アルゴリズム 12－27
(1989. 11. 22)

Closure Properties of Alternating One-way Multihead Finite Automata with Constant Leaf-sizes

Hiroshi MATSUNO* Katsushi INOUE** and Itsuo TAKANAMI**

* Oshima National College of Maritime Technology
** Faculty of Engineering, Yamaguchi University

Abstract

In previous papers, we introduced alternating multihead finite automata with constant leaf-sizes (AMHFACLs) and investigated several properties of these automata. Leaf-size, in a sense, reflects the number of processors which run in parallel in scanning a given input word. AMHFACLs are more realistic parallel computation models than ordinary alternating multihead finite automata because of restriction of the number of processors which run in parallel to constant. In this paper, we will examine closure properties of the classes of sets accepted by one-way AMHFACLs and one-way alternating simple multihead finite automata with constant leaf-sizes under the operations of taking union, intersection, complementation, concatenation, Kleene closure, reversal, or ε-free homomorphism.

1. Introduction

In [1] we introduce alternating multihead finite automata with constant leaf-sizes (AMHFACLs) and investigate several properties of these automata. The main result of [1] are as follows. (1) two-way sensing AMHFACLs can be simulated by two-way nondeterministic simple multihead finite automata, (2) for one-way AMHFACLs, k+1 heads are better than k, and (3) for one-way alternating simple multihead finite automata with constant leaf-sizes (ASPMHFACLs), sensing versions are more powerful than non-sensing versions.

In this paper, we will examine closure properties of the class of sets accepted by AMHFACLs and ASPMHFACLs under the operations of taking union, intersection, complementation, concatenation, Kleene closure, reversal, or ε-free homomorphism.

Section 2 gives terminologies and notations necessary for this paper. In Section 3,4, we investigate closure properties of AMHFACLs, ASPMHFACLs, respectively.

2. Preliminaries

The reader is referred to [2] for formal definitions of an alternating multihead finite automaton (AMHFA). A alternating *simple* multihead finite automaton (ASPMHFA) is an AMHFA with the restriction that one head (called the 'reading head') can sense input symbols, while the others (called the 'counting heads') can only detect the left endmarker "¢" and the right endmarker "\$". When the heads of AMHFA (ASPMHFA) are allowed to sense the presence of other heads on the same input position, we call such AMHFA (ASPMHFA) a 'sensing' AMHFA (ASPMHFA).

A one-way AMHFA are defined as usual. A semi-one-way ASPMHFA is an ASPMHFA whose reading head can move only in one direction, but whose counting heads can move in two directions. A one-way ASPMHFA is an ASPMHFA whose reading and counting heads can move in one direction.

A *step* of an AMHFA (ASPMHFA) M consists of reading a symbol from the input string by each head, moving the heads in specified directions (note that any of the heads can remain stationary during a move), and entering a new state, in accordance with the transition function. If one of the heads falls off the input string, then M can make no further move.

In this paper, to represent the different kinds of one-way ASPMHFAs (resp. AMHFAs, sensing AMHFAs) systematically, we use the notation XYk-HFA (resp. Xk-HFA, XSNk-HFA), $k \geq 1$, where

(1) $X \in \{D,N,A,U\}$
  D : deterministic
  N : nondeterministic
  A : alternation
  U : alternating automaton with only universal states

(2) $Y \in \{SP,SNSP\}$
  SP : simple
  SNSP : sensing simple

(3) k-H : k-head (the number of heads is k).

Furthermore,

$\mathscr{L}[XYk\text{-HFA}]=\{T \mid T=T(M) \text{ for some } XYk\text{-HFA } M\}$
$\mathscr{L}[XSNk\text{-HFA}]=\{T \mid T=T(M) \text{ for some } XSNk\text{-HFA } M\}$
$\mathscr{L}[Xk\text{-HFA}]=\{T \mid T=T(M) \text{ for some } Xk\text{-HFA } M\}$.

In [1] we introduce leaf-size for AMHFAs and ASPMHFAs, and define that 'leaf-size' is the number of leaves of an accepting computation tree with the fewest leaves. We modify this definition as follows.

Definition 2.1 Let $L: N \to R$ be a function, where N denotes the set of all positive integers and R denotes the set of all nonnegative real numbers. For each tree t, let LEAF(t) denote the leaf-size of t (i.e., the number of leaves of t). We say that an XYk-HFA (XSNk-HFA, Xk-HFA) M is L(n) leaf-size bounded, if we give an input x of length n to M, there is no computation tree of M on x such that $\mathrm{LEAF}(t) > \lceil L(n) \rceil$.

For each $X \in \{A,U\}$, $Y \in \{SP, SNSP\}$, $k \geq 1$, we let XYk-HFA(L(n)) (resp. Xk-HFA(L(n))), XSNk-HFA(L(n))) denote L(n) leaf-size bounded XYk-HFA (resp. Xk-HFA, XSNk-HFA). $\mathscr{L}$[XYk-HFA(L(n))] $\mathscr{L}$[Xk-HFA(L(n))], and $\mathscr{L}$[XSNk-HFA(L(n)] are defined similarly above.

3. Multihead Finite Automata

In this section, we will investigate closure properties of the class of sets accepted by AMHFACLs.

Lemma 3.1. Let $T_1 = \{w_1 2 w_2 2 \cdots 2 w_{2b} \mid (b \geq 1) \ \& \ \forall i (1 \leq i \leq 2b)[w_i \in \{0,1\}^* 3 \{0,1\}^*] \ \& \ \exists i,j [w_i = x3y \ \& \ w_j = x3z \ \& \ y \neq z]\}$. Then,

(1) $T_1 \in \mathscr{L}$[N2-HFA], and

(2) $\overline{T}_1 \notin \bigcup_{1 \leq k < \infty} \mathscr{L}$[NSNk-HFA].

Proof. (1): The proof of (1) is omitted, since it is easily seen.

