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線形計画問題に対する乗法的罰金関数法の多項式時間性について

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線形計画問題に対する内点法の一つである乗法的罰金関数法 (Iri, Imai [1]) については、算法の最終段階での局所 2 次収束性については示されていたものの、全域的な 1 次収束性については従来厳密には示されていなかった。最近、Zhang, Shi [2] は、直線探索がほぼ正確に行なえるという仮定の下で、全域的な 1 次収束性を示した。しかしながら、正確な直線探索が多項式時間で行なえるかどうかについては言及されておらず、これだけでは算法の多項式時間性を示したことにはならない。また、そこで与えられている 1 次収束の縮小率もかなりわるい。本稿では、Iri, Imai [1] での全域的 1 次収束に関して考察した命題に基づき、それに Zhang, Shi [2] の手法を加味することにより、乗法的罰金関数法の多項式時間性を示す。また、縮小率に関しても、かなり改善されたものを示す。

On the Polynomiality of the Time Complexity of the Multiplicative Penalty Function Method for Linear Programming

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This paper proves that the multiplicative penalty function method for linear programming may run in polynomial time. It is shown that the multiplicative penalty function is reduced by a factor of $1 - \frac{1}{16m(m+2)^2}$ per iteration by directly applying the Newton method, where m is the number of constraints of a given linear programming problem in inequality form. This reduction may be attained without performing any exact line search.

1. Introduction

The Interior method is now recognized as a powerful method to solve a large-scale linear programming problem. The multiplicative penalty function method (Iri, Imai [1]) is an interior method which minimizes the convex multiplicative penalty function defined for a given linear programming problem by the Newton method. In [1], the local quadratic convergence of the method was shown, while the global convergence property was left open. Recently, Zhang and Shi [2] proved the global linear convergence of the method under an assumption that the line search can be performed rigorously. However, it is not shown that the exact line search can be executed in polynomial time, and so this result does not imply the polynomiality of the multiplicative penalty function method. Also, the reduction factor they showed was comparatively large.

This paper proves that the multiplicative penalty function method for linear programming may run in polynomial time. We make use of a key proposition given in Iri and Imai [1] about the global convergence of the method, and partly employ some result in Zhang and Shi [2] to bound the quadratic term in Taylor expansion of the function. It is shown that the multiplicative penalty function is reduced by a factor of $1-\frac{1}{16m(m+2)^2}$ per iteration by directly applying the Newton method, where m is the number of constraints all in inequality form. This reduction may be attained without performing any exact line search, and its rate is much better than the rate in [2]. This implies that the multiplicative penalty function method solves the linear programming problem in $O(m^4L)$ iterations, where L is the size of the input problem, and the most time-consuming part of each iteration is to solve a system of $O(m)$ linear equations.

2. Preliminaries

We consider the following linear programming problem:

$$\begin{aligned} &\min\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ &\text{s.t. } A\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

where $\boldsymbol{c}, \boldsymbol{x} \in \mathbf{R}^n$, $\boldsymbol{b} \in \mathbf{R}^m$ and $A \in \mathbf{R}^{m\times n}$. In the sequel, we assume the following:

1) The feasible region $X = \{\, \boldsymbol{x} \mid A\boldsymbol{x} \geq \boldsymbol{b} \,\}$ is bounded.
2) The interior Int X of the feasible region X is not empty.
3) The minimum value of $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ is zero.

Consider the multiplicative penalty function for this linear programming problem:

$$F(\boldsymbol{x}) = (\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x})^{m+1} / \prod_{i=1}^{m} (\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} - b_i) \qquad (\boldsymbol{x} \in \mathrm{Int}\, X)$$

where $\boldsymbol{a}_i \in \mathbf{R}^n$ is the i-th row vector of A. This function is introduced in [1].

