アルゴリズム 11－6
（1989. 10. 24）

コグラフ的多種フローアルゴリズムについて

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コグラフ的多種フロー問題は最近M.MiddendorfとF.PfeifferによってNP-困難であることが証明された。本論文ではいくつかの可解な場合について概説する。マイナー演算で閉じている、より一般的な可解なクラスを見つけることをめざしている。

On cographic multicommodity flow algorithms

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ABSTRACT

The "cographic multicommodity flow problem" has recently been proved to be NP-complete by M. Middendorf and F. Pfeiffer (1989). In this paper we give a survey of some solvable cases. We are also interested in finding more general solvable classes closed under minor containment.

1. Introduction

One of the best-known particular cases of the multicommodity flow problem is the "plane multicommodity flow problem", the special case where the graph defined by the union of the demand- and supply-edges forms a planar graph. We shall define this problem somewhat more generally, in terms of binary matroids (following Seymour (1981b)): this setting makes easier to look into the reasons why some methods work in some graphs and not in the others. Let $M = (E, \mathcal{F})$ be a binary matroid, $R \subseteq E$, $r : R \longrightarrow \mathbb{N}$ and $c : E \backslash R \longrightarrow \mathbb{N}$ ($\mathbb{N}$ is the set of positive integers). A set $\mathcal{C}$ of circuits and a function $f : \mathcal{C} \longrightarrow \mathbb{N}$ has to be found so that

(i) $$\forall C \in \mathcal{C} : |C \cap R| = 1$$

(ii) $$\forall e \in R : \sum_{e \in C \in \mathcal{C}} f(C) = r(e),$$

(iii) $$\forall e \in E \backslash R : \sum_{e \in C \in \mathcal{C}} f(C) \leq c(e).$$

(Think of f as a function telling the *multiplicities* of elements of $\mathcal{C}$.) We shall say that (M, R, r, c) is a *network*. $r(xy)$ is called the *demand* of the pair x, y, $xy \in R$, and $c(e)$, $e \in E \backslash R$ is the *capacity* of e. f, or more precisely $(\mathcal{C}, f)$ will be called a *flow*, if it satisfies (i), (ii) and (iii). If f is integer valued we shall say that the flow is *integer*. R is called *demand-graph* and $E \backslash R$ *supply-graph* .

In this paper we are mainly concerned with the case when M is cographic, which contains the above-mentioned planar multicommodity flow problem. This "cographic multiflow problem" has been investigated a lot, due also to the underlying nice combinatorial structure and its appealing relation to matching theory. Seymour (1981a) discovered its relation to the Chinese Postman Problem and used it to solve the plane multicommodity flow problem for Eulerian graphs, and to settle the case of two demand-edges. We shall give a survey of some results about this problem in Section 2.

Another feature of the cographic multicommodity flow problem is that some structural descriptions implying most of the results concerning them are strongly related to distances and potentials in undirected graphs (cf. Sebő (1987b), (1989)). I would like to spend some minutes of my talk to explain this connection.

In another celebrated piece of work, Seymour(1981b) studied the characterization of binary matroids for which a certain trivially necessary condition, the *cut condition* (for every cut the sum of the demands is inferior or equal to the sum of the capacities) is sufficient to have an (integer) multicommodity flow. We are interested in characterizing binary matroids in which some more general conditions are necessary and sufficient, and in characterizing the classes closed under minor containment for which the multicommodity flow problem is polynomially solvable or NP-complete respectively. I would like to explain this direction of research in Section 3., and in my talk.

2. A survey

The cographic multicommodity flow problem is NP-complete in general (cf. Middendorf, Pfeiffer (1989)), even for planar graphs, even under some more restrictions.

We are mainly interested in the polynomiality or NP-completeness of the different problems, and the type of good-characterization they use: in the *combinatorial content*, and not in small differences in complexity.

A. Results restricting the underlying graph

Lovász (1975) proved that a half integer solution exists if and only if the cut condition is satisfied. Seymour (1981a) proved that if in addition the sum of the demands and capacities is even for every cut – in this case the problem is called *Eulerian* –, the cut condition is necessary and sufficient for the existence of an integer flow. These papers were not concerned with algorithms.

