凸分割の射影像について

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アブストラクト

d 次元空間の凸分割 S を、$d-k$次元の部分空間に射影して得られる分割を $Pr_k(S)$ と書こう。本論文では、S が n 個の d 次元領域を持つ時、$Pr_k(S)$ の複雑度を考察する。自明な上界 $O(n^{(k+2)(d-k)})$ に対して、本論文で与える上界は、$d-k$が偶数の時 $O(n^{(k+1)(d-k)})$、奇数の時 $O(n^{(k+1)(d-k)-1})$ である。特に $k=1$ の場合には、下界として、$\Omega(n^{\lfloor(3d-3)/2\rfloor})$ が示され、これは、$k \leq 4$ で最適である。応用として、点位置決定問題などが考えられる。

Complexity of Projected Images of Convex Subdivisions *

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Abstract

In this paper, we give asymptotic bounds for the complexity of the projective image $Pr_k(S)$ of a convex subdivision S of d-dimensional space to a subspace of codimension k. If S has n regions, the complexity of $Pr_k(S)$ is $O(n^{(k+1)(d-k)})$ if $d-k$ is even, and $O(n^{(k+1)(d-k)-1})$ if $d-k$ is odd. This bound is better than naive $O(n^{(k+2)(d-k)})$ bound by the factor of n^{d-k} if $d-k$ is even, and n^{d-k+1} if $d-k$ is odd. Further, we give a lower bound for $Pr_1(S)$, which is $\Omega(n^{\lfloor(3d-3)/2\rfloor})$. This bound is tight if $d \leq 4$.

Applications to the point location problem and related problems are discussed.

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3 Introduction

Projection and the projected images often play important roles in algorithms of computational geometry. For example, a Voronoi diagram is realized as the projected image of a convex polytope. Also, if we consider the data structures for point location in a subdivision of a space [Co, DL, PT, THI], its efficiency usually depends on the complexity of its projected images to lower dimensional spaces.

In three dimensional case, two point location structures in a convex subdivision S are known. One can query with $O(\log n)$ time, and need the space complexity as same as the complexity of the projected image $Pr_1(S)$ of S on a plane [THI]. The other needs $O(\log^2 n)$ query time, but reduces the space complexity to $O(K log^2 n)$ [PT]. In order to compare these two method, we must estimate the complexity of $Pr_1(S)$. If we estimate the complexity of $Pr_1(S)$ with respect to K, it can be $\Theta(K^2)$. Since K is $\Theta(n^2)$, it seems that $Pr(S)$ has $\Theta(n^4)$ faces. On the other hand, if we estimate $Pr(S)$ with respect to n, it is not trivial to construct a subdivision S such that $Pr(S)$ contains more than $O(n^2)$ faces. In fact, the complexity of $Pr(S)$ is $\Theta(n^3)$.

It is important to express the complexity by using n. Suppose we construct the convex hull P of n points in the $(d+1)$-dimensional space. P contains at most n vertices and $O(n^{\lfloor (d+1)/2 \rfloor})$ faces. We would like to construct a data structure, such that given a hyperplane H, query the nearest point of P to H efficiently. In the dual space, this problem is the point location problem in a convex subdivision of the d-dimensional space into n convex regions. Here, it is desirable to estimate the complexity of the point location structure by using n.

In this paper, we consider the complexity of the projected image of a convex subdivision of d-dimensional space onto a lower dimensional affine space. Let S be a convex subdivision of the d-dimensional space R^d. The number of regions in S is n. We project the $(d-k-1)$-skeleton of S onto a $(d-k)$-dimensional subspace L. The projection image is denoted by $Pr_k(S)$.

The trivial upper bound for $Pr_k(S)$ is $O(n^{(k+2)(d-k)})$. Our upper bound is the following:

Theorem 4.4 If we project the $(d-k-1)$ skeleton to a $(d-k)$ dimensional subspace, the complexity is $O(n^{(k+1)(d-k)})$ if $k < \lfloor (d-1)/2 \rfloor$ and $d-k$ is even, $O(n^{(k+1)(d-k)-1})$ if $k < \lfloor (d-1)/2 \rfloor$ and $d-k$ is odd, $O(n^{(k+1)(d-k)-2})$ if $k = \lfloor (d-1)/2 \rfloor$, $O(n^{\lfloor (d-1)/2 \rfloor (d-k)})$ if $k > \lfloor (d-1)/2 \rfloor$.

