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連続モジュール設計問題に対する新アルゴリズム

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この論文では、よく知られている連続モジュール設計問題について考える。対応する幾何計画問題とその双対問題の特殊な構造を十分利用することにより、新しいアルゴリズムを与える。このアルゴリズムは、効率的な探索方向を与える一般化された König の問題の解を用いる。最適なステップサイズは、容易に計算することができる。

許容方向降下法のよく知られている結果より、効率の良い収束性を示すことができる。

A NEW ALGORITHM FOR THE CONTINUOUS MODULAR DESIGN PROBLEM

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Abstract

The well known continuous modular design problem is considered in this paper. Fully utilizing the special structure of the corresponding geometric programming problem and its dual a new algorithm is derived. This algorithm is based on the subsequent solution of generalized König problems which provides an efficient feasible direction. The optimal step size is easily computed.

An efficient convergent implementation follows from the well known results on the feasible direction methods.

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1.INTRODUCTION.

Evans [3,4] was the first who considered the modular design problem (MD). He proposed an algorithm as well. The continuous modular design problem mathematically can be stated as:

$$\min \left(\sum_{i=1}^{m} \eta_i \beta_i'\right)\left(\sum_{j=1}^{n} \tau_j \gamma_j'\right) \qquad (1.1)$$

$$\eta_i \tau_j \geq \alpha_{ij}' \qquad i=1,\dots,m;\ j=1,\dots,n \qquad (1.2)$$

$$\eta_i, \tau_j \geq 0 \qquad i=1,\dots,m;\ j=1,\dots,n \qquad (1.2)$$

The economic interpretation is as follows: Parts $i=1,\dots,m$ are to be grouped into a module and in applications $j=1,\dots,n$ several of this module are to be used. η_i denotes the number of part i in the module (decision variable), τ_j denotes the number of modules required in application j (decision variable), α_{ij}' is the number of part i units required in application j, so the inequality $\eta_i \tau_j \geq \alpha_{ij}'$ represents, that the demand for part i in all application j have to be satisfied. β_i' is the unit cost of part i, so $\sum_{i=1}^{m} \eta_i \beta_i'$ is the unit cost of the module. γ_j' is the demand for application j, so $\sum_{j=1}^{n} \tau_j \gamma_j'$ is the total demand for the module. Obviously the objective is to minimize the total cost. According to the above interpretation, we assume that $\alpha_{ij}'>0$, $\beta_i'>0$, $\gamma_j'>0$, that implies $\tau_j>0$ and $\eta_i>0$ for $\forall$ i,j.

This problem was extensively studied in the last decades Evans [3], Charnes and Kirby [1] applied general methods to solve this specially structured nonlinear programming problem. Although Charnes and Kirby [1] recognized that MD is a geometrical programming problem (GP), they did not utilized this fact. They transformed MD into a separable convex programming problem. Passy [8] explore the potential of the geometrical programming structure, but the convergence of his algorithm is still an open question. Smeers [11] made some suggestions on improving Passy's approach. To the best of our knowledge, last time Schaftel [9] and Thompson [10] studied this problem. Designing simplex-like algorithms they examined both the mathematical programming properties and applications of this problem as well.

We will utilize the special GP programming structure of MD, so our approach is close to Passy's approach. We will use the exponential formulation of GP instead of the posinomial format. In the previous papers the $\sum_{j=1}^{n} \tau_j = 1$ assumption (not restrictive) was introduced, and so a linear objective was obtained. We will not use this assumption.

The special structure of the corresponding primal and dual GP problems are explored in the next section. Finally our algorithm is presented and its convergence is proved.

2. MD AS A GP PROBLEM.

By some simple transformation MD can be transformed into GP problem. By means of positivity assumptions the following notations can be introduced for the new variables μ, ν, μ_i, ν_j and parameters β_i, γ_j, α_{ij}.

$$\eta_i = e^{-\mu_i} ; \quad \beta'_i = e^{-\beta_i} ; \quad \sum_{i=1}^{m} \eta_i \beta_i \le e^{-\mu} ;$$

$$\tau_j = e^{-\nu_j} ; \quad \gamma'_j = e^{-\gamma_j} ; \quad \alpha'_{ij} = e^{-\alpha_{ij}} ; \quad \sum_{j=1}^{n} \tau_j \gamma_j \le e^{-\nu} ;$$

So the MD problem (1,1),(1,2),(1,3) is as follows:

$$\min \left(e^{-\mu} e^{-\nu}\right)$$

$$\sum_{i=1}^{m} e^{-\mu_i - \beta_i} \le e^{-\mu}$$

$$\sum_{j=1}^{n} e^{-\nu_j - \gamma_j} \le e^{-\nu}$$

$$e^{-\mu_i} e^{-\nu_j} \ge e^{-\alpha_{ij}}$$

Denote $\nu_j = \nu_j + \gamma_j$, $\mu_i = \mu_i + \beta_i$, $\alpha_{ij} = \alpha_{ij} + \beta_i + \gamma_j$ for all i,j.