Barahona (1980) and Korach (1982) develop primal versions of Edmonds and Johnson's (1973) algorithm to the Chinese Postman Problem that can be improved to find integer solutions for the Eulerian problem. Korach (1982) finds this integer solution via a postoptimality method. Korach and Penn (1986) prove that a flow "almost satisfying" all the demands can be found in an arbitrary graph. Barahona (1987) finds an integer flow in a simple direct way: we recommend this paper for a quick understanding of this Edmonds-Johnson type approach.

For the planar case Matsumoto, Nishizeki and Saito (1985), (1986) apply planar matching algorithms with low worst-case complexity to find integer flows in the Eulerian case (and half-integers in general). The best complexity is obtained by Barahona (1987), who builds into his primal algorithm the efficient data-structures used in planar matching algorithms.

Sebő (1988) proves that there is a simple "translation" between dual solutions to minimum weight perfect matching problems and multicommodity flows, which is in addition integrality-preserving. This allows a mechanical way of translating algorithms from one of these problems to the other. I would like to say more about this translation in my talk: this implies the polynomial solvability of cographic problems if the number of demands is bounded (see B.).

All the above-mentioned good-characterizations can be obtained in polynomial time via some "magic numbers" assigned to the vertices. These seem to contain the main structural information about the problem (cf. Sebő (1987b), (1989) .) like the Gallai-Edmonds structure theorem or the Kotzig-Lovász theorem for matchings, cf. Lovász and Plummer (1986). (In fact, the main result of Sebő (1987b) contains these theorems too.) This structural insight permits to find an integer solution under various restrictions on the graph G, including all the previously known integrality results, (Korach and Penn's result integral "almost flows" as well). More about this in the talk.

A different type of integrality result is proved by Seymour(1977) : in series-parallel graphs the cut condition is necessary and sufficient for the existence of integer flows (and actually much more is proved).

The most general theorem I know in this direction of restricting the graph is the main result of A. Gerards (1988) generalizing both Seymour's Eulerian and series-parallel case.

B. Results restricting the number of demands

In the special case where all capacities are 1, the multicommodity flow problem specializes of course to the problem of finding edge–disjoint paths between a given set of pairs of vertices. Note that even this problem is NP–complete for graphs in general, (Even, Itai, Shamir (1976)) but if the number of demand-edges is fixed, it is polynomially solvable according to the celebrated papers of Robertson and Seymour (1988). The problems we are considering here is independent of this result : we allow *arbitrary demands and capacities* but we have constraints for the matroid defined on the union of the demand and supply edges.

It is implicit in Seymour (1981a) (cf. Sebő (1987a)) that in case of 2 demand edges the cut condition together with the following *parity condition* (P.C) are necessary and sufficient for the existence of an integer flow:

(P.C) There is no odd cut contained in the union of tight cuts,

where a *tight* cut is one for which the cut condition holds with equality, and a cut is *odd* if the sum of the capacities and demands in the cut is odd. (The necessity of (P.C) is easy to see.) Seymour's solution implies a polynomial algorithm as well.

For the case of 3 demand edges Korach(1982) found a decomposition method that reduced the problem to some number of small graphs. Korach and

Newmann (1986) claim that the generalization of this approach works for 4 demand edges, though the number of "bricks" to check is over 200.

Sebő (1988) proves that the cographic multicommodity flow problem is polynomially solvable for any fixed number of demands (cf. Section 2). Actually, a polynomial solution for a more general class of matroids is implied (cf. Section 3).

Frank (1988a) generalized the 2-demand case in the following way: *In a planar graph, if the demand edges are on two faces of the supply-graph*, then again the cut condition and the parity condition are necessary and sufficient for the existence of an integer multicommodity flow. (Frank (1988b) proved several different multicommodity flow theorems (the others are not "matroid flow problems") which can be put in the following form: "there exists an integer flow if and only if we cannot find m odd cuts whose sum can be majorized by a set of cuts with total surplus less than m". (The ***surplus*** of a cut is the difference between the sums of the capacities and the demands of the cut.) He conjectured this condition to be necessary and sufficient in the cographic case as well, which was disproved by Middendorf and Pfeiffer (1989). It would now be interesting to characterize the matroids for which this conjecture still holds. In Section 3. we raise a simply-stated general problem, which gives a compact unified reformulation of this and other problems, and present some first results about these.)