Define $\boldsymbol{\eta} = \boldsymbol{\eta}(x)$ and $H = H(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathrm{Int}\, X$ by

$$\boldsymbol{\eta}(x) = \frac{\nabla F(\boldsymbol{x})}{F(\boldsymbol{x})} = (m+1)\frac{\boldsymbol{c}}{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}} - \sum_{i=1}^{m} \frac{\boldsymbol{a}_i}{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} - b_i}$$

and

$$H(\boldsymbol{x}) = \frac{\nabla^2 F(\boldsymbol{x})}{F(\boldsymbol{x})}.$$

The multiplicative penalty function method directly minimizes the penalty function $F(\boldsymbol{x})$ by the Newton method, starting from some initial interior point. This paper estimates the reduction in the penalty function per iteration by this method. Roughly, we show the following:

$$\frac{F(\boldsymbol{x}^{(\nu+1)})}{F(\boldsymbol{x}^{(\nu)})} \leq 1 - \frac{1}{16m(m+2)^2}$$

where $\boldsymbol{x}^{(\nu)}$ and $\boldsymbol{x}^{(\nu+1)}$ are the ν-th and $(\nu+1)$-st solution by the method.

3. Main Propositions

Claim 3.1.

$$F(\boldsymbol{x}+t\boldsymbol{\xi}) = F(\boldsymbol{x}) + t\boldsymbol{\xi}^{\mathrm{T}}\nabla F(\boldsymbol{x}) + t^2\int_0^1\int_0^1 \boldsymbol{\xi}^{\mathrm{T}}\nabla^2 F(\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi})\boldsymbol{\xi}\lambda \mathrm{d}\lambda \mathrm{d}\mu. \quad \square$$

Claim 3.2. If $F(\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi}) \leq F(\boldsymbol{x})$ for $0 \leq \lambda, \mu \leq 1$,

$$\frac{F(\boldsymbol{x}+t\boldsymbol{\xi})}{F(\boldsymbol{x})} \leq 1 + t\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\eta} + t^2\int_0^1\int_0^1 \boldsymbol{\xi}^{\mathrm{T}}H(\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi})\boldsymbol{\xi}\lambda \mathrm{d}\lambda \mathrm{d}\mu. \quad \square$$

Lemma 3.1. For t and K satisfying

$$\frac{1}{2}t^2\boldsymbol{\xi}^{\mathrm{T}}H(\boldsymbol{x})\boldsymbol{\xi} \leq K^2, \qquad K^2 \leq \frac{1}{16m(m+2)},$$

$\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi} \in \mathrm{Int}\, X$ for $0 \leq \lambda, \mu \leq 1$ and

$$t^2\int_0^1\int_0^1 \boldsymbol{\xi}^{\mathrm{T}}H(\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi})\boldsymbol{\xi}\lambda \mathrm{d}\lambda \mathrm{d}\mu \leq 4m(m+2)^2K^2. \quad \square$$

Let $\tilde{\boldsymbol{\xi}} = \tilde{\boldsymbol{\xi}}(\boldsymbol{x})$ be the Newton direction of $F(\boldsymbol{x})$ at $\boldsymbol{x}$:

$$H(\boldsymbol{x})\tilde{\boldsymbol{\xi}} = -\boldsymbol{\eta}(\boldsymbol{x})$$

Lemma 3.2. (Iri, Imai [1]) For t and K satisfying

$$\frac{1}{2}t^2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}H(\boldsymbol{x})\tilde{\boldsymbol{\xi}} = K^2, \qquad K^2 \leq \frac{1}{16m(m+2)},$$

we have

$$t\tilde{\boldsymbol{\xi}}^{\mathrm{T}}\boldsymbol{\eta} < -K. \quad \square$$

Theorem 3.1. For t and K satisfying

$$\frac{1}{2}t^2\tilde{\boldsymbol{\xi}}^{\mathrm{T}}H(\boldsymbol{x})\tilde{\boldsymbol{\xi}} = K^2, \qquad K^2 \leq \frac{1}{16m(m+2)},$$