The exponential function is monotone, so this problem is equivalent to the following primal GP problem.

<u>GP primal</u>

$$\max \mu + \nu$$

$$e^{\mu_i + \nu_j - \alpha_{ij}} \le 1 \qquad \forall\, i,j \tag{2.1}$$

$$\sum_{i=1}^{m} e^{-\mu_i + \mu} \le 1 \tag{2.2}$$

$$\sum_{j=1}^{n} e^{-\nu_j + \nu} \le 1 \tag{2.3}$$

This primal GP problem obviously fulfills the Slater regularity condition. Its dual and the equilibrium conditions (see e.g.[7,2]) is as follows:

$$\underline{\text{GP dual}} \qquad \min\Big(\sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij}\alpha_{ij} + \log \frac{\prod_{i=1}^{m} \zeta_i^{\zeta_i}}{\left(\sum_{i=1}^{m} \zeta_i\right)^{\sum_{i=1}^{m} \zeta_i}} + \log \frac{\prod_{j=1}^{n} \vartheta_j^{\vartheta_j}}{\left(\sum_{j=1}^{n} \vartheta_j\right)^{\sum_{j=1}^{n} \vartheta_j}} \Big)$$

$$\sum_{j=1}^{n} \xi_{ij} - \zeta_i = 0 \qquad \forall i \tag{2.4}$$

$$\sum_{i=1}^{m} \xi_{ij} - \vartheta_j = 0 \qquad \forall j \tag{2.5}$$

$$\sum_{i=1}^{m} \zeta_i = 1 \tag{2.6}$$

$$\sum_{j=1}^{n} \vartheta_j = 1 \tag{2.7}$$

$$\xi_{ij} \ge 0, \ \zeta_i \ge 0, \ \vartheta_j \ge 0 \qquad \forall i,j$$

Obviously the dual GP problem satisfies Slater regularity as well.

<u>Equilibrium conditions:</u>

$$\xi_{ij} = \xi_{ij} e^{\mu_i + \nu_j - \alpha_{ij}} \qquad \forall i,j \tag{2.8}$$

$$\zeta_i = e^{-\mu_i + \mu} \sum_{i=1}^{m} \zeta_i \qquad \forall i \tag{2.9}$$

$$\vartheta_j = e^{-\nu_j + \nu} \sum_{j=1}^{n} \vartheta_j \qquad \forall j \tag{2.10}$$

Conditions (2.6), (2.9) imply equality in (2.2). The same way (2.7) and (2.10) imply equality in (2.3). So by substitution variables ϑ_j and ζ_i can be eliminated.

Since both the primal and the dual problem is Slater regular (so feasible) the duality theory of GP guarantee optimal solutions for both of these problems, which can be obtained by solving the following nonlinear inequality (equality) system of the primal, dual and equilibrium constraints.

<u>Primal:</u>

$$\mu_i + \nu_j \le \alpha_{ij} \qquad \forall\ i,j \tag{2.11}$$

$$\sum_{i=1}^{m} e^{-\mu_i + \mu} = 1 \qquad \forall i \tag{2.12}$$

$$\sum_{j=1}^{n} e^{-\nu_j + \nu} = 1 \qquad \forall j \tag{2.13}$$

<u>Dual:</u>

$$\sum_{j=1}^{n} \xi_{ij} = e^{-\mu_i + \mu} \qquad \forall i \tag{2.14}$$

$$\sum_{i=1}^{m} \xi_{ij} = e^{-\nu_j + \nu} \qquad \forall j \tag{2.15}$$

$$\xi_{ij} \ge 0 \qquad \forall i,j \tag{2.16}$$

Equilibrium: $\xi_{ij}(\mu_i+\nu_j-\alpha_{ij}) = 0 \qquad \forall i,j \qquad (2.17)$

Remark 2.1. Based on the observations taken above an equivalent form of the primal is as follows:

$$\max -\log \sum_{i=1}^{m}\sum_{j=1}^{n} e^{-\mu_i-\nu_j} \qquad (*)$$
$$\mu_i+\nu_j \le \alpha_{ij}$$

From this form it is obvious that primal problem depends only on the sum of $\mu_i+\nu_j$ and so the primal feasible set (μ_i,ν_j) is unbounded, but it is bounded if we consider the primal feasible set in terms of $(\mu_i+\nu_j)$.

