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パス幅が限られたグラフの族に関する閉路を含まない極小禁止マイナー

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グラフのパス幅及び真のパス幅に関して，それらが制限されたグラフの族について考察する．これら２つのグラフの族はマイナーに関して閉じている．マイナーに関して閉じている族に関する極小禁止マイナーの数は有限であり，極小禁止マイナーがすべて列挙されたならば，与えられたグラフがその族に属すか否かが多項式時間で判定できることが知られている．小文では上の２つの族に関する閉路を含まない極小禁止マイナーを特徴付ける．さらに，これらに族に関する極小禁止マイナーの数と点数を評価する．また，真のパス幅が限られたグラフの族に関して，一般的な極小禁止マイナーの一系列を構成する方法を示す．

Minimal Acyclic Forbidden Minors for the Family of Graphs with Bounded Path-Width

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The graphs with bounded path-width, introduced by Robertson and Seymour, and the graphs with bounded proper-path-width, introduced in this paper, are investigated. These families of graphs are minor-closed. We characterize the minimal acyclic forbidden minors for these families of graphs. We also give estimates for the numbers of minimal forbidden minors and for the numbers of vertices of the largest minimal forbidden minors for these families of graphs.

1 Introduction

Graphs we consider are finite and undirected, but may have loops and multiple edges. A graph H is a minor of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A family $\mathcal{F}$ of graphs is said to be minor-closed if the following condition holds: If $G \in \mathcal{F}$ and H is a minor of G then $H \in \mathcal{F}$. A graph G is a minimal forbidden minor for a minor closed family $\mathcal{F}$ of graphs if $G \notin \mathcal{F}$ and any proper minor of G is in $\mathcal{F}$.

Robertson and Seymour proved the following deep theorems.

Theorem A [15]: Every minor-closed family of graphs has a finite number of minimal forbidden minors.

Theorem B [14]: The problem of deciding if a fixed graph is a minor of an input graph can be solved in polynomial time.

The combination of these theorems suggests the existence of a polynomial time algorithm for the problem of testing membership for any minor-closed family $\mathcal{F}$ of graphs. Although many important problems are known to be reduced to the problem, we cannot have a polynomial time algorithm unless we can find all the minimal forbidden minors for $\mathcal{F}$. Unfortunately, it was proved that there is no general method to find all the minimal forbidden minors for any minor-closed family of graphs [6][7], as suspected. However, special arguments could be applied for individual minor-closed family. In fact, the minimal forbidden minors were found for families of planar graphs [18], graphs embeddable on the projective plane [9][1], partial 2-trees [3], partial 3-trees [3][4], and graphs with path-width at most 2 [5].

We investigate the family $\mathcal{F}_k$ of graphs with path-width at most k for any $k \geq 0$. We introduce the proper-path-width of graphs and investigate the family $\mathcal{P}_k$ of graphs with proper-path-width at most k for any $k \geq 1$. $\mathcal{F}_k$ and $\mathcal{P}_k$ are minor-closed families. We show that every minimal acyclic forbidden minor for $\mathcal{F}_k$ (respectively $\mathcal{P}_k$) can be obtained from those for $\mathcal{F}_{k-1}$ (respectively $\mathcal{P}_{k-1}$) by a simple composition. We also give estimates for the numbers of minimal forbidden minors for $\mathcal{F}_k$ and $\mathcal{P}_k$, and the numbers of vertices of the largest minimal forbidden minors for $\mathcal{F}_k$ and $\mathcal{P}_k$.

2 Minimal Acyclic Forbidden Minors for Graphs with Bounded Path-Width

The path-width of a graph was introduced by Robertson and Seymour [13].

Definition 1: Given a graph G, a sequence $X_1, X_2, \cdots, X_r$ of subsets of $V(G)$ is a *path-decomposition* of G if the following conditions are satisfied.

(i) For every edge $e \in E(G)$, some X_i $(1 \leq i \leq r)$ contains both ends of e.

(ii) For $1 \leq l \leq m \leq n \leq r$, $X_l \cap X_n \subseteq X_m$.

The *path-width* of G, denoted by $pw(G)$, is the minimum value of $k \geq 0$ such that G has a path-decomposition $X_1, X_2, \cdots, X_r$ with $|X_i| \leq k+1$ for $i = 1, 2, \cdots, r$. □

The following lemmas mentioned in [13] will be used later.