Middendorf and Pfeiffer (1989) note that a polynomial solution for the edge-disjoint paths problem (i.e. for the case when all demands and capacities are 1) follows from a homotopic routing theorem of Schrijver(1988) provided all the demand-edges are on a fixed number of faces.

All of the three previous cases would be contained in one general algorithm if the following question had a positive answer:

Problem: Is the planar multicommodity flow problem polynomially solvable if the demand edges are contained in a fixed number of faces of the supply-graph.

3. Matroids and multicommodity flows

In the (not too big amount of) space left I would like to raise some problems. Some first results about these problems will be presented in my talk.

Seymour(1981b) characterized those matroids for which the cut condition is necessary and sufficient for the solvability of some kinds of multicommodity flow problems. We ask two questions that grow out naturally from Seymour's work:

Definition: We shall say that a class of binary matroids closed under minor containment is *solvable* if the integer flow problem can be solved in polynomial time for an arbitrary set of demand edges, demands and capacities. We shall say it is *finitely solvable* if we require polynomial solvability only for the choice of a prefixed number of demand edges, and *evenly solvable* if we restrict the demand and capacity function to define an Eulerian problem. (We have chosen these three but there is actually an infinite choice of restrictions or generalizations to consider. Eg. even instead of considering only "matroid flow problems" we can also ask for the demand-matroids that define polynomial problems for arbitrary supply-graphs, and vice-versa.)

Problem: What are the (finitely, evenly) solvable classes of matroids ?

To find such classes is not difficult: some compositions of solvable classes are solvable, and the known solvable classes yield more general ones eg. through 1- and 2-sum. But what about finding exactly the border of NP-harness and polynomiality ? Can the solvable classes be characterized ? What is *the complexity of deciding whether for given matroids (as input) the class of matroids not containing these as minors is (finitely, evenly) solvable* ?

If we want to expect "nice" good-characterizations, we have to restrict the notion of solvable classes:

Problem: What is the class of matroids for which the DISTANCE CRITERION is necessary and sufficient for the existence of an integer flow for arbitrary Eulerian weighting and set of demand edges ?

Since this note appears in Japan, we suppose the reader is familiar with the DISTANCE CRITERION, a necessary and sufficient condition for the existence of fractional multicommodity flows –claimed by the "Japanese Theorem" of Iri (1970) and Onaga, Kakusho (1971)–, or can easily look for a reference. Then it is an easy exercise to generalize it to matroids. The DISTANCE CRITERION is probably the most general necessary condition under which we can be interested in the existence of integer multicommodity flows.

Equivalently, we are interested in "routing" matroids defined below. Our only reason to involve the DISTANCE CRITERION was to point out that the "routing" property (countrary to the first impression it gives) really ensures a good-characterization for the existence of a flow:

Definition: A binary matroid will be called *routing* if for an arbitrary set of demands and an arbitrary Eulerian choice of demands and capacities, the existence of a fractional flow implies the existence of an integer flow.

Proposition: *If a matroid is routing, then all its minors are also routing.*

Problem: Characterize routing matroids in terms of excluded minors.

I will show in my talk that routing matroids provide a general context into which different theorems and conjectures for non-Eulerian matroids especially those using (P.C) fit. Frank's theorems and conjecture (cf. Section 2.) also fit into this context, moreover the last problem above is mainly motivated by the problem of characterizing matroids for which Frank's conjecture is true.

The reader may wonder how a non-Eulerian problem with a parity condition can be related to an Eulerian one: the answer in my talk will lie in the notion of *Eulerian extension.* (This notion is implicit in several constructions of Seymour's.) It is easy to see that every matroid has a one-element extension which is Eulerian, and the added element is strictly related with the parities of cuts in the original matroid.

I believe that routing matroids and Eulerian extensions are a very promising direction of research with many applications. I will use my talk for the purpose of advertising some results and questions about them.

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