(2): Suppose that for some $k \geq 1$, there exists an NSNk-HFA M accepting $\overline{T}_1$. Let $T_1' = \{w_1 2 w_2 2 \cdots 2 w_{2b} \mid \forall (1 \leq i \leq 2b)[w_i = w_{2b+1-i} = B(\min(i, 2b+1-i)) 2y \ \& \ y \in \{0,1\}^*\}$.‡ It is easily seen that $T_1' \subseteq \overline{T}_1$. Thus, from assumption above, we can see that T_1' is accepted by M. On the other hand, we can prove $T_1 \notin \bigcup_{1 \leq k < \infty} \mathscr{L}$[NSNk-HFA] by using same technique as in the proof of Theorem 1 in [3]. This is a contradiction. Q.E.D.

Theorem 3.1. *For each $k \geq 2$ and each $s \geq 1$, $\mathscr{L}$[Ak-HFA(s)], $\mathscr{L}$[ASNk-HFA(s)], $\bigcup_{1 \leq r < \infty} \bigcup_{1 \leq t < \infty} \mathscr{L}$[Ar-HFA(t)], and $\bigcup_{1 \leq r < \infty} \bigcup_{1 \leq t < \infty} \mathscr{L}$[ASNr-HFA(t)] are not closed under complementation.*

Proof. It is shown in Theorem 4.4 in [1] that $\mathscr{L}$[ASNk-HFA(s)] $\subsetneq \mathscr{L}$[NSN(ks)-HFA] ($k \geq 2$, $s \geq 1$). From this fact and Lemma 3.1, we can get that $\overline{T}_1 \notin \bigcup_{1 \leq k < \infty} \bigcup_{1 \leq s < \infty} \mathscr{L}$[ASNk-HFA(s)]. This completes the proof of the theorem. Q.E.D.

Lemma 3.2. For each $r \geq 2$ and each $i (1 \leq i \leq r(r+1)/2)$, $T_2(r,i) = \{w_1 2 w_2 2 \cdots 2 w_p \mid p = r(r+1) \ \& \ \forall i (1 \leq i \leq p)[w_i \in \{0,1\}^* \ \& \ w_i = w_{p+1-i}]\}$. Then,

(1) for each $r \geq 2$ and each $i (1 \leq i \leq r(r+1)/2)$, $T_2(r,i) \in \mathscr{L}$[D2-HFA] and

(2) $T_2(ks,1) \cap T_2(ks,2) \cap \cdots \cap T_2(ks,(ks+1)ks/2) = T_2((ks+1)ks/2) \notin \mathscr{L}$[ASNk-HFA(s)] ($k \geq 2$, $s \geq 1$).

Proof. (1): The proof of (1) is omitted, since it is easy to prove.

(2): It is shown in [3], that $T_2((k+1)k/2) \notin \mathscr{L}$[NSNk-HFA]. From this fact and Theorem 4.4 in [1] (see above), we can get this lemma. Q.E.D.

We shall formulate two sufficient conditions for a language L in order to be in $\mathscr{L}$[USNk-HFA(s)] for $k \geq 2$ and $s \geq 1$. We shall need the following languages for arbitrary natural f.

$C_f(n) = \{u c w_1 c w_2 c \cdots c w_f c w_f c \cdots c w_2 c w_1 \mid |u| = |w_i| = n \ \& \ u, w_i \in \{a,b\}^*\}$ for each $i (1 \leq i \leq f)\}$;‡‡

$D_f(n) = \{v_1 d v_1 d \mid v_1 \in C_f(n)\}$;

$E_f(n) = \{u c w_1 c w_2 c \cdots c w_f c w_{f+1} c \cdots c w_{2f-1} c w_{2f} \mid |u| = |w_i| = n \ \& \ u, w_i \in \{a,b\}^*$ for each $i (1 \leq i \leq 2f) \ \& \ \exists j (1 \leq j \leq f)[w_j \neq w_{2f+1-j}]\}$ and

$F_f(n) = \{v_1 c w_1 c w_2 c \cdots c w_f c w_f c \cdots c w_2 c w_1 d v_2 c w_1 c w_2 c \cdots c w_f c w_f c \cdots c w_2 c w_1 d \mid |v_1| = |v_2| = |w_i| = n, \ v_1, v_2, w_i \in \{a,b\}^*$ for each $i (1 \leq i \leq f($ and $v_1 \neq v_2\}$.

Let $C_f = \bigcup_{1 \leq n < \infty} C_f(n)$; $D_f = \bigcup_{1 \leq n < \infty} D_f(n)$; $E_f = \bigcup_{1 \leq n < \infty} E_f(n)$ and $F_f = \bigcup_{1 \leq n < \infty} F_f(n)$ for arbitrary $f = 1, 2, 3, \ldots$.

Lemma 3.3. Let L be an arbitrary set fulfilling the following conditions:

(1) $L \supseteq C_f \cup D_f$

(2) $L \cap (E_f \cup F_f) = \phi$.

Let $f = k(k-1)s/2$, where $k \geq 2$ and $s \geq 1$. Then L is not in $\mathscr{L}$[USNk-HFA(s)].

Proof. The proof is extension of the proof of Theorem 1 in [4]. Let us assume that there exists a USNk-HFA(s') which recognize a set L satisfying (1) and (2) where $1 \leq s' \leq s$. We need the following notations.

A *configuration* of M working on the input word w is (k+1)-tuple $(q, i_1, i_2, \ldots, i_k)$ where q is the state of the finite state control and i_j is the position of the j-th head on the input word w.

A *prominent configuration* is a configuration of the computation tree on the input word x in $C_f \cup D_f \cup E_f \cup F_f$, from which M moves one of its head on the symbol c,d or \$.

The subsequence of prominent configuration of the j-th path of the computation tree on the word x is called *j-pattern* of x (denoted by $P_j(x)$).

For each word x given to USNk-HFA(s) M, we let $(P_1(x), P_2(x), \ldots, P_{s'}(x))$ denote the *pattern* of M.