Lemma A [13]: If every connected component of G has path-width $\leq k$, then G has path-width $\leq k$.

Lemma B [13]: If $X \subseteq V(G)$ and $pw(G/X) \leq k$, then $pw(G) \leq k + |X|$, where G/X denotes the graph obtained from G by deleting the vertices in X.

Let $\mathcal{F}_k$ be the family of graphs with path-width at most k. It is easy to see that $\mathcal{F}_k$ is minor-closed. Let $\Omega(\mathcal{F}_k)$ be the set of all minimal forbidden minors for $\mathcal{F}_k$, and $\Omega_a(\mathcal{F}_k)$ be the set of all minimal acyclic forbidden minors for $\mathcal{F}_k$. Obviously, $\Omega_a(\mathcal{F}_k) \subseteq \Omega(\mathcal{F}_k)$.

We introduce the star-composition of graphs which plays an important role in the following.

Definition 2: Let H_1, H_2, and H_3 be connected graphs. A graph obtained from H_1, H_2, and H_3 by the following construction is called a *star-composition* of H_1, H_2, and H_3:

(i) Choose a vertex $v_i \in H_i$ for $i = 1, 2$, and 3.

(ii) Let v be a new vertex not in H_1, H_2, or H_3.

(iii) Connect v to v_i by an edge (v, v_i) for $i = 1, 2$, and 3.

The vertex v is called the *center* of the star-composition. □

We first prove the following theorem.

Theorem 1: Let $k \geq 1$. A tree T is in $\Omega_a(\mathcal{F}_k)$ if and only if T is a star-composition of (not necessarily distinct) three trees in $\Omega_a(\mathcal{F}_{k-1})$.

Proof: We prove this theorem by a series of lemmas.

Lemma 1: Let $X_1, X_2, \cdots, X_r$ be a path-decomposition of G and $v \in V(G)$. If $v \in X_i$ and $v \in X_j$ $(1 \leq i \leq j \leq r)$, then $v \in X_l$ for any l $(i \leq l \leq j)$. That is, v appears in consecutive X_i's.

Proof: Suppose that $v \notin X_l$ for some l $(i < l < j)$. Then $X_i \cap X_j \not\subseteq X_l$, contradicting (ii) of Definition 1. □

Lemma 2: Let H be a connected subgraph of G, and $X_1, X_2, \cdots, X_r$ be a path-decomposition of G. If $X_i \cap V(H) \neq \phi$ and $X_j \cap V(H) \neq \phi$ $(1 \leq i \leq j \leq r)$, then $X_l \cap V(H) \neq \phi$ for any l $(i \leq l \leq j)$. That is, the vertices of H appear in consecutive X_i's.

Proof: Suppose that $X_l \cap V(H) = \phi$ for some l $(i < l < j)$. Each vertex of H appears in consecutive X_i's by Lemma 1. Thus, if $P = \bigcup_{i=1}^{l-1}(X_i \cap V(H))$ and $Q = \bigcup_{i=l+1}^{r}(X_i \cap V(H))$, $P \cap Q = \phi$. Since $V(H)$ is partitioned into P and Q, and H is connected, there exist $u \in P$ and $v \in Q$ such that $(u, v) \in E(H)$. However, $\{u, v\} \not\subseteq X_i$ for any i $(1 \leq i \leq r)$, contradicting (i) of Definition 1. □

Lemma 3: Let T be a tree and k be a positive integer. Suppose that for any $v \in V(T)$, $T/\{v\}$ has no connected component with path-width $k+1$ or more and at most two connected components with path-width k. Then $pw(T) \leq k$.

Proof: Let T_0 be T, and let v_0 be a vertex such that $T_0/\{v_0\}$ has the maximum number of connected components with path-width k.

If $T_0/\{v_0\}$ has no connected component with path-width k, then $pw(T) \leq k$ by Lemma B.