In particular, the upper bound of $Pr_1(S)$ is $O(n^{2d-3})$ (for d even) and $O(n^{2d-2})$ (for d odd) if $d \geq 5$. The trivial lower bound for $Pr_1(S)$ is $\Omega(n^{d-1})$. Our lower bound, which is tight if $d \leq 4$, is the following:

Theorem 4.3 The complexity of the projected image $Pr_1(S)$ of S onto a space of codimension 1 is $\Omega(n^{\lfloor (3d-3)/2 \rfloor})$.

The paper is arranged as follows: In Section 2, the tight bound for three dimensional case is given. In Section 3, the number of topological changes in the projected image is shown to be $\Theta(n^4)$ if we rotate S around an axis parallel to the projection plane. In Section 4, the higher dimensional case is treated, and main results are stated. In Section 5, algorithms for computing the projected images are given. Finally, some applications are given in Section 6.

4 3-Dimensional Case

Let S be a convex subdivision of 3-dimensional space into n polytopes. The number of edges in S is denoted by K. $Pr(S)$ is the projected image of (the 1-skeleton of) S onto a plane H.

Theorem 4.1 *The number of vertices in $Pr(S)$ is $O(nK)$.*

Proof The regions are numbered as Q_i for $i = 1, 2, .., n$. For each region Q_i in S, the boundary of the convex hull of the projected image $Pr(Q_i)$ is denoted by $C(Q_i)$, and number of edges of $C(Q_i)$ is denoted by k_i. Every edge of $C(Q_i)$ is a projection of and edge of Q_i, and each edge contributes to at most two $C(Q_i)$, so $\sum_{i=1}^{n} k_i \leq 2K$. For each edge e in S, let $H(e)$ be the plane containing e and perpendicular to H. Because of the convexity, there exists a region Q containing e which does not intersect $H(e)$. Then, $Pr(e)$ is an edge of $C(Q)$. Therefore, every projected image of edge is contained in the convex boundary of a projected image of a suitable region. Since $C(Q_i)$ and $C(Q_j)$ intersect at most $2\min(k_i, k_j)$

points, there are at most $\sum_{i,j=1}^{n} 2\min(k_i, k_j) \leq \sum_{i,j=1}^{n}(k_i + k_j) = 2n\sum_{i=1}^{n} k_i \leq 4nK$ intersections in $Pr(S)$. □

Next, we give the lower bound of the complexity of the projection of the skeleton of a three dimensional convex subdivision satisfying that K, the number of edges of the convex subdivision S, is at least n. We fix a plane H in the space.

Theorem 4.2 *There exists a convex subdivision S of the space into n convex polytopes such that projection of it on H has $\Theta(nK)$ vertices.*

Proof It suffices to show the lower bound. Because of the Dehn-Sommerville equation, $K \leq n^2$. For given arbitrary number $18n \leq k \leq n^2$, we set $m = k/6n$, and $s = (n - 2m)/2$. It is easy to see that $s > n/3$. We consider a circle C on a plane H_1 parallel to H. Let l be a line through the center O_1 of C perpendicular to H. Let A_1 denote the set of vertices of a regular m-gon inscribed into the circle C. Further, we consider a point set B_1. consisting of s points on l. We consider the Voronoi diagram V_1 of $A_1 \cup B_1$. Then the projection of the cap boundary of each region of a point of B_1 in V_1 is a regular m-gon with center O, which is the projection of O_1. Further, each of these polygon can be transformed into another by a scaling transformation. If the whole set of B_1 is close enough to O_1, the scaling factor is larger than $\cos\frac{\pi}{2m}$ (and smaller than 1) for each pair.