If ν_j, μ_i is given, then ν and μ is easily computed, then the dual constraints define a generalized König problem (constraints of a transportation problem) that can be solved effectively by network flow methods (see e.g. [5,6]). Our algorithm is based on this property.

3. THE ALGORITHM

Initialization: Let μ_i^0 and ν_j^0 arbitrary "small" numbers such that (2.11) holds. (e.g. $\mu_i^o=\min_j\{\alpha_{ij}\}$ and $\nu_j^o=\min_i\{\alpha_{ij}-\mu_i^o\}$)

General iteration: Primal feasible μ_i^k and ν_j^k is given. Using (2.12) and (2.13) we have $\mu^k=-\log\sum_{i=1}^{m} e^{-\mu_i^k}$ and $\nu^k= -\log\sum_{j=1}^{n} e^{-\nu_j^k}$. Let $I^k(0)=\{\ (i,j)\ :\ \mu_i^k+\nu_j^k = \alpha_{ij}\ \}$ and denote $\delta_i^k=e^{-\mu_i^k+\mu^k}=\dfrac{e^{-\mu_i^k}}{\sum_{i=1}^{m} e^{-\mu_i^k}}$ and $\sigma_j^k=e^{-\nu_j^k+\nu^k}=\dfrac{e^{-\nu_j^k}}{\sum_{j=1}^{n} e^{-\nu_j^k}}$.

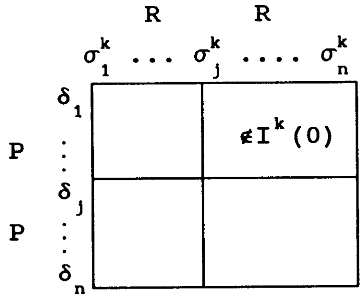
Step 1. Solve the following generalized König problem

$$\sum_{j=1}^{n}\xi_{ij} = \delta_i^k \qquad \forall i$$
$$\sum_{i=1}^{m}\xi_{ij} = \sigma_j^k \qquad \forall j$$
$$\xi_{ij} \ge 0 \qquad \forall i,j$$
$$\xi_{ij} = 0 \qquad \text{if } (i,j)\notin I^k(0)$$

Case a. If ξ_{ij}^k is a solution for the König problem, then ξ_{ij}^k, μ_i^k, ν_j^k, μ^k, ν^k are optimal solutions.

If the König problem is infeasible, then there are sets

$P \subset \{1,\ldots,m\}$ and $R \subset \{1,\ldots,n\}$ such that $\Delta^k = \sum_{i\in P} \delta_i^k - \sum_{j\in R} \sigma_j^k > 0$ and $(i,j) \notin I^k(0)$ if $i\in P$ and $j\notin R$.



(Figure 1.)

Step 2. Let $\lambda' = \min\{\alpha_{ij} - \mu_i^k - \nu_j^k :\ i\in P,\ j\notin R\} > 0$

$$\lambda'' = \frac{1}{2}\log\frac{(\sum_{i\in P}\delta_i^k)(\sum_{j\notin R}\sigma_j^k)}{(\sum_{i\notin P}\delta_i^k)(\sum_{j\in R}\sigma_j^k)} > 0$$

$$\lambda^k = \min(\lambda', \lambda'')$$

$$\mu_i^{k+1} = \begin{cases} \mu_i^k + \lambda^k & i\in P \\ \mu_i^k & i\notin P \end{cases} ; \qquad \nu_j^{k+1} = \begin{cases} \nu_j^k - \lambda^k & j\in R \\ \nu_j^k & j\notin R \end{cases}$$

$k=k+1$ (start the next iterational cycle)

Remark 3.1: By efficient network flow methods it is easy to solve the generalized König problem. Either the optimal solution ξ_{ij} or the sets P and R are the output of this module.

Remark 3.2: The value λ' denotes the maximal step size within the primal feasible region in the direction defined by P and R.

Remark 3.3: The primal objective $\mu+\nu$ attains its maximal value at step size λ'' in the above mentioned direction. This can be verified by some elementary computation.