If $T_0/\{v_0\}$ has two connected components with path-width k, let T_1 be one of these components and $v_1 \in V(T_1)$ be a vertex adjacent to v_0 in T_0. We recursively define T_i and $v_i \in V(T_i)$ $(1 < i \leq a)$ while $T_{i-1}/\{v_{i-1}\}$ has a component with path-width k as follows: Let T_i be a connected component of $T_{i-1}/\{v_{i-1}\}$ with path-width k and $v_i \in V(T_i)$ be a vertex adjacent to v_{i-1} in T_{i-1}. T_a has no connected component with path-width k. Let T_{a+1} be the other connected component of $T_0/\{v_0\}$ with path-width k, and $v_{a+1} \in V(T_{a+1})$ be a vertex adjacent to v_0 in T_0. Define recursively T_i and $v_i \in V(T_i)$ $(a+1 < i \leq b)$ as above. Notice that $T_i/\{v_i\}$ $(1 \leq i \leq b)$ has at most one connected component with path-width k, for otherwise $T_0/\{v_i\}$ has three or more connected components with path-width k, contradicting the assumption of the lemma.

Let $H_i^!$ $(0 \leq i \leq b)$ be the union of components of $T_i/\{v_i\}$ with path-width $< k$, and H_i $(0 \leq i \leq b)$ be the induced subgraph of T on $V(H_i^!) \cup \{v_i\}$. By Lemma A, $pw(H_i^!) < k$ $(0 \leq i \leq b)$. By Lemma B, $pw(H_i) \leq k$ $(0 \leq i \leq b)$. Let $X^{(i)} = (X_1^{(i)}, X_2^{(i)}, \cdots, X_{r_i}^{(i)})$ be a path-decomposition of H_i $(0 \leq i \leq b)$ such that every $X_j^{(i)}$ contains v_i and $|X_j^{(i)}| \leq k+1$ $(1 \leq j \leq r_i)$. We define sequences L and R as follows.

$$\begin{aligned} L &= X^{(a)}, \{v_a, v_{a-1}\}, X^{(a-1)}, \{v_{a-1}, v_{a-2}\}, \cdots, X^{(2)}, \{v_2, v_1\}, X^{(1)}, \{v_1, v_0\} \\ R &= \{v_0, v_{a+1}\}, X^{(a+1)}, \{v_{a+1}, v_{a+2}\}, X^{a+2}, \cdots, \{v_{b-2}, v_{b-1}\}, X^{(b-1)}, \{v_{b-1}, v_b\}, X^{(b)} \end{aligned}$$

It is easy to see that the following concatenation of sequences

$$L, X^{(0)}, R$$

is a path-decomposition of T, and $pw(T) \leq k$.

If $T_0/\{v_0\}$ has just one connected component with path-width k, the sequence R above is empty, and $L, X^{(0)}$ is a path-decomposition of T and we also have $pw(T) \leq k$. □

Lemma 4: Let T be a tree and k be a positive integer. If T has a vertex v such that $T/\{v\}$ has a connected component with path-width $\geq k+1$, then T has a vertex w such that $T/\{w\}$ has at least three connected components with path-width $\geq k$.

Proof: Let T' be a minimal subgraph of T with $pw(T') \geq k+1$. For any $v \in V(T')$, $T'/\{v\}$ has no connected component with path-width $\geq k+1$ by definition. Thus there exists a vertex $w \in V(T')$ such that $T'/\{w\}$ has at least three connected components with path-width k, for otherwise $pw(T') \leq k$ by Lemma 3. Thus $T/\{w\}$ has at least three connected components with path-width $\geq k$. □

Lemma 5: For any tree T and integer $k \geq 1$, $pw(T) \geq k+1$ if and only if T has a vertex v such that $T/\{v\}$ has at least three connected components with path-width k or more.

Proof: Suppose that $pw(T) \geq k+1$. If there exists a vertex $v \in V(T)$ such that $T/\{v\}$ has a connected component with path-width $\geq k+1$, then there exists a vertex $w \in V(T)$ such that $T/\{w\}$ has at least three connected components with path-width $\geq k$ by Lemma 4. If there exists no such vertex v, then there exists a vertex w such that $T/\{w\}$ has at least three connected components with path-width k by Lemma 3.

Conversely, suppose that T has a vertex v such that $T/\{v\}$ has at least three connected components with path-width $\geq k$. We may assume that the path-widths of these connected components are exactly k, for otherwise trivially $pw(T) \geq k+1$. Let T_1, T_2, and T_3 be connected components of $T/\{v\}$ with path-width k, and $v_1 \in V(T_1)$, $v_2 \in V(T_2)$, and $v_3 \in V(T_3)$ be vertices adjacent to v in T.