Next, we consider another plane H_2 parallel to H, and project A_1 onto H_2 to get a point set on the circle C_2 with center O_2. We rotate these points on C_2 by an angle $\frac{\pi}{m}$ to obtain a point set A_2. We let a point set B_2 be a translate of B_1 by the vector $\vec{O_1O_2}$. The Voronoi diagram V_2 of $A_2 \cup B_2$ is congruent to V_1. However, since the point set is rotated, the projection of the cap boundary of the region of a point of B_2 is rotated by $\frac{\pi}{m}$ with respect to the corresponding one in V_1.

Now, we place H_2 sufficiently far from H_1, and we consider the Voronoi diagram V of $A_1 \cup A_2 \cup B_1 \cup B_2$. Then, the projection of the cap boundary of the region of a point of B_1 intersects at $2m$ points with that of any point of B_2. Since there are s^2 pairs of such cap boundaries, the total number of intersections is at least $2ms^2 \geq \frac{1}{27}nk$.

On the other hand, the Voronoi diagram V has n regions and $K = \frac{k}{9} + O(n)$ edges. Thus, we obtain the theorem. □

Corollary 4.3 *The number of vertices in $Pr(S)$ is $\Theta(n^3)$.*

5 Rotation and topological change

It is also interesting to investigate the topological change of $Pr(S)$ if S is rotated with respect to the angle around a line. If we can cheaply rotate the subdivision, we can find the angle such that the complexity of the projective image is minimized. We say that a rotation is parallel if the rotation axis is parallel to the projection plane. Otherwise, it is called a skew rotation.

During the rotation, a topological change occurs when three projected edges meets. Since there are $O(n^2)$ edges, a naive bound of the number of topological changes is $O(n^6)$. We can improve this naive bound for parallel rotations.

Theorem 5.1 *The number of topological changes is $\Theta(n^4)$ for a parallel rotation.*

Proof First, we prove the upper bound. We assume that the projection plane H is parallel to the $x-y$ plane. The plane H_θ is the plane obtained by rotating H by an angle θ around x-axis. The projection Pr_θ is the orthogonal projection to H_θ. Suppose the projected images of three edges e_1, e_2, and e_3 meet at a point $p_0 = (x_0, y_0, z_0)$ of H_θ. Then, we consider the plane $H(x_0)$ intersecting with the x-axis orthogonally at x_0. Then, the intersecting points of e_1, e_2, e_3 are located on a line on $H(x_0)$.

Let $X(S) = \{x_1, x_2, .., x_N\}$ be the sorted list of the x-coordinate values of vertices of S. We define $x_0 = -\infty$ and $x_{N+1} = \infty$. Let us count the topological changes by using a space sweep with respect to the x-axis. We consider a sweep plane $H(t)$, which intersects with the x-axis at $(t, 0, 0)$ orthogonally. Let $S(t)$ be the intersection of S with $H(t)$. Obviously, $S(t)$ is a planar convex subdivision with $O(n)$ regions; thus it has $O(n)$ vertices. We move the sweep plane $H(t)$ from $t = x_0$ to $t = x_{N+1}$, and count the number of colinear triples of the vertices of $S(t)$. Let x_i and x_{i+1} be two consecutive elements of $X(S)$. For any two values t and t' in (x_i, x_{i+1}), the graph structure of $S(t')$ is same as that of $S(t)$. For each triple e_1, e_2, e_3 of edges there is at most one t' such that the intersecting points with $H(t')$ are located on a line. Thus, at most n^3 topological changes are found during the sweep from $x_0 = -\infty$ to x_1. When the sweep plane passes through $x = x_i$, k_i edges (incident to the corresponding vertex to x_i) are newly cut by the sweep plane. Thus, $k_i n^2$ triples are newly created. Since $\sum_{i=1}^{N+1} k_i = O(n^2)$, the total number of topological changes is $O(n^4)$.