$x + t\tilde{\xi} \in \text{Int } X$, and

$$\frac{F(x + t\tilde{\xi})}{F(x)} \le 1 - K + 4m(m+2)^2K^2.$$

Corollary 3.1. Setting K to $1/(8m(m+2)^2)$, and then determining t by

$$\frac{1}{2}t^2\tilde{\xi}^{\mathrm{T}}H(x)\tilde{\xi} = K^2,$$

$x + t\tilde{\xi} \in \text{Int } X$, and

$$\frac{F(x + t\tilde{\xi})}{F(x)} \le 1 - \frac{1}{16m(m+2)^2}. \quad \square$$

4. Proofs

Proof of Claim 3.1: Omitted. $\square$

Proof of Claim 3.2: Omitted. $\square$

To prove Lemmas, we provide several additional claims, including some of the results in Zhang and Shi [2].

Define $L(x)$ by

$$L(x) = \left[\frac{(m+1)c}{c^{\mathrm{T}}x}, -\frac{a_1}{a_1^{\mathrm{T}}x - b_1}, \ldots, -\frac{a_m}{a_m^{\mathrm{T}}x - b_m}\right]$$

Lemma 4.1. (Zhang and Shi [2]) For any ξ,

$$\frac{1}{2m(m+2)}\xi^{\mathrm{T}}L(x)L(x)^{\mathrm{T}}\xi \le \xi^{\mathrm{T}}H(x)\xi \le (m+2)\xi^{\mathrm{T}}L(x)L(x)^{\mathrm{T}}\xi. \quad \square$$

In fact, it is shown in Zhang and Shi [2] that

$$\frac{1}{2m(m+2)}\xi^{\mathrm{T}}L(x)L(x)^{\mathrm{T}}\xi \le \frac{1}{2m}\frac{m+2-\sqrt{m^2+4m}}{2}\xi^{\mathrm{T}}L(x)L(x)^{\mathrm{T}}\xi \le \xi^{\mathrm{T}}H(x)\xi$$

and

$$\xi^{\mathrm{T}}H(x)\xi \le \frac{m^2+3m+1}{m+1}\xi^{\mathrm{T}}L(x)L(x)^{\mathrm{T}}\xi \le (m+2)\xi^{\mathrm{T}}L(x)L(x)^{\mathrm{T}}\xi$$

Lemma 4.2. For t and K satisfying

$$\frac{1}{2}t^2\xi^{\mathrm{T}}H(x)\xi \le K^2, \qquad K^2 \le \frac{1}{16m(m+2)},$$

we have

$$t^2\xi^{\mathrm{T}}L(x)L(x)^{\mathrm{T}}\xi \le 4m(m+2)K^2$$

and hence

$$(m+1)^2t^2\left(\frac{c^{\mathrm{T}}\xi}{c^{\mathrm{T}}x}\right)^2 \le \frac{1}{4}, \text{ and } t^2\left(\frac{a_i^{\mathrm{T}}\xi}{a_i^{\mathrm{T}}x - b_i}\right)^2 \le \frac{1}{4} \text{ for each } i. \quad \square$$

Proof: From Lemma 4.1,

$$\frac{1}{4m(m+2)}\xi^{\mathrm{T}} L(x)L(x)^{\mathrm{T}}\xi \leq \xi^{\mathrm{T}} H(x)\xi \leq K^2.$$

Since

$$t^2\boldsymbol{\xi}^{\mathrm{T}} L(x)L(x)^{\mathrm{T}}\boldsymbol{\xi} = (m+1)^2t^2\left(\frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}}\right)^2 + t^2\left(\frac{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}-b_i}\right)^2 \leq 4m(m+2)K^2 \leq \frac{1}{4},$$

the latter half follows. □

Claim 4.1. For s, t with $t > 0$ and $|s/t| \leq 1/2$,

$$\int_0^1\int_0^1 \frac{s^2\lambda}{(t+s\lambda\mu)^2}\mathrm{d}\lambda\mathrm{d}\mu \leq (s/t)^2.$$