Suppose contrary that $pw(T) \leq k$ and T has a path-decomposition $X_1, X_2, \cdots, X_r$ with $|X_i| \leq k+1$ for any i. There exists some i_j such that $X_{i_j} \subseteq V(T_j)$ for $j = 1, 2$, and 3, for otherwise $X_1 \cap V(T_j), X_2 \cap V(T_j), \cdots, X_r \cap V(T_j)$ is a path-decomposition of T_j with $|X_i \cap V(T_j)| < k+1$ $(1 \leq i \leq r)$, and $pw(T_j) < k$. Without loss of generality we assume that $i_1 < i_2 < i_3$. It is trivial that $T/V(T_2)$ is a connected subgraph of T. However $X_{i_1} \cap V(T/V(T_2)) \neq \phi$ and $X_{i_2} \cap V(T/V(T_2)) = \phi$, and $X_{i_3} \cap V(T/V(T_2)) \neq \phi$, contradicting Lemma 2. Thus $pw(T) \geq k+1$. □

Lemma 6: If $k \geq 1$ and T_1, T_2, and T_3 are (not necessarily distinct) trees in $\Omega_a(\mathcal{F}_{k-1})$ then any star-composition of T_1, T_2, and T_3 is in $\Omega_a(\mathcal{F}_k)$.

Proof: Let T be a star-composition of T_1, T_2, and T_3, and v be the center of the star-composition. Since $T_i \in \Omega_a(\mathcal{F}_{k-1})$ $(i = 1, 2$, and $3)$, $T/\{v\}$ has three connected components with path-width k. Thus $pw(T) \geq k+1$ by Lemma 5. On the other hand, $pw(T/\{v\}) \leq k$ by Lemma A, and so $pw(T) \leq k+1$ by Lemma B. Hence we have $pw(T) = k+1$.

Because $T_i \in \Omega_a(\mathcal{F}_{k-1})$ $(i = 1, 2$, and $3)$, any proper minor of T_i is in $\mathcal{F}_{k-1}$. Thus for any vertex w in any proper minor T' of T, $T'/\{w\}$ has at most two connected components with path-width $\geq k$, and so $pw(T') \leq k$ by Lemma 5. Thus T is minimal. □

Lemma 7: If $k \geq 1$ and T is any tree in $\Omega_a(\mathcal{F}_k)$ then T is a star-composition of some (not necessarily distinct) trees T_1, T_2, and T_3 in $\Omega_a(\mathcal{F}_{k-1})$.

Proof: There is no vertex w such that $pw(T/\{w\}) = k+1$ because T is minimal. Thus there is a vertex v such that $T/\{v\}$ has three or more connected components with path-width k by Lemma 5. Because T is minimal, $T/\{v\}$ has exactly three connected components with path-width k and no connected component with path-width $< k$. Let T_1, T_2, and T_3 be connected components of $T/\{v\}$ with path-width k. Suppose $T_1 \notin \Omega_a(\mathcal{F}_{k-1})$. Let T_1' be a proper minor of T_1 with path-width k and T' be a star-composition of T_1', T_2, and T_3. Then $pw(T') = k+1$, contradicting that $T \in \Omega_a(\mathcal{F}_k)$. Thus $T_1 \in \Omega_a(\mathcal{F}_{k-1})$. Similarly T_2 and T_3 are in $\Omega_a(\mathcal{F}_{k-1})$. □

By Lemma 6 and Lemma 7, we obtain the theorem. □

It is easy to see that $\Omega(\mathcal{F}_0) = \{K_2\}$. The graphs in $\Omega(\mathcal{F}_1)$ and $\Omega_a(\mathcal{F}_2)$ are shown in Fig.1 and 2, respectively. The following corollary can easily be proved by induction on k.

Corollary 1: (1) The number of vertices of a tree in $\Omega_a(\mathcal{F}_k)$ is $\frac{5\cdot 3^k - 1}{2}$ $(k \geq 0)$.
(2) $|\Omega_a(\mathcal{F}_k)| \geq k!^2$ $(k \geq 0)$.

We counted $|\Omega_a(\mathcal{F}_k)|$ for $k = 0, 1, 2, 3$, and 4 as follows: $|\Omega_a(\mathcal{F}_0)| = |\Omega_a(\mathcal{F}_1)| = 1$, $|\Omega_a(\mathcal{F}_2)| = 10$, $|\Omega_a(\mathcal{F}_3)| = 117{,}480$, $|\Omega_a(\mathcal{F}_4)| = 14{,}403{,}197{,}619{,}396{,}707{,}660$.