Let USNk-HFA(s') M recognizing a set L which satisfies (1) and (2) have t states. We shall consider the initial part of computation tree of M on the word y in $C_f(n) \cup D_f(n)$, that begins in the initial configuration and ends in prominent configuration, in which one of the heads had read the whole subword $y_1 = u c w_1 c w_2 c \cdots c w_f c w_f c \cdots c w_2 c w_1$ of the input word y. [I.e., the ini-

‡ B(n) denotes the binary representation of i.

‡‡ For any word w, |w| denote the length of w.

tial part of the computation tree is that part of the computation tree which is the same for words y_1 and y_1dy_1d bacause M does not know whether it is working on the word y_1 in $C_f(n)$ or on the word y_1dy_1d in $D_f(n)$.]

Now, let us consider the number of all patterns of the initial part of the computation tree on the word y in $C_f(n) \cup D_f(n)$ which we denote p(n). If we note that $|y_1|=k(k-1)s(n+1)+n$, we can easily see that the number of all configurations on word y_1 is bounded by

$$t[(k(k-1)s+1)(n+1)]^k.$$

And so we obtain the following inequality

$$p(n) \leqq [[t[(k(k-1)s+1)(n+1)]^k]^{k((k-1)ks+1)}]^s,$$

bacause no j-pattern of initial part of computation tree can consist more than $k((k-1)ks+1)$ prominent configurations and leaf-size of computation tree is bounded by $s'(\geqq 1)$.

Since the number of all words y_1 from $C_f(n)$ is

$$2^{[(k-1)ks/2+1]n}$$

there exists a pattern σ of the initial part of the computation tree such that at least

$$2^{[(k-1)ks/2+1]n}/p(n)$$

different words y_1 from $C_f(n)$ have the same pattern σ of the initial part of the computation tree.

Now we distinguish two following cases according to the last prominent configuration $(q, i_1, i_2, \ldots, i_k)$ of the pattern σ.

(1) $i_j > n$ for all j in $\{1,2,\ldots,k\}$, i.e., all heads had read the initial subword $u \in \{a,b\}^*$.

(2) There exists some j in $\{1,2,\ldots,k\}$ such that $i_j \leqq n$, i.e., at least one head had not read the initial subword $u \in \{a,b\}^*$.

We shall below show that both (1) and (2) lead to a contradiction.

(1) In this case we shall consider input words y_1dy_1d in $D_f(n)$ where y_1 has the pattern σ. Noting that there exist at least $2^{((k-1)ks/2+1)n}/p(n)$ different words y_1 with the pattern σ, we obtain that there exist at least

$$m=[2^{((k-1)ks/2+1)n}/p(n)] \cdot [1/2^{(k-1)ks/2}]$$

different words y_1 with the pattern σ differing from each other only in the initial subword u, i.e., there exist at least m words $y_1=ucx$, where $x=w_1cw_2c\cdots cw_fcw_fc\cdots cw_2cw_1$ is fixed, with the pattern σ.

It is obvious that $m=2^n/p(n)$ is greater than 2 for sufficient large n, since p(n) is bounded by a polynomial. It means that for sufficiently large n there exist two words from $C_f(n)$ $v_1=u_1cx$ and $v_2=u_2cx$, where $u_1 \neq u_2$, with the same pattern σ of the initial part of the computation tree. Since M accepts the word $y=v_1dv_1d=u_1cxdu_1cxd$ in $D_f(n)$ and the state set of M consists of only universal states, it follows that M must also accept the word $y'=u_2cxdu_1cxd$ which clearly belongs to $F_f(n)$.

(2) We shall consider the input word y in $C_f(n)$ in this case. Let us consider the whole accepting computation trees on all

$$2^{(k(k-1)s/2+1)n}/p(n)$$

different words $y=y_1$ in $C_f(n)$ having the same pattern σ of the initial part of the computation tree.

Let p'(n) be the number of all possible patterns of accepting computation trees on words y in $C_f(n)$. We obtain the following inequality,

$$p'(n) \leqq [[t[(k(k-1)s+1)(n+1)]^k]^{k((k-1)ks+1}]^s.$$

From this fact it follows that there exist at least

$$2^{(k(k-1)s/2+1)n}/p'(n)$$

different words y in $C_f(n)$ with the same pattern σ' containing the pattern σ as an initial subsequence.

From Lemma 4.1 in [1] and the assumption (2), we can see that for each input word in $C_f(n)$, there must be an i_0 such that both subword w_i of the words $y= ucw_1cw_2c\cdots cw_{i_0}c\cdots cw_fcw_fc\cdots cw_{i_0}c\cdots cw_2cw_1$ are never read by any couple of heads at the same time. It means that there exist at least

$$m=[2^{(f+1)n}/p'(n)] \cdot [1/2^{(f-1)n}2^n]=2^n/p'(n)$$

different words in $C_f(n)$ having the same pattern σ', which differ from each other in the subword w_{i_0} only.

It can be seen that $m \geqq 2$ for sufficiently large n and so there exist two words in $C_f(n)$

$$v_1=ucw_1cw_2c\cdots cw_{i_0}c\cdots cw_f cw_fc\cdots cw_{i_0}c\cdots cw_2cw_1$$ and

$$v_2=ucw_1cw_2c\cdots cw'_{i_0}c\cdots cw_f cw_fc\cdots cw'_{i_0}c\cdots cw_2cw_1$$

with the same pattern σ' of the accepting computation tree, where $w_{i_0} \neq w'_{i_0}$.

By an argument similar to that in the proofs of Theorem 1 in [3] and Theorem 1 in [4], it can be shown that M must also accept the word

$$y'=ucw_1cw_2c\cdots cw_{i_0}c\cdots cw_f cw_fc\cdots cw'_{i_0}c\cdots cw_2cw_1,$$

which belongs to $E_f(n)$. This is a contradiction.

Q.E.D.

Lemma 3.4. Let L be an arbitrary set fulfilling the following conditions:

(3) $L \supseteq \{e\} \cdot C_f \cup \{e\} \cdot D_f$

(4) $L \cap (\{e\} \cdot E_f \cup \{e\} \cdot F_f) = \phi$.

Let $f=k(k-1)s/2$, where $k \geqq 2$ and $s \geqq 1$. Then the set L is not in $\mathscr{L}$[USNk-HFA(s)].

