) \leq f(X)(X \in \mathcal{F})\},$$
$$B(f) = \{x \in R^E \mid x(X) \leq f(X)(X \in \mathcal{F}) \text{ and } x(E) = f(E)\},$$

where coordinates indexed by $E$ and $x(e) \in R(e \in E)$ and $x(X) := \sum_{e \in X} x(e)$. Define $\overline{\mathcal{F}} := \{E - X \mid X \in \mathcal{F}\}$, $\overline{f}(E - X) := f(E) - f(X)(E - X \in \overline{\mathcal{F}})$. Then $\overline{\mathcal{F}} = \{X \subseteq E \mid X$ is an upper ideal of $(E, \leq)\}$ with $\emptyset, E \in \overline{\mathcal{F}}, \overline{f}(\emptyset) = 0$ and $\overline{f}$ is *suppermodular* on $\overline{\mathcal{F}}$, i.e. for each pair of $X, Y \in \overline{\mathcal{F}}$

$$\overline{f}(X \cup Y) + \overline{f}(X \cap Y) \geq \overline{f}(X) + \overline{f}(Y).$$

$(E, \overline{\mathcal{F}}, \overline{f})$ is called *dual supermodular system* of $(E, \mathcal{F}, f)$. Define a *suppermodular polyhedron* $P(\overline{f})$ and *suppermodular base polyhedron* $B(\overline{f})$ by

$$P(\overline{f}) := \{x \in R^E \mid x(X) \geq \overline{f}(X)(X \in \overline{\mathcal{F}})\},$$
$$B(\overline{f}) := \{x \in R^E \mid x(X) \geq \overline{f}(X)(X \in \overline{\mathcal{F}}), x(E) = \overline{f}(E)\}$$

respectively (Fujishige, S. [1984]). Then we have

$$\overline{f}(\emptyset) = f(\emptyset) = 0,\ \overline{f}(E) = f(E),\ B(\overline{f}) = B(f).$$

Any vector $x \in B(f) = B(\overline{f})$ is called a *base* of $B(f) = B(\overline{f})$. Let $\chi_u$ be a *characteristic function* of u, i.e. $\chi_u(e) = 1$ for $e = u$ and $\chi_u(e) = 0$ for $e \in E \setminus \{u\}$. Define a *dual saturation function* $\overline{sat}() : P(\overline{f}) \to 2^E$ by $\overline{sat}(x) = \{u \in E \mid \forall d > 0, x - d\chi_u \notin P(\overline{f})\}$. Then we have the following lemmas, where $\overline{\mathcal{A}}(x) := \{A \in \overline{\mathcal{F}} \mid x(A) = \overline{f}(A)\}$ (Iwamura, K. [1987], Fujishige, S. [1980]).

**Lemma 2.1.** *Let $X \in P(\overline{f})$ and $A, B \in \overline{\mathcal{F}}$. If $x(A) = \overline{f}(A), x(B) = \overline{f}(B)$, then $x(A \cap B) = \overline{f}(A \cap B)$ and $x(A \cup B) = \overline{f}(A \cup B)$ hold.*

□

**Lemma 2.2.** *Let $x \in P(\overline{f})$. Then $\overline{sat}(x)$ satisfies*

$$\overline{sat}(x) \in \overline{\mathcal{F}}, x(\overline{sat}(x)) = \overline{f}(\overline{sat}(x)).$$

*Furthermore $\overline{\mathcal{A}}(x)$ is a distributive lattice with a partial order relation defined by the set inclusion and $\overline{sat}(x)$ is the maximum element of $\overline{\mathcal{A}}(x)$.*