Since we did not use the condition that T_i is a tree in the proof of Lemma 6, Lemma 6 can be generalized as follows: If $k \geq 1$ and H_1, H_2, and H_3 are (not necessarily distinct) graphs in $\Omega(\mathcal{F}_{k-1})$ then any star-composition of H_1, H_2, and H_3 is in $\Omega(\mathcal{F}_k)$.

Fig.1 The graphs in $\Omega(\mathcal{F}_1)$

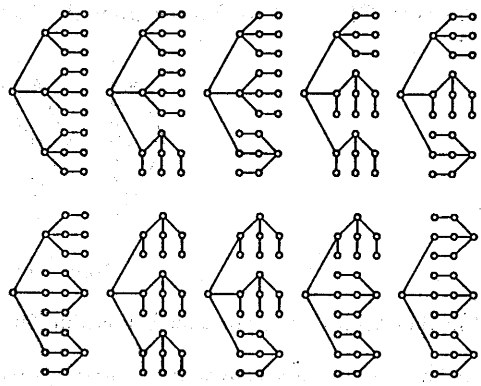

Fig.2 The trees in $\Omega_a(\mathcal{F}_2)$

3 Minimal Acyclic Forbidden Minors for Graphs with Bounded Proper-Path-Width

We introduce in this section the proper-path-width of graphs.

Definition 3: The path-decomposition $X_1, X_2, \cdots, X_r$ of G with $|X_i| \leq k+1$ $(k \geq 1)$ for any i is called a *proper-path-decomposition* of G if the following condition holds: For any X_l, X_m, and X_n such that each one is not a subset of the others $(1 \leq l < m < n \leq r)$, $|X_l \cap X_m \cap X_n| < k$. The *proper-path-width* of G, denoted by $ppw(G)$, is the minimum value of $k \geq 1$ such that G has a proper-path-decomposition $X_1, X_2, \cdots, X_r$ with $|X_i| \leq k+1$ for any i. □

Let $\mathcal{P}_k$ be the family of graphs with proper-path-width at most k. It is easy to see that $\mathcal{P}_k$ is minor-closed. Let $\Omega(\mathcal{P}_k)$ be the set of all minimal forbidden minors for $\mathcal{P}_k$, and $\Omega_a(\mathcal{P}_k)$ be the set of all minimal acyclic forbidden minors for $\mathcal{P}_k$.

For a positive integer k, *k-trees* are defined recursively as follows: (i) The complete graph with k vertices is a k-tree; (ii) Given a k-tree Q with n vertices $(n \geq k)$, a graph obtained from Q by adding a new vertex adjacent to the vertices of a complete subgraph of Q with k vertices is a k-tree with $n+1$ vertices. A k-tree Q is called a *k-path* [12] or *k-chordal path* [2] if $|V(Q)| \leq k+1$ or Q has exactly two vertices of degree k. For a positive integer k, *interior k-caterpillars* [12] are defined recursively as follows: (i) A k-path is an interior k-caterpillar; (ii) Given an interior k-caterpillar

Q with n vertices ($n \geq k+2$), a graph obtained from Q by adding a new vertex adjacent to a k-separator of Q is an interior k-caterpillar with $n+1$ vertices.

It is not difficult to see that $pw(G) \leq k$ ($k \geq 1$) if and only if G is a partial interior k-caterpillar, and $ppw(G) \leq k$ ($k \geq 1$) if and only if G is a partial k-path.

We have the following theorem for $\mathcal{P}_k$ corresponding to Theorem 1. The proof of the theorem is almost same as that of Theorem 1, and is omitted.

Theorem 2 Let $k \geq 2$. A tree T is in $\Omega_a(\mathcal{P}_k)$ if and only if T is a star-composition of (not necessarily distinct) three trees in $\Omega_a(\mathcal{P}_{k-1})$.

It is easy to see that $\Omega(\mathcal{P}_1) = \{K_3, K_{1,3}\}$. The trees in $\Omega_a(\mathcal{P}_2)$ are shown in Fig.3.

Corollary 2: (1) The number of vertices of a tree in $\Omega_a(\mathcal{P}_k)$ is $\frac{3^{k+1}-1}{2}$ ($k \geq 1$).
(2) $|\Omega_a(\mathcal{P}_k)| \geq k!^2$ ($k \geq 1$).