Proof. It is a matter of easy technical considerations to show that if there exists a set L satisfying the conditions of Lemma 3.1 such that $L \in \mathscr{L}$[USNk-HFA(s)], then there exist a set L' fulfilling the conditions of Lemma 3.3 such that $L' \in \mathscr{L}$[USNk-HFA(s)]. Q.E.D.

The following theorem can be directly gotten from Lemma 3.2 above.

Theorem 3.2. *For each $k \geqq 2$ and $s \geqq 1$, $\mathscr{L}$[Ak-HFA(s)] and $\mathscr{L}$[ASNk-HFA(s)] are not closed under intersection.*

Theorem 3.3. *For each $k \geqq 2$ and $s \geqq 1$, $\mathcal{L}$[Uk-HFA(s)] and $\mathcal{L}$[USNk-HFA(s)] are not closed under the following operations.*

(1) intersection
(2) concatenation
(3) reversal
(4) Kleene closure
(5) union
(6) ε-free homomorphism.

Proof. (1): Obvious from Lemma 3.2.

(2): Let us consider the following languages:

$L_1=\{a,b\}^*c \cup \{\varepsilon\}$,

$L_2=\{udud \mid u \in \{a,b,c\}^*\} \cup \{\varepsilon\}$,

$G_f=\{w_1cw_2c\cdots cw_fcw_fc\cdots cw_2cw_1 \mid w_i \in \{a,b\}^*$ for $1 \leqq i \leqq f\} \cup \{\varepsilon\}$ for $i=1,2,3,\ldots$

Clearly, $L_1 \in \mathcal{L}$[D1-HFA], $L_2 \in \mathcal{L}$[D2-HFA] and $G_f \in \mathcal{L}$[Uk-HFA(s)] for $f \leqq k(k-1)s/2$. On the other hand, by an argument similar to that in the proof of Theorem 2 in [4], we can see that the set $L_1L_2G_{k(k-1)s/2}$ is not in $\mathcal{L}$[USNk-HFA(s)].

(3): The set $L_2 \cup \{a,b\}^*cG_{(k-1)ks/2}$ does not belong to $\mathcal{L}$[USNk-HFA(s)], since it fulfills the conditions of Lemma 3.3, but $L_2^R \cup \{a,b\}^*cG_{(k-1)ks/2}$ belongs to $\mathcal{L}$[Uk-HFA(s)].

(4): Let us consider the set $L_3=\{e\}\cdot L_2 \cup \{a,b\}^*cG_{(k-1)ks/2} \cup \{e\}$ which belongs to $\mathcal{L}$[Uk-HFA(s)]. On the other hand, by an argument similar to that in the proof of Theorem 4 in [4], we can see that L_3^* satisfies the condition (3) and (4) of Lemma 3.4 which implies that L_3 is not in $\mathcal{L}$[USNk-HFA(s)].

(5): It can be easily seen that the set L_2 and $\{a,b\}^* cG_{(k-1)ks/2}$ belongs to $\mathcal{L}$[Uk-HFA(s)] for each $k \geqq 2$ and $s \geqq 1$ and that the set $L_2 \cup \{a,b\}^*cG_{(k-1)ks/2}$ fulfills the conditions of Lemma 3.3.

(6): Clearly, the set $L_4=\{e\}\cdot L_2 \cup \{g\}\cdot\{a,b\}^*cG_{(k-1)ks/2}$ belongs to $\mathcal{L}$[Uk-HFA(s)] for each $k \geqq 2$ and $s \geqq 1$. Let us define a ε-free homomorphism h as follows: h(e)=h(g)=e, h(a)=a, h(b)=b, h(c)=c, h(d)=d. Then $T(L_4)$ satisfies the condition (3) and (4) of Lemma 3.4. Q.E.D.

When leaf-size is not restricted, the following result holds.

Theorem 3.4. *For each $k \geqq 2$, $\mathcal{L}$[Uk-HFA] and $\mathcal{L}$[USNk-HFA] are not closed under complementation.*

Proof. For otherwise, we suppose that $\mathcal{L}$[Uk-HFA] is closed under complementation. From Theorem 1 in [5], we get that $\mathcal{L}$[Uk-HFA]=co-$\mathcal{L}$[Nk-HFA] for $k \geqq 1$. It follows that for some set L, if $L \in \mathcal{L}$[Nk-HFA] then $\bar{L} \in \mathcal{L}$[Uk-HFA] and $\bar{\bar{L}}=L \in \mathcal{L}$[Uk-HFA] from assumption above. Thus, $\mathcal{L}$[Nk-HFA] $\subseteq \mathcal{L}$[Uk-HFA]. On the other hand, from Corollary 3 (3) in [5], we can get that $\mathcal{L}$[Uk-HFA] is incomparable with $\mathcal{L}$[Nk-HFA] for each $k \geqq 2$. This is a contradiction. The case of $\mathcal{L}$[USNk-HFA] is proved by using similar argument as above.

Q.E.D.

4. Simple Multihead Finite Automata

The closure properties under Boolean operations of ASPMHFAs are given in [6]. In this section, we first summarise the closure properties under Boolean operation of ASPMHFACLs derived from the results in [6]. The following theorem is obvious.

Theorem 4.1. *For each $Y \in$ {SP,SNSP}, $k \geqq 1$, and $s \geqq 1$, $\mathcal{L}$[AYk-HFA(s)] is closed under union.*

Theorem 4.2. *For each $Y \in$ {SP,SNSP}, $k \geqq 2$, and $s \geqq 1$, $\mathcal{L}$[UYk-HFA(s)] is not closed under union.*

Proof. The proof is given by Lemma 6.5 in [6]. Q.E.D.

Theorem 4.3. *For each $X \in$ {A,U}, $Y \in$ {SP,SNSP}, $k \geqq 2$, and $s \geqq 1$, $\mathcal{L}$[XYk-HFA(s)] is not closed under complementation and intersection.*

Proof. The proof is given by Lemmas 6.2, 6.3, and 6.4 in [6]. Q.E.D.