□

**Lemma 2.3.** *Let $x \in P(\overline{f})$. Then $x \in B(\overline{f})$ iff $\overline{sat}(x) = E$.*

□

Let $n := |E|$. For any real sequences $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of length $n$, $a$ is called *lexicographically greater than or equal to* $b$ if for some $j \in \{1, \ldots, n\}$

$$a_i = b_i \ (i = 1, \ldots, j-1)$$
$$a_j > b_j$$

or

$$a_i = b_i \ (i = 1, \ldots, n).$$

A vector $w \in R^E$ such that $w(e) > 0$ $(e \in E)$ is called a *weight vector.* For a vector $x \in R^E$, denote by $T(x)$ the $n$-tuple (or sequence) of the numbers $x(e)(e \in E)$ arranged in order of increasing magnitude. Given a weight vector $w$, a base $x$ of $P(f)$ is called a *lexicographically maximum base with respect to the weight vector* $w$ if the $n$-tuple $T((x(e)/w(e))_{e \in E})$ is lexicographically maximum among all $n$-tuples $T((y(e)/w(e))_{e \in E})$ for all bases $y$ of $P(f)$. The mathematical programming problem to get $x \in B(f)$ such that

$$T((x(e)/w(e))_{e \in E}) = \text{Lexicographically maximum } T((y(e)/w(e))_{e \in E})$$
$$\text{subject to } y \in B(f)$$

is called *wl max b* (weighted lexicographically maximum base) *problem* for a submodular base polyhedron $B(f)$.

For a vector $x \in R^E$, denote by $\overline{T}(x)$ the $n$-tuple (or sequence) of the numbers $x(e)(e \in E)$ arranged in order of decreasing magnitude. Given a weight vector $w$, a base $x$ of $B(\overline{f})$ is called a *lexicographically minimum base with respect to the weight vector* $w$ if the $n$-tuple $\overline{T}((x(e)/w(e))_{e \in E})$ is lexicographically minimum among all $n$-tuples $\overline{T}((y(e)/w(e))_{e\in E})$ for all bases $y$ of $P(\overline{f})$. The mathematical programming problem to get $x \in B(\overline{f})$ such that

$$\overline{T}((x(e)/w(e))_{e\in E}) = \text{Lexicographically minimum } \overline{T}((y(e)/w(e))_{e\in E}) \\ \text{subject to } y \in B(\overline{f})$$

is called *wl min b* (*w*eighted *l*exicographically *min*imum *b*ase) *problem* for a submodular base polyhedron $B(\overline{f})$.

## 3. Primal dual algorithms for the lexicographically optimal base of a submodular polyhedron and its relation to a poset greedoid

In Iwamura, K. [1987], we have developed an algorithm to get the (unique) lexicographically maximum base with respect to the weight vector $w$.

### Algorithm to get the lexicographically maximum base (primal)

Step 1. *Set* $i := 1$ and *compute* $c_i := min\{\frac{f(A)}{w(A)} \mid \emptyset \neq A \in \mathcal{F}\}$ and *set* $U_{c_i}(e) := c_i w(e)(e \in E)$.

Step 2. *If* $sat(U_{c_i}) = E$, *then STOP.*

Step 3. *Compute* $\epsilon_i := min\{\frac{f(A)-U_{c_i}(A)}{w(A\setminus sat(U_{c_i}))} \mid A \in \mathcal{F},\ A \setminus sat(U_{c_i}) \neq \emptyset\}$ and *set* $c_{i+1} := c_i + \epsilon_i$ and *set*

$$U_{c_{i+1}}(e) := \begin{cases} U_{c_i}(e) & \text{for } e \in sat(U_{c_i}), \\ U_{c_i}(e) + \epsilon_i w(e) & \text{for } e \in E - sat(U_{c_i}). \end{cases}$$

*Set* $i := i+1$ and *go to* Step 2.

With Lemma 2.1.-2.3., similar arguments as that of Iwamura, K. [1987] show that the following algorithm produces the lexicographically minimum base with respect to the weight vector $w$.

**Algorithm to get the lexicographically minimum base (dual)**

Step 1. *Set* $i := 1$ and *compute* $\overline{c_i} := max\{\frac{\overline{f}(A)}{w(A)} \mid \emptyset \neq A \in \overline{\mathcal{F}}\}$ and *set* $U_{\overline{c_i}}(e) := \overline{c_i} w(e)(e \in E)$.