We counted $|\Omega_a(\mathcal{P}_k)|$ for $k = 1, 2, 3$, and 4 as follows: $|\Omega_a(\mathcal{P}_1)| = 1$, $|\Omega_a(\mathcal{P}_2)| = 4$, $|\Omega_a(\mathcal{P}_3)| = 1,330$, $|\Omega_a(\mathcal{P}_4)| = 2,875,919,312,080$.

It should be noted that "if" part of Theorem 2 can be generalized as follows: If $k \geq 2$ and H_1, H_2, and H_3 are (not necessarily distinct) graphs in $\Omega(\mathcal{P}_{k-1})$ then any star-composition of H_1, H_2, and H_3 is in $\Omega(\mathcal{P}_k)$. This follows from the fact that Lemma 6 can be generalized to the case of $\Omega(\mathcal{F}_k)$.

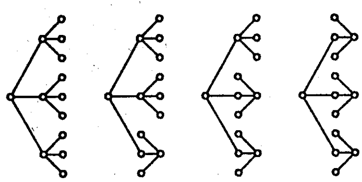

Fig.3 The trees in $\Omega_a(\mathcal{P}_2)$

Another kind of composition is possible for $\Omega(\mathcal{P}_k)$.

Definition 4: A *delta-composition* of connected graphs H_1, H_2, and H_3 is a graph obtained from H_1, H_2, and H_3 by the following construction.

(i) Choose a vertex $v_i \in H_i$ for $i = 1, 2$, and 3.

(ii) Connect v_1 to v_2, v_2 to v_3, and v_3 to v_1 by edges (v_1, v_2), (v_2, v_3), and (v_3, v_1), respectively. □

Theorem 3: If $k \geq 2$ and H_1, H_2, and H_3 are (not necessarily distinct) graphs in $\Omega(\mathcal{P}_{k-1})$ then any delta-composition of H_1, H_2, and H_3 is in $\Omega(\mathcal{P}_k)$.

Proof: Let H be a delta-composition of H_1, H_2, and H_3. Let $v_i \in V(H_i)$ be the chosen vertex for $i = 1, 2$, and 3. Because $H_i \in \Omega(\mathcal{P}_{k-1})$, $ppw(H_i/\{v_i\}) = k-1$. Let $X^{(i)} = (X_1^{(i)}, X_2^{(i)}, \cdots, X_{r_i}^{(i)})$ be the proper-path-decomposition of $H_i/\{v_i\}$ with $|X_j^{(i)}| \leq k$ ($i = 1, 2$, and 3, $1 \leq j \leq r_i$). In the following, the sequence $X_1 \cup \{v\}, \cdots, X_r \cup \{v\}$ is denoted by $X \cup \{v\}$ for simplicity.

First, we show that $ppw(H) = k+1$. Consider the following sequence.

$$X^{(1)} \cup \{v_1\}, X^{(2)} \cup \{v_1, v_2\}, \{v_1, v_2, v_3\}, X^{(3)} \cup \{v_3\}$$

It is easy to see that this sequence is a proper-path-decomposition of H, and $ppw(H) \leq k+1$. Suppose that $ppw(H) \leq k$ and H has a proper-path-decomposition $X_1, X_2, \cdots, X_r$ with $|X_i| \leq k+1$ for any i. There exists some i_j such that $X_{i_j} \subseteq V(H_j)$ for $j = 1, 2$, and 3, for otherwise $X_1 \cap V(H_j), X_2 \cap V(H_j), \cdots, X_r \cap V(H_j)$ is a path-decomposition of H_j with $|X_i \cap V(H_j)| < k+1$ ($1 \leq i \leq r$), and $pw(H_j) < k$. Without loss of generality we assume that $i_1 < i_2 < i_3$. It is trivial that $H/V(H_2)$ is a connected subgraph of H. However $X_{i_1} \cap V(H/V(H_2)) \neq \phi$ and

$X_{i_2} \cap V(H/V(H_2)) = \phi$, and $X_{i_3} \cap V(H/V(H_2)) \neq \phi$, contradicting Lemma 2. (Notice that a proper-path-decomposition is also a path-decomposition.) Hence we have $ppw(H) = k+1$.