Proof:

$$\int_0^1\int_0^1 \frac{s^2\lambda}{(t+s\lambda\mu)^2}\mathrm{d}\lambda\mathrm{d}\mu = \frac{s}{t} - \ln(1+\frac{s}{t})$$

For $x = s/t$, $x^2 - x + \ln(1+x)$ attains the minimum value 0 at $x = 0$ for $|x| \leq 1/2$. The lemma follows. □

Proof of Lemma 3.1: From Lemma 4.2, we have

$$\begin{aligned}\frac{1}{4} \geq t^2\left(\frac{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}-b_i}\right)^2 &= \left(\frac{(\boldsymbol{a}_i^{\mathrm{T}}(\boldsymbol{x}+t\boldsymbol{\xi})-b_i)-(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}-b_i)}{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}-b_i}\right)^2 \\ &= \left(\frac{(\boldsymbol{a}_i^{\mathrm{T}}(\boldsymbol{x}+t\boldsymbol{\xi})-b_i)}{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}-b_i} - 1\right)^2\end{aligned}$$

Hence,

$$0 < \frac{1}{2} \leq \frac{(\boldsymbol{a}_i^{\mathrm{T}}(\boldsymbol{x}+t\boldsymbol{\xi})-b_i)}{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}-b_i} \leq \frac{3}{2}$$

and so $\boldsymbol{a}_i^{\mathrm{T}}(\boldsymbol{x}+t\boldsymbol{\xi}) > b_i$ for each i. This implies that $\boldsymbol{x}+t\boldsymbol{\xi}$ is in the interior of the feasible region X.

Using Lemma 4.1,

$$\begin{aligned}&t^2\int_0^1\int_0^1 \xi^{\mathrm{T}} H(\boldsymbol{x}+t\lambda\mu\xi)\xi\lambda\mathrm{d}\lambda\mathrm{d}\mu \\ &\leq (m+2)t^2\int_0^1\int_0^1 \xi^{\mathrm{T}} L(\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi})L(\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi})^{\mathrm{T}}\boldsymbol{\xi}\lambda\mathrm{d}\lambda\mathrm{d}\mu \\ &= (m+2)\int_0^1\int_0^1\left(\left((m+1)^2\frac{t\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{c}^{\mathrm{T}}(\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi})}\right)^2 + \sum_{i=1}^{m}\left(\frac{t\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{a}_i^{\mathrm{T}}(\boldsymbol{x}+t\lambda\mu\boldsymbol{\xi})-b_i}\right)^2\right)\lambda\mathrm{d}\lambda\mathrm{d}\mu\end{aligned}$$

From Lemma 4.2,

$$t\left|\frac{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}}\right| \leq \frac{1}{2(m+1)} < \frac{1}{2}, \text{ and } t\left|\frac{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}-b_i}\right| \leq \frac{1}{2} \text{ for each } i,$$

and we can apply Claim 2.1, and then have the following.

$$
\begin{aligned}
&t^2 \int_0^1 \int_0^1 \boldsymbol{\xi}^{\mathrm{T}} H(\boldsymbol{x} + t\lambda\mu\boldsymbol{\xi})\boldsymbol{\xi}\lambda \mathrm{d}\lambda \mathrm{d}\mu \\
&\leq (m+2)\left(\left((m+1)^2 \frac{t\boldsymbol{c}^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{c}^{\mathrm{T}}(\boldsymbol{x} + t\lambda\mu\boldsymbol{\xi})}\right)^2 + \sum_{i=1}^{m}\left(\frac{t\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{\xi}}{\boldsymbol{a}_i^{\mathrm{T}}(\boldsymbol{x} + t\lambda\mu\boldsymbol{\xi}) - b_i}\right)^2\right) \\
&= (m+2)at^2\boldsymbol{\xi}^{\mathrm{T}} L(\boldsymbol{x})L(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{\xi}.
\end{aligned}
$$

Applying Lemma 4.2 to this, we obtain the lemma. □

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Rererences

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