Step 2. *If* $\overline{sat}(U_{\overline{c_i}}) = E$*, then STOP.*

Step 3. *Compute* $\overline{\epsilon_i} := min\{\frac{U_{\overline{c_i}}(A) - \overline{f}(A)}{w(A \setminus \overline{sat}(U_{\overline{c_i}}))} \mid A \setminus \overline{sat}(U_{\overline{c_i}}) \neq \emptyset, A \in \overline{\mathcal{F}}\}$ and *set* $\overline{c_{i+1}} := \overline{c_i} - \overline{\epsilon_i}$ and *set*

$$U_{\overline{c_{i+1}}}(e) := \begin{cases} U_{\overline{c_i}} & \text{for } e \in \overline{sat}(U_{\overline{c_i}}), \\ U_{\overline{c_i}}(e) - \overline{\epsilon_i} w(e) & \text{for } e \in E \setminus \overline{sat}(U_{\overline{c_i}}). \end{cases}$$

*Set* $i := i + 1$ and *go to* Step 2.

Suppose that the above algorithm stops after $d$ iterations, then we have

$$U_{\overline{c_d}}(e) = \begin{cases} \overline{c_1} w(e)(e \in \overline{sat}(U_{\overline{c_1}})) \\ \overline{c_2} w(e)(e \in \overline{sat}(U_{\overline{c_2}}) \setminus \overline{sat}(U_{\overline{c_1}})) \\ \vdots \\ \overline{c_i} w(e)(e \in \overline{sat}(U_{\overline{c_i}}) \setminus \overline{sat}(U_{\overline{c_{i-1}}})) \\ \vdots \\ \overline{c_d} w(e)(e \in \overline{sat}(U_{\overline{c_d}}) \setminus \overline{sat}(U_{\overline{c_{d-1}}}) = E \setminus \overline{sat}(U_{\overline{c_{d-1}}})), \end{cases}$$

$U_{\overline{c_d}} \in B(\overline{f}) = B(f), \emptyset \subset^{*)} \overline{sat}(U_{\overline{c_1}}) \subset \ldots \subset \overline{sat}(U_{\overline{c_d}}) = E$ which are all in $\overline{\mathcal{F}}$, $U_{\overline{c_d}}(\overline{sat}(U_{\overline{c_i}})) = \overline{f}(\overline{sat}(U_{\overline{c_i}}))$ $(1 \leq i \leq d)$ and $\overline{c_1} > \overline{c_2} > \ldots > \overline{c_d}$.

**Theorem 3.1. (Primal–dual theorem).** *The above* $U_{\overline{c_d}}$ *is the lexicographically maximum base with respect to the weight vector* $w$.

*Proof.* We use Theorem 3.3 of Iwamura, K. [1987]. Define $\hat{c}(e) := U_{\overline{c_d}}(e)/w(e)$ $(e \in E)$. Then we see that $\hat{p} = d$ with $\hat{c}_1 = \overline{c_d}, \hat{c}_2 = \overline{c_{d-1}}, \ldots, \hat{c}_{d-1} = \overline{c_2}, \hat{c}_d = \overline{c_1}$. Using $U_{\overline{c_d}} \in B(\overline{f}) = B(f)$, we get

$$U_{\overline{c_d}}(E) = \overline{f}(E) = f(E),$$

$$\begin{aligned} U_{\overline{c_d}}(E - \overline{sat}(U_{\overline{c_i}})) &= f(E) - \overline{f}(\overline{sat}(U_{\overline{c_i}})) \\ &= f(E) - \{f(E) - f(E - \overline{sat}(U_{\overline{c_i}}))\} \\ &= f(E - \overline{sat}(U_{\overline{c_i}})), \end{aligned}$$

*) $X \subset Y$ means $X$ is a proper subset of $Y$.