Next we show that H is minimal. Let H' be a proper minor of H. Suppose first that H' is a minor of H obtained by contracting edge (v_1, v_2). Then the following sequence

$$X^{(1)} \cup \{v_1\}, X^{(2)} \cup \{v_1\}, \{v_1, v_3\}, X^{(3)} \cup \{v_3\}$$

is a proper-path-decomposition of H' and $ppw(H') \leq k$. Suppose next that H' is a minor of H obtained by deleting edge (v_1, v_2). Then the following sequence

$$X^{(1)} \cup \{v_1\}, \{v_1, v_3\}, X^{(3)} \cup \{v_3\}, \{v_3, v_2\}, X^{(2)} \cup \{v_2\}$$

is a proper-path-decomposition of H' and $ppw(H') \leq k$. Hence if H' is a minor of H obtained by contracting or deleting edge (v_1, v_2), then $ppw(H') \leq k$. This is also true for edges (v_2, v_3) and (v_3, v_1).

Now we assume that H' be a delta-composition of H_1, H_2', and H_3 such that H_2' be a proper minor of H_2 and v_1, v_2, and v_3 be the chosen vertices in H_1, H_2', and H_3, respectively. Let $X' = (X_1', X_2', \cdots, X_{r'}')$ be a proper-path-decomposition of H_2' with $|X_i'| \leq k$ $(1 \leq i \leq r')$.

We show that we can assume that $v_2 \in X_a'$ and $|X_a'| \leq k-1$ for some a. Suppose that $|X_i'| = k$ for any X_i' that contains v_2. Without loss of generality we assume that $X_i' \not\subseteq X_j'$ for any distinct i, j. If $v_2 \in X_1'$ (respectively $v_2 \in X_{r'}'$) then insert $\{v_2\}$ before X_1' (respectively after $X_{r'}'$) in X'. If $v_2 \in X_p' \cap X_{p+1}'$ for some p then insert $X_p' \cap X_{p+1}'$ between X_p' and X_{p+1}' in X'. (Notice that $|X_p' \cap X_{p+1}'| \leq k-1$ by the assumption above.) In either case, the new sequence thus obtained is a proper-path-decomposition of H' and $v_2 \in X_a'$ and $|X_a'| \leq k-1$ for some a. Then assume that $v_2 \in X_p'$ and $v_2 \notin X_{p-1}' \cup X_{p+1}'$ $(1 < p < r')$. Because $|X_{p-1}' \cap X_p' \cap X_{p+1}'| \leq k-2$ by Definition 3, we have $|X_{p-1}' \cap X_p'| \leq k-2$ or $|X_p' \cap X_{p+1}'| \leq k-2$. If $|X_{p-1}' \cap X_p'| \leq k-2$ (respectively $|X_p' \cap X_{p+1}'| \leq k-2$) then insert $X_{p-1}' \cap X_p' \cup \{v_2\}$ (respectively $X_p' \cap X_{p+1}' \cup \{v_2\}$) between X_{p-1}' and X_p' (respectively X_p' and X_{p+1}') in X'. This new sequence is a proper-path-decomposition of H' and $v_2 \in X_a'$ and $|X_a'| \leq k-1$ for some a. Thus we may assume that $v_2 \in X_a'$ and $|X_a'| \leq k-1$ for some a.

If $v_2 \in X_a'$ and $|X_a'| \leq k-1$ for some a, the following sequence

$$X^{(1)} \cup \{v_1\}, X_1' \cup \{v_1\}, \cdots, X_{a-1}' \cup \{v_1\}, X_a' \cup \{v_1, v_3\}, X_{a+1}' \cup \{v_3\}, \cdots, X_{r'}' \cup \{v_3\}, X^{(3)} \cup \{v_3\}$$

is a proper-path-decomposition of H' and $ppw(H') \leq k$. Thus proper-path-widths of proper minors of H are at most k, and H is minimal.

Hence $H \in \Omega(\mathcal{P}_k)$. □

Notice that the above theorem does not hold for $\Omega(\mathcal{F}_k)$. A graph shown in Fig.4 that is a delta-composition of graphs in $\Omega(\mathcal{F}_1)$ is not in $\Omega(\mathcal{F}_2)$, because its minor shown in Fig.5 is in $\Omega(\mathcal{F}_2)$. Notice also that the star- and delta-compositions are not sufficient to characterize minimal forbidden minors for $\mathcal{P}_k$. A graph in $\Omega(\mathcal{P}_2)$ shown in Fig.6 is neither a star-composition nor a delta-composition of graphs in $\Omega(\mathcal{P}_1)$.