Remark 3.4: The primal objective strictly monotonically increases at each iteration. The algorithm improves step by step the primal objective, while preserves primal feasibility and equilibrium conditions. Optimality is reached, when a dual feasible solution is obtained.

Remark 3.5.: Optimality is really obtained at case a. since the actual solutions are both primal and dual feasible, and equilibrium conditions also holds.

Remark 3.6.: Δ^k is the value of the maximal flow at the König problem.

4. THE ALGORITHM AS A ZOUTENDIJK'S FEASIBLE DIRECTION METHOD

Primal feasible μ_i, ν_j is given (as at an iteration). Consider the primal problem as in Remark 2.1.

$$\min \log \sum_{i=1}^{m} e^{-\mu_i} + \log \sum_{j=1}^{n} e^{-\nu_j}$$
$$\mu_i+\nu_j \le \alpha_{ij} \tag{4.1}$$

As we have seen the objective is a convex function. So the methods of feasible directions can be applied. The subproblem to locate a feasible direction is as follows.

$$\begin{aligned} \bar\mu_i+\bar\nu_j &\le 0 \qquad (i,j)\in I(0)\\ -1 \le \bar\mu_i &\le 1 \qquad i;\ (i,j)\in I(0)\\ -1 \le \bar\nu_j &\le 1 \qquad j;\ (i,j)\in I(0) \end{aligned} \tag{4.2}$$

$$\min -\sum_{i=1}^{m} \bar\mu_i \frac{e^{-\mu_i}}{\sum_{t=1}^{m} e^{-\mu_t}} - \sum_{j=1}^{n} \bar\nu_j \frac{e^{-\nu_j}}{\sum_{t=1}^{n} e^{-\nu_t}} .$$

The objective can be stated as max $\sum_{i=1}^{m} \bar\mu_i\delta_i + \sum_{j=1}^{n} \bar\nu_j\sigma_j$ (δ_i and σ_j is defined in the general step of the algorithm).

<u>Remark</u> <u>4.1.:</u> Since $\delta_i>0$ and $\sigma_j>0$ so lower bound assumptions can be left out, since at optimality these are consequences of the upper bound constraints, So problem (4.2) is as follows.

$$\begin{aligned} \bar\mu_i+\bar\nu_j &\le 0 \qquad (i,j)\in I(0)\\ \bar\mu_i &\le 1 \qquad i;\ (i,j)\in I(0)\\ \bar\nu_j &\le 1 \qquad j;\ (i,j)\in I(0) \end{aligned} \tag{4.3}$$
$$\max \sum_{i=1}^{m} \bar\mu_i\delta_i + \sum_{j=1}^{n} \bar\nu_j\sigma_j$$

Its dual problem can easily be formulated.

$$\begin{aligned} \sum_{j=1}^{n} \xi_{ij} + \zeta_i &= \delta_i \qquad \forall i\\ \sum_{i=1}^{m} \xi_{ij} + \vartheta_j &= \sigma_j \qquad \forall j\\ \xi_{ij}\ge 0,\ \zeta_i\ge 0,\ \vartheta_j&\ge 0 \qquad \forall i,j\\ \xi_{ij} &= 0 \qquad \text{if } (i,j)\notin I(0) \end{aligned} \tag{4.4}$$
$$\max \sum_{i=1}^{m} \zeta_i + \sum_{j=1}^{n} \vartheta_j$$

the minimization of $\sum_{i=1}^{m} \zeta_i + \sum_{j=1}^{n} \vartheta_j$ is equivalent to the maximization of $\sum_{\substack{i=1 \\ (i,j)\notin I(0)}}^{m} \sum_{j=1}^{n} \xi_{ij}$. So it is easy to see, that dual problem (4.4) is a max flow problem which is equivalent to the generalized König problem of the algorithm. This way the solution of König problem either gives an optimal solution for MD problem or the sets P and R gives optimal solution for problem (4.3). So the following theorem is proved.

Theorem 4.1. The algorithm presented in section 3. is a special implementation of Zoutendijk's [13] feasible direction method.

The fearly well developed theory of the feasible directions method can be applied to guarantee and prove convergence. It is well known, that feasible direction methods in the simple form that it is presented do not necessarily converge in general [12]. This drawback can be avoided e.g. by ε tolerance technique. The detailed description of the ε tolerance variant and a direct proof of convergence is a subject of another paper.

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