Next, we consider the lower bound. We use the Voronoi diagram V defined in Section 2. We adopt the notations in Section 2. We assume that the distance D between H and H_1 is sufficiently large, and the distance between H_1 and H_2 is very small. Let us assume the rotation axis α on H meets the line l at the origin O. Let $\bar{l}$ be a line on H orthogonal to α, such that $\bar{l}$ meets l and α at O. We define a set X of n points on $\bar{l}$, such that the maximal distance between them is bounded by a small constant δ. Let $\bar{V}$ be the Voronoi diagram of the point set $A_1 \cup A_2 \cup B_1 \cup B_2 \cup X$. We consider the subdivision S which arises by adding the plane H to $\bar{V}$. Evidently, S is a convex subdivision consisting of $O(n)$ regions. Since H_1 is far enough from H, almost all regions of V survive in S (actually, only the lower envelope of V is changed). We call this part $\tilde{V}$. There exist a maximal angle ϕ such that the topological structure of $Pr(\tilde{V})$ is not changed if we rotate it by any angle between $-\phi$ and ϕ. This angle ϕ is independent of the distance between H and H_2. On the other hand, S contains the set $\mathcal{L}$ of $n-1$ segments parallel to α, which are intersections between the plane H and the Voronoi boundary of the points of X. Each of these segments is long enough, and it intersects $\bar{l}$. Further, the maximal distance between them is bounded by δ. We can assume that $D \tan \phi > d$. Then, during rotating S from $-\phi$ to ϕ, each of $\Theta(n^3)$ vertices of $Pr(\tilde{V})$ meets each of $n-1$ segments of $\mathcal{L}$ at an angle. Thus, there exist $\Omega(n^4)$ topological changes. □

Any rotation is written as a product of the three rotations around x-axis, y-axis, and z axis. Obviously, the rotation around z-axis makes no topological change to $Pr(S)$. However, if we deal with a skew rotation, we should consider the intersection of the subdivision with a circular cone instead of the intersection with a hyperplane. Unfortunately, the complexity of an intersection of S with a circular cone is $\Theta(n^2)$. Therefore, it is open to obtain a nontrivial bound for the number of topological changes for a skew rotation.

6 Higher dimensional extension

We have shown that the complexity of the projection image of a convex subdivision in 3-dimensional space to a plane is $\Theta(n^3)$. In this section, we generalize the result for the convex subdivisions in R^d, and show an upper bound and a lower bound for the complexity of its projected image.

Let S be a convex subdivision of R^d into n polytopes. It is well-known [E] that the worst-case complexity of S is $\Theta(n^{\lfloor (d+1)/2 \rfloor})$. The projection of the $(d-k+1)$-skeleton of S onto a $(d-k)$-dimensional subspace L is denoted by $Pr_k(S)$.

A face of S is called *facet* if it has codimension 1. A face of S is called *ridge* if it has codimension 2. A face of dimension j is called a j-face.

Any face of $Pr_k(S)$ is an intersection of projected images of at most $d-k$ faces of S. The projection is called nondegenerate if there is no degeneration in $Pr_k(S)$ except those originally in S. It is easy to observe the following lemma:

Lemma 6.1 *The complexity of $Pr_k(S)$ is asymptotically bounded by the number of vertices in $Pr_k(S)$ and the number of original faces if the projection is nondegenerate.*

To obtain an upper bound of the complexity of $Pr_k(S)$, we can assume that the projection is nondegenerate without loss of generality.

Let us begin with the upper bound for $Pr_1(S)$. There are $O(n^3)$ ridges in S, and a vertex of $Pr_1(S)$ is an intersection of $d-1$ projected ridges. Hence, the complexity of $Pr_1(S)$ is naively $O(n^{3(d-1)})$, which is far from optimal. Our upper bound is the following:

Theorem 6.2 *The complexity of $Pr_1(S)$ is $O(n^{2d-3})$ for d even, and $O(n^{2d-2})$ for d odd. Moreover, the complexity is $O(n^d)$ if $d \leq 4$.*

Proof The theorem is a corollary of Theorem 4.4 shown below. □

Next, we generalize the lower bound of Theorem 2.2 for the higher dimensional cases.

Theorem 6.3 *The complexity of $Pr_1(S)$ is $\Omega(n^{\lfloor(3d-3)/2\rfloor})$.*

Proof Let us consider the moment curve $\Gamma : x(t) = (t, t^2