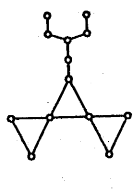

Fig.4 A graph not in $\Omega(\mathcal{F}_2)$ that is a delta-composition of minors in $\Omega(\mathcal{F}_1)$

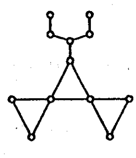

Fig.5 A graph in $\Omega(\mathcal{F}_2)$

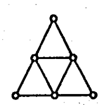

Fig.6 A graph in $\Omega(\mathcal{P}_2)$

We conclude with the following remarks: Similar results can be found in the literature. Parsons [11] gave a recursive forbidden subgraph characterization of trees with the edge-search number more than k, for each $k \geq 1$, by proving a similar result as Lemma 5. Möhring [10] mentioned without proof that similar results can be obtained for the node-search number. We learned recently that Scheffler [16] obtained independently the same result as Lemma 5 in this paper. Scheffler also gave a linear-time algorithm to determine the path-width of a given tree. We can also give a linear-time algorithm to determine the proper-path-width of a given tree. A special case of Theorem 2 when $k = 2$ was proved by Takeuchi [17] and independently by Fukuhara [8].

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References

[1] D. Archdeacon, A Kuratowski theorem for the projective plane, J. Graph Theory vol. 7(1983)325-334.

[2] S. Arnborg, D. G. Corneil, and A. Proskurowski, Complexity of finding embeddings in a k-tree, SIAM J. Alg. Disc. Meth. Vol.8, No.2(1987)277-284.

[3] S. Arnborg, A. Proskurowski, and D. G. Corneil, Forbidden minors characterization of partial 3-trees, Discrete Mathematics 80(1990)1-19.

[4] W. W. Dai, and M. Sato, Minimal forbidden minor characterization of planar partial 3-trees and application to circuit layout, Proceedings of ISCAS (1990)2677-2681.

[5] M. R. Fellows, N. G. Kinnersley, and M. A. Langston, Finite-basis theorems and a computational-integrated approach to obstruction set isolation, Proceedings of Computers and Mathematics Conference (1989).

[6] M. R. Fellows, and M. A. Langston, On search, decision and the efficiency of polynomial-time algorithms, Proc. 21st ACM Symposium on Theory of Computing (1989)501-512.

[7] H. Friedman, N. Robertson, and P. D. Seymour, The metamathematics of the graph minor theorem, Applications of Logic to combinatorics, American Math. Soc., Providence, RI (1987)to appear.

[8] T. Fukuhara, Master Thesis, Tokyo Institute of Technology (1990).

[9] H. H. Glover, P. Huneke, and C. Wang, 103 graphs that are irreducible for the projective plane, J. Combin. Theory Series B 27(1979)332-370.

[10] R. H. Möhring, Graph problems related to gate matrix layout and PLA folding, in: G. Tinhofer, E. Mayr, H. Noltemeier, and M. Syslo, eds., Computational Graph Theory, Computing suppl. 7 (Springer-Verlag, Wien New York 1990)17-51.

[11] T. D. Parsons, Pursuit-evasion in a graph, in: Y. Alavi and D. Lich, eds., Theory and applications of graphs (Springer-Verlag, Berlin 1976)426-441.

[12] A. Proskurowski, Separating subgraphs in k-trees: cables and caterpillars, Discrete Mathematics 49(1984)275-285.

[13] N. Robertson, and P. D. Seymour, Graph minors. I. excluding a forest, Journal of Combinatorial Theory Series B 35(1983)39-61.

[14] N. Robertson, and P. D. Seymour, Graph minors. XIII. the disjoint paths problem, preprint, 1986.

[15] N. Robertson, and P. D. Seymour, Graph minors. XVI. Wagner's conjecture, preprint, 1987.

[16] P. Scheffler, A linear algorithm for the pathwidth of trees, in: R. Bodendiek and R. Henn, eds., Topics in combinatorics and graph theory (Physica-Verlag, Heidelberg 1990)613-620.

[17] M. Takeuchi, private communication.

[18] K. Wagner, Bemerkung zu hadwigers vermutung, Math. Ann. vol.141(1960)433-451.