where $\emptyset \subset E - \overline{sat}(U_{\overline{c_{d-1}}}) \subset E - \overline{sat}(U_{\overline{c_{d-2}}}) \subset \ldots \subset E - \overline{sat}(U_{\overline{c_1}}) \subseteq E$, all in $\mathcal{F}$. Furthermore $E - \overline{sat}(U_{\overline{c_i}}) = \{e \in E \mid \hat{c}(e) \leq \overline{c_{i+1}}\}(0 \leq i \leq d-1)$. Hence by Theorem 3.3 of Iwamura, K. [1987] we get that $U_{\overline{c_d}}$ is the lexicographically maximum base with respect to weight vector $w$.

□

A careful reader would have noticed that the proof for Theorem 3.1 of Iwamura, K. [1987] remains valid for $z \in P(f)$. Hence the following mathematical programming problems

Lexicographically maximum $T((y(e)/w(e))_{e \in E})$,
subject to $y \in P(f)$

Lexicographically minimum $\overline{T}((y(e)/w(e))_{e \in E})$,
subject to $y \in P(\overline{f})$

have the same solution as that of $wl\,min\,b$ – and $wl\,max\,b$ – problem. Hence, we call these problems *wlo* (*w*eighted *l*exicographically *o*ptimal)-problems for a submodular system.

Let $(E, \mathcal{F}, f)$ be a submodular system and let $(E', \mathcal{F}', f')$ be its simplification. Let $w(e) > 0$ $(e \in E)$ be a weight vector and let $w'(A_i) := \sum_{e \in A_i} w(e) > 0$ $(A_i = e'_i \in E'$ $(1 \leq i \leq k))$.

**Theorem 3.2.** *Let $x'(e')(e' \in E')$ be the lexicographically maximum base of $(E', \mathcal{F}', f')$ with respect to the weight vector $w'$ just above. Let $x(e) = (w(e)/w'(e'_i))x'(e'_i)$ for any $e \in e'_i, e'_i \in E'$. Then $x(e)(e \in E)$ is the lexicographically maximum base of $(E, \mathcal{F}, f)$ with respect to the weight vector $w$.*

*Proof.* Submodular polyhedrons corresponding to $(E', \mathcal{F}', f')$ and $(E, \mathcal{F}, f)$ become

$$P(f') = \{x' \in R^{E'} \mid x'(A) \leq f'(A)(A \in \mathcal{F}')\}$$

and

$$P(f) = \{x \in R^{E} \mid x(X) \leq f(X)(X \in \mathcal{F})\}$$

respectively.

Let $c'(e') = (x'(e')/w'(e'))(e' \in E')$ and let $c(e) = (x(e)/w(e))(e \in E)$. Let the distinct numbers of $c'(e')(e' \in E')$ be given by $c'_1 < \ldots < c'_{p'}$ and define $S'_i = \{e' \in E' \mid c'(e') \leq c'_i\}$, $S_i = \{e \in E \mid c(e) \leq c'_i\}$. Then the distinct numbers of $c(e)(e \in E)$ are just the same as that of $c'(e')(e' \in E')$. By theorem 3.3 in Iwamura, K. [1987], we see that

$$S'_i \in \mathcal{F}' \text{ and } x'(S'_i) = f'(S'_i) \; (1 \leq i \leq p').$$

$x \in B(f)$ because for any $X \in \mathcal{F}$, $X = \cup\{A_i \mid A_i \in I\}$, $I \in \mathcal{F}'$ and $x(X) = \sum_{A_i \in I} \sum_{e \in A_i} x(e) = \sum_{A_i \in I} x'(A_i) = x'(I) \leq f'(I) = f(X)$ with $x(E) = f(E)$. By the definition of $\mathcal{F}'$ and $S'_i \in \mathcal{F}'$, we get $\mathcal{F} \ni \cup\{A_j \mid A_j \in S'_i\} = \{e \in E \mid c(e) \leq c'_i\} = S_i$ with $x(S_i) = f(S_i)$ for $1 \leq i \leq p'$.
Again by theorem 3.3 in Iwamura, K. [1987], we get the conclusion.