We next investigate the closure properties of ASPMHFA with only universal states (USPMHFA) under operations of Kleene closure, reversal, and ε-free homomorphism.

Lemma 4.1. Let $T_3=\{x \in \{0,1\}^+ \mid (|x| \geqq 3)$ & $(|x|$ is odd) & (the center symbol in x is '1'))} and $T_4=\{a\}^*$. Then,

(1) T_3, T_4, $T_4T_3 \in \mathcal{L}$[DSP2-HFA] and

(2) $T_3T_4=(T_4T_3)^R \notin \mathcal{L}$[USNSPk-HFA(s)].

Proof. (1): We omit the proof of (1), since it is readily proved.

(2): Let $T_5=T_3T_4$. Suppose that there exists a USNSPk-HFA(s) M which accepts T_5. Let u be the number of states (of the finite control) of M and R be the reading head of M. For each $n \geqq 1$, let

$V(n)=\{0^nw0^{r_1}a^{r_2} \mid (w \in \{0,1\}^*)$ & $(|w|=n)$ & $(r_1,r_2 \geqq 1)$ & $(r_1+r_2=2n)\}$.

For each $x=0^nw0^{r_1}a^{r_2}$ in V(n), let SC(z) be the multi-set of semi-configurations‡ of M defined as follows.

$SC(z)=\{(q,i_1,i_2,\ldots,i_{k-1}) \mid c=(z,2n+1,(q,i_1,i_2,\ldots,i_{k-1}))$ is a configuration of M just after the point where R reads the initial segment 0^nw of ¢z$}.

Then, the following proposition must hold.

Proposition 4.1. For any two words z, z' in V(n) whose initial segments 0^nw's (of length 2n) are different, $SC(z) \neq SC(z')$.

[For otherwise, suppose that $z=0^nw0^{r_1}a^{r_2}$, $z'=0^nw'0^{r_1}a^{r_2}$ $(w \neq w')$ and $SC(z)=SC(z')$. Let

‡ $(q,i_1,i_2,\ldots,i_{k-1})$ represents the state of the finite control and positions of k-1 counting heads of M, and is called semi-configuration of M.

$w=w_1 1 w_2$, $w'=w_1 0 w_2$ ($|w_1|=|w_2|=t: 0\leq t\leq n-1$). We then consider the following two words $z=0^n w_1 1 w_2 0^p a^r$ and $z_1=0^n w_1 0 w_2 0^p a^r$ ($p=n+t-|w_2|$, $r=2(n-t)-1$) in $V(n)$. Clearly, $z\in T_1$, and so z is accepted by M. It follows that z_1 must be also accepted by M. This contradicts the fact that z_1 is not in T_1.]

Clearly $t(n)<u(4n+2)^{k-1}$, where $t(n)$ is the number of possible semi-configurations of M just after the point where R reads the initial segments $0^n w$'s (of length 2n) of words in $V(n)$. For each z in $V(n)$, the leaf-size of computation tree of M on z is at most $s(\geq 1)$. Thus, for each z in $V(n)$, $|SC(n)|\leq s$. Therefore, letting $S(n)=\{SC(z)\mid z\in V(n)\}$, it follows that for some constants c and c'

$$|S(n)| < ct(n)^s < c'n^{(k-1)s}.$$

As is easily seen, $|V(n)|=2^n$. From these facts, it follows that for large n, $|S(n)|<|V(n)|$. Therefore for large n, there must be two words z,z' in $V(n)$ whose initial segments $0^n w$'s are different such that $SC(z)=SC(z')$. This contradicts Proposition 4.1. Q.E.D.

Theorem 4.5. *For each $k\geq 2$, $Y\in\{SP,SNSP\}$, $\mathscr{L}[UYk\text{-}HFA(s)]$ is not closed under the following operations.*

(1) reversal

(2) Kleene closure

(3) ε -free homomorphism

Proof. (1): Obvious from Lemma 4.1.

(2): Let $T_6=T_3\cup T_4$. It is easy to see that $T_6\in\mathscr{L}$[DSP2-HFA]. On the other hand, $T_6\cap(\{0,1\}^+\{a\}^*)=T_3T_4\notin\bigcup_{1\leq k<\infty}\bigcup_{1\leq s<\infty}\mathscr{L}$[USNSPk-HFA(s)] (from Lemma 4.1). It follows from this fact and the fact that $\mathscr{L}$[USNSPk-HFA(s)] is closed under union with a regular set (It is easy to prove.), that $T_6\notin\bigcup_{1\leq k<\infty}\bigcup_{1\leq s<\infty}\mathscr{L}$[USNSPk-HFA(s)]. This completes the proof of (2).

(3): Let $T_3'=\{x\in\{0,1\}^+ \mid (|x|\geq 3)$ & ($|x|$ is odd) & (x has exactly one '2' as the center symbol of x$\}$. Then it is readily proved that $T_3T_5\in\mathscr{L}$[DSP2-HFA]. On the other hand, let h be the ε -free homomorphism defined by $h(0)=0$, $h(1)=1$, $h(2)=1$, and $h(a)=a$. Then $h(T_3'T_5)=T_3T_5\notin\bigcup_{1\leq k<\infty}\bigcup_{1\leq s<\infty}\mathscr{L}$ [USNSPk-HFA(s)] ($s\geq 1$). This completes the proof of (3). Q.E.D.

Theorem 4.6. *For each $X\in\{A,U\}$, $k\geq 2$, and $s\geq 1$, $\mathscr{L}[XSPk\text{-}HFA(s)]$ is not closed under concatenation.*

Proof. For each $\ell\geq 2$, let $L_\ell=\{a^n b^n \mid n\geq 1\}^\ell$. It is easily seen that $L_{k-1}\in\mathscr{L}$[DSPk-HFA]. If $L_{((k-1)s+2)((k-1)s+1)s}\notin\mathscr{L}$[ASPk-HFA(s)] is shown, then we completes the proof of the theorem. The proof is extention of Theorem 1 in [7].

For otherwise we suppose that there exists an ASPk-HFA(s) M ($k\geq 2$, $s\geq 1$) accepting $L_{((k-1)s+2)((k-1)s+1)s}$ which has m states. (Without loss of generality we assume that the input tape of M has no left endmarker.) For each input word w in $L_{((k-1)s+2)((k-1)s+1)s}$, there exists an accepting computation tree of M denoted by $T_M(w)$. We divide input word w into s subword. That is,

$$w=w_1 w_2\cdots w_s.$$

Without loss of generality, we assume that each node of $T_M(w)$ which is labeled by a configuration with a universal state has exactly two children. Then, because of the bounded leaf-size s, there are at most s-1 nodes labeled by configurations with universal state in $T_M(w)$. From this fact and the word w has s subwords w_i's, there is a subword w_i in the word w such that on each computation path of $T_M(w)$, there is a sequence of steps which implies that M never enters a universal state while reading the subword w_i. We let such subword w_i be w_f, let $e(1\leq e\leq s)$ be the number of sequence of steps during M reads the subword w_f ,and let $S(1), S(2),\ldots, S(e)$ be these e sequences of steps.

Let the subword $w_i(1\leq i\leq s)$ be as follows.

$$w_i=y_1y_2\cdots y_{(k-1)s+1}\ (1\leq i\leq s)$$
$$y_i=y=x_1x_2\cdots x_{(k-1)e+2}\ (1\leq i\leq (k-1)s+1)$$
$$x_i=x=a^n b^n\ (1\leq i\leq (k-1)e+2).$$

For each $i(1\leq i\leq e)$ and each $j(1\leq j\leq (k-1)s+1)$, let $N_i(j)$ be the number of counting heads that reach the right endmarker \$ while the reading head R reads the y_j in w_f, in the i-th sequence $S(i)$ of $T_M(w)$. Since M has only (k-1) counting heads and leaf-size s, it follows that $N_1(j_0)=N_2(j_0)=\cdots=N_e(j_0)=0$ for some $j_0(1\leq j_0\leq (k-1)e+2)$.

Consider the case when in $T_M(w)$ R reads the subword y_{j_0} such that $N_1(j_0)=N_2(j_0)=\cdots=N_e(j_0)=0$. We fix an arbitrary number i_0 $(1\leq i_0\leq (k-1)e+2)$ and let

$$x_{i_0}=a_{i_0 1}a_{i_0 2}\cdots a_{i_0 n}b^n\ (a_{i_0 j}=a,\ 1\leq j\leq n).$$

For each $j(1\leq j\leq e)$ and each symbol $a_{i_0\ell}$, let $q^j_{i_0\ell}$ be the state in which M is when R moves onto $a_{i_0\ell}$ on the j-th sequence $S(j)$. For each symbol $a_{i_0\ell}$, we consider the e-tuple of states as follows.

$$(q^1_{i_0\ell}, q^2_{i_0\ell},\ldots, q^e_{i_0\ell})=Q_{i_0\ell}.$$

We call $Q_{i_0\ell}$ above a *multi-state* of M.

A *j-configuration* of M is a (k+1)-tuple $(q^j_i, b^j_{i1},\ldots,b^j_{ik})$ (denoted by c^j_i), where q^j_i is the state of finite control of M and $b^j_{i\ell}$ is the position of the ℓ-th head in $S(j)$. An *j-increment* is a (k+1)-tuple $(q^j_i, h^j_{i1},\ldots,h^j_{ik})$, where q^j_i is the state of finite control of M and each $h^j_{i\ell}$ is either 0 or 1. (Informally, the j-increment describes moving the heads at one step of computation in j-th sequence $S(j)$.) Let $c^j_f, c^j_{f+1},\ldots,c^j_\ell$ be the subsequence of $S(j)$, where c^j_f is a j-configuration when R reads the symbol $a_{i_0 1}$ and c^j_ℓ is a j-configuration when R reads the symbol $a_{i_0 n}$ of $x_{i_0}=a_{i_0 1}a_{i_0 2}\cdots a_{i_0 n}b^n$.

‡ A configuration of M on w is a (k+1)-tuple of a state of finite control and k heads positions.

We say that the sequence of j-increments $d^j_f, d^j_{f+1}, \ldots, d^j_{g-1}$, where

$$d^j_i = (q^j_i, b^{i+1}_{j1} - b^i_{j1}, \ldots, b^{i+1}_{jk} - b^i_{jk})$$

if $c^j_i = (q^j_i, b^i_{j1}, \ldots, b^i_{jk})$ and

$$c_{i+1} = (q^j_{i+1}, b^{i+1}_{j1}, \ldots, b^{i+1}_{jk})$$

for each $i(1 \leqq i \leqq g-1)$, is the sequence of j-increments of $c^j_f, c^j_{f+1}, \ldots, c^j_g$.

We let

$$\begin{array}{l} d^1_{f_1}, d^1_{f_1+1}, \ldots, d^1_{g_1-1} \\ d^2_{f_2}, d^2_{f_2+1}, \ldots, d^2_{g_2-1} \\ \quad\vdots \\ d^e_{f_e}, d^e_{f_e+1}, \ldots, d^e_{g_e-1} \end{array} \qquad (1)$$

where $d^j_{f_j}, d^j_{f_j+1}, \ldots, d^j_{g_j-1}$ is a subsequence of $S(j)$ and $d^j_{f_j}$ $(d^j_{g_j-1})$ is j-increment when M reads the symbol $a_{i_0 1}$ $(a_{i_0 n})$ in $S(j)$.

Let

$$\begin{array}{l} d^1_{\alpha_1}, d^1_{\alpha_1+1}, \ldots, d^1_{\beta_1} \\ d^2_{\alpha_2}, d^2_{\alpha_2+1}, \ldots, d^2_{\beta_2} \\ \quad\vdots \\ d^e_{\alpha_e}, d^e_{\alpha_e+1}, \ldots, d^e_{\beta_e} \end{array} \qquad (f_i \leqq \alpha_i < \beta_i \leqq g_i - 1 \; : \forall i (1 \leqq i \leqq e))$$

be a subsequence of (1) , and is denoted by *segment*. If the length of $d^3_{\alpha_3}, \ldots, d^3_{\beta_3}$ is shortest among $d^i_{\alpha_i}, \ldots, d^i_{\beta_i}$'s then we let the length of $d^3_{\alpha_3}, \ldots, d^3_{\beta_3}$, be the *length of the segment*.

For each symbol $a_{i_0 \ell_1}$ $(a_{i_0 \ell_2})$ $(\ell_1 < \ell_2)$, let $d^j_{\ell_{1j}}$ $(d^j_{\ell_{2j}})$ be j-increment when M reads symbol $a_{i_0 \ell_1}$ $(a_{i_0 \ell_2})$, and let $Q_{i_0 \ell_1} = Q_{i_0 \ell_2}$. Then we say that the segment

$$\begin{array}{l} d^1_{\ell_{11}}, d^1_{\ell_{11}+1}, \ldots, d^1_{\ell_{21}} \\ d^2_{\ell_{12}}, d^2_{\ell_{12}+1}, \ldots, d^2_{\ell_{22}} \\ \quad\vdots \\ d^e_{\ell_{1e}}, d^e_{\ell_{1e}+1}, \ldots, d^e_{\ell_{2e}} \end{array} \qquad (f \leqq \ell_{1j} < \ell_{2j} \leqq g-1 \; : \forall i(1 \leqq i \leqq e))$$

is *Q-cycle*. Futhermore, we say the following (k-1)e+1-tuple

$$\left(\sum_{\ell=\ell_{11}}^{\ell_{21}} h^\ell_{11}, \sum_{\ell=\ell_{11}}^{\ell_{21}} h^\ell_{12}, \ldots, \sum_{\ell=\ell_{11}}^{\ell_{21}} h^\ell_{1k}, \sum_{\ell=\ell_{12}}^{\ell_{22}} h^\ell_{22}, \sum_{\ell=\ell_{12}}^{\ell_{22}} h^\ell_{23}, \ldots, \sum_{\ell=\ell_{12}}^{\ell_{22}} h^\ell_{2k}, \ldots, \sum_{\ell=\ell_{1e}}^{\ell_{2e}} h^\ell_{e2}, \sum_{\ell=\ell_{1e}}^{\ell_{2e}} h^\ell_{e3}, \ldots, \sum_{\ell=\ell_{1e}}^{\ell_{2e}} h^\ell_{ek} \right)$$

is *parameter* of this Q-cycle.

Fact 4.1. If the (1) above can be written in the form s_1, p_1, s_2, p_2, s_3, where s_1, s_2, s_3 are the segments and p_1, p_2 are the Q-cycle (for some multi-state Q), then there is an accepting computation tree of M which is constructed by replacing s_1, p_1, s_2, p_2, s_3 of $T_M(w)$ by s_1, p_1, p_2, s_2, s_3.

Since every segment with length at least m^e+1 contains a Q-cycle, by Fact 4.1, we have the following.

Fact 4.2. There is a permutation of (1) which can be written in the form

$$s_1, p_1, s_2, p_2, \ldots, s_r, p_r, s_{r+1}, \qquad (2)$$

where $r \leqq m^e$, each s_i is a segment with length at most m^e, each p_i can be written in the form $p_i = p^1_i, p^2_i, \ldots, p^{g_i}_i$, where each p^j_i is a Q-cycle with length at most m^e and there is an accepting computation tree of M on w which is constructed by replacing (1) of $T_M(w)$ by $s_1, p_1, s_2, p_2, \ldots, s_r, p_r, s_{r+1}$.

Fact 4.3. Let p^j_i's be the Q-cycles from Fact 2. For (2), there is a parameter $v = (v_1, v_2, \ldots, v_{(k-1)e+1})$ with $v_1 > 0$ and $0 \leqq v_i \leqq m^e$ for each $i(1 \leqq i \leqq (k-1)e+1)$, such that the number of Q-cycles p^j_i with parameter v is at least $(n-(m^e+1)m^e)/(m^e(m^e+1)^{(k-1)e+1})$.

Proof. Since the reading head crosses the i_0-th subword a^n of word y_{j_0} during the part of the computation corresponding to (2), there are n increments (in (2)) at which the reading head is moved to the right. Clearly, at least $n-(m^e+1)m^e$ increments from these n increments are contained in the cycles p^j_i, because $r \leqq m^e$ and the length of each s_i is at most m^e (see Fact 4.2.). This implies that the number of Q-cycles p^j_i with parameters whose first component is greater than zero is at least $(n-(m^e+1)m^e)/m^e$. Since the number of all different parameters, for the cycles with length at most m^e, is at most $(m^e+1)^{(k-1)e+1}$, there is a parameter v such that the number of cycles p^j_i with parameter v is at least $(n-(m^e+1)m^e)/(m^e(m^e+1)^{(k-1)e+1})$. ■

Since the number i_0 $(1 \leqq i_0 \leqq (k-1)e+2)$ was selected arbitrarily, by Fact 4.3, we have that there is an accepting computation tree of M on w with the sequence

$$u_1, z_1, u_2, z_2, \ldots, u_{(k-1)e+2}, z_{(k-1)e+2}, u_{(k-1)e+3}, \qquad (3)$$

where, for each $i(1 \leqq i \leqq (k-1)e+2)$, z_i is the segment corresponding to the part of this accepting computation tree at which the reading head reads the i-th subword a^n of word y_{j_0}, and z_i is of the form (2), and each u_i is a segment. Further, by Fact 4.3, there are parameters $v^i = (v^i_1, v^i_2, \ldots, v^i_{(k-1)e+1})$ for each $i(1 \leqq i \leqq (k-1)e+2)$, with $v_1 > 0$ and $0 \leqq v_j \leqq m^e$ for each $i(1 \leqq i \leqq (k-1)e+2)$ and each $j(1 \leqq j \leqq (k-1)e+1)$, such that the number of cycles with parameter v^i is at least $(n-(m^e+1)m^e)/(m^e(m^e+1)^{(k-1)e+1})$ in segment z_i for each $i(1 \leqq i \leqq (k-1)e+2)$. Clearly there are rational numbers $r_1, r_2, \ldots, r_{(k-1)e+2}$ such that

$$\sum_{i=1}^{(k-1)e+2} r_i v^i = 0, \qquad (4)$$

where

$0 = (0, 0, \ldots, 0)$ and $r_i \neq 0$ for some $i(1 \leqq i \leqq (k-1)e+2)$,

bacause the vectors $v^i \neq 0$ are linearly independent. Without loss of generality we can assume that the r_i's in (4) are integers.

Let w be the word as above. Now we consider the word

$$w' = y_1 \cdots y_{j_0-1} y_{j_0} y_{j_0+1} \cdots y_{(k-1)s+1},$$

where

$$w = y_1 \cdots y_{(k-1)s+1},$$

$$y_{j_0} = a^{n_1} b^n a^{n_2} b^n \cdots a^{n_{(k-1)e+2}} b^n \text{ and } n_i = n + r_i v^i_1$$

for each $i(1 \leq i \leq (k-1)e+2)$. (Note that all $n_i>0$ for n large enough.) Since $r_i \neq 0$ for some i and $v^i_i>0$ for all i (see above), we have that $n_i \neq n$ for some i and therefore $w' \notin L_{((k-1)e+1)((k-1)s+1)s}$. On the other hand, by (3), (4) and

$$|y'_{j_0}|=2((k-1)e+2)n + \sum_{i=1}^{(k-1)e+2} r_i v^i_i = 2((k-1)e+2)+0 = |y_{j_0}|,$$

we have that there is an accepting computation tree of M on w' with the sequence $u_1, z'_1, u_2, z'_2, \ldots, u_{(k-1)e+2}, z'_{(k-1)e+2}, u_{(k-1)e+3}$ where segment z'_i is obtained by inserting (if $r_i>0$) or by deleting (if $r_i<0$) r_i cycles wiht parameter v^i from segment z_i. Therefore, w' is accepted by M. This is a contradiction. Q.E.D.

Theorem 4.7. *For $k \geq 2$ and $s \geq 1$, $\mathcal{L}$[ASPk-HFA(s)] and $\bigcup_{1 \leq r < \infty} \bigcup_{1 \leq t < \infty} \mathcal{L}$[ASPr-HFA(t)] is not closed under Kleene closure.*

Proof. Let $T_7=\{a^n b^n \mid n \geq 1\}$. It is easily seen that $T_7 \in \mathcal{L}$[DSP2-HFA]. On the other hand, it is shown Lemma 4.4 in [1] that $T_7^* \notin \bigcup_{1 \leq k < \infty} \bigcup_{1 \leq s < \infty} \mathcal{L}$[ASPk-HFA(s)]. This completes the proof of the theorem. Q.E.D.

5. Conclusions

The closure properties of AMHFACLs and ASPMHFACLs are summarized in Tables 5.1 and 5.2. Symbol ○ (×, ?) stands for closed (not closed, unknown).

Almost results in Table 5.2 hold for semi-one-way alternating simple multihead finite automata with constant leaf-sizes. That is, the symbol ○ with superscript 2 indicates the result is only valid for one-way.

Acknowledgment

One of authers, H.Matsuno would like to thank Professor Hiroshi Taniguchi of Tottori University for his hearty encouragement.

References

[1] H.Matsuno, K.Inoue, I.Takanami, and H.Taniguchi, Alternating multihead finite automata with constant leaf-sizes, *Trans. IEICE of Japan (Section E)*, E71, 10, 1006-1011 (1988).
[2] K.N.King, Alternating multihead finite automata, *Theoret. Comput. Sci.* 61, 149-174 (1988).
[3] A.C.Yao and R.L.Rivest, K+1 heads are better than k, *J.ACM.*, 25, 2, 337-340 (1978).
[4] J.Hromkovič, One-way multihead deterministic finite automata, *Acta Inf.*, 19, 377-384 (1983).
[5] S.Sakurayama, H.Matsuno, K.Inoue, I.Takanami, and H.Taniguchi, Alternating one-way multihead Turing machines with only universal states, *Trans. IECE of Japan (Section E)*, E68, 10, 705-711 (1985).
[6] H.Matsuno, K.Inoue, H.Taniguchi, and I. Takanami, Alternating simple multihead finite automata, *Theoret. Comput. Sci.* 36, 291-308 (1985).
[7] P.Ďuriš and J.Hromkovič, One-way simple multihead finite automata are not closed under concatenation, *Theoret. Comput. Sci.* 27, 121-125 (1983).

Table 5.1. Multihead Finite Automata

	U	USN	A	ASN
comple.	? [1]	? [1]	×	×
union	×	×	○	○
inter.	×	×	×	×
concate.	×	×	?	?
Kleene	×	×	?	?
revers.	×	×	?	?
ε-free	×	×	?	?

U : $\mathcal{L}$[Uk-HFA(s)], USN : $\mathcal{L}$[USNk-HFA(s)]
A : $\mathcal{L}$[Ak-HFA(s)], ASN : $\mathcal{L}$[ASNk-HFA(s)]
1. $\mathcal{L}$[Uk-HFA] and $\mathcal{L}$[USNk-HFA] are not closed under complementation.

Table 5.2. Simple Multihead Finite Automata

	USP	USNSP	ASP	ASNSP
comple.	×	×	×	×
union	×	×	○	○
inter.	×	×	×	×
concate.	× [2]	?	× [2]	?
Kleene	×	×	× [2]	?
revers.	×	×	?	?
ε-free	×	×	?	?

USP: $\mathcal{L}$[USPk-HFA(s)], ASP: $\mathcal{L}$[ASPk-HFA(s)]
USNSP: $\mathcal{L}$[USNSPk-HFA(s)]
ASNSP: $\mathcal{L}$[ASPSNk-HFA(s)]
2. Only valid for one-way (not hold for semi-one-way).