□

Let $(E', \mathcal{F}')$ be an arbitrary poset greedoid on $E' = \{e'_1, \ldots, e'_m\}$. Let $f'$ be a submodular function on $\mathcal{F}'$ with $f'(\emptyset) = 0$. Then $(E', \mathcal{F}', f')$ is a simple submodular system. For each $e'_i \in E'$, assign a subset $E'_i$ of $E$ such that $E'_i \cap E'_j = \emptyset \; (1 \leq i < j \leq m)$ and $\bigcup_{i=1}^{m} E'_i = E$. Let $|E| = \sum_{i=1}^{m} |E'_i| = n$, and let $\mathcal{F} := \{\bigcup_{i \in I} E'_i \mid \{e'_i \mid i \in I\} \in \mathcal{F}'\}$. Then clearly $(E, \mathcal{F})$ is a distributive lattice with set union and intersection as the lattice operations, and $\emptyset, E \in \mathcal{F}$. Define $f : \mathcal{F} \to \mathrm{R}$ by $f(\bigcup_{i \in I} E'_i) = f'\{e'_i \mid i \in I\}$ for any $\{e'_i \mid i \in I\} \in \mathcal{F}'$. Then $f$ is a submodular function with $f(\emptyset) = 0$ and so $(E, \mathcal{F}, f)$ is a general submodular system, which we call the *expansion* of $(E', \mathcal{F}', f')$. In fact, the simplification of $(E, \mathcal{F}, f)$ is $(E', \mathcal{F}', f')$. Given a positive weight vector $w(e)(e \in E)$, define $w'(e'_i) := w(E'_i) = \sum_{e \in E'_i} w(e)$. Then $w'(e'_i) > 0$ for any $e'_i \in E'$.

**Corollary 3.3. (Expansion theorem).** *Let $x'(e')(e' \in E')$ be the lexicographically maximum base of $(E', \mathcal{F}', f')$ with respect to the weight vector $w'$. Let $x(e) = \frac{w(e)}{w'(e'_i)} x'(e'_i)$ for any $e \in E'_i, e'_i \in E'$. Then $x(e)(e \in E)$ is the lexicographically maximal base of $(E, \mathcal{F}, f)$ with respect to the weight vector $w$.*

*Proof.* Same as Theorem 3.2.

□

References

Birkhoff, G. (1937): Rings of sets. Duke Mathematical Journal 3, 443-454.

Fujishige, S. (1980): Lexicographically optimal base of a polymatroid with respect to a weight vector. Math. of O.R. 5, 186-196.

Fujishige, S., Tomizawa, N.(1983): A note on submodular functions on distributive lattices. J. of the Operations Research Society of Japan 26, 309-317.

Fujishige, S. (1984): Submodular systems and related topics. Mathematical Programming Study 22, 113-131.

Fujishige, S. (1987): Private communication.

Iwamura, K. (1985): Greedy algorithm for primal dual matroids(Japanese). Proceeding of Symposium in Language and Automata Theory.

Iwamura, K.(1987): Lexicographically optimal base of a submodular system with respect to a weight vector. Report No. 87482-OR, Institut für Operations Research, Universität Bonn.

Korte, B., Lovász, L. (1983): Structural properties of greedoids. Combinatorica 3, 359-374.

Korte, B., Lovász, L. (1984a): Greedoids, a structural framework for the greedy algorithm. in: Pulleyblank, W.R.(ed.):Progress in Combinatorial Optimization, 221-243, Academic Press.

Korte, B., Lovász, L.(1984c): Greedoids and linear objective functions. SIAM Journ. Algeb. Discr. Methods 5, 229-238.

Morton, G., von Randow, R., Ringwald, K. (1985): A greedy algorithm for solving a class of convex programming problems and its connection with polymatroid theory. Math. Programming 32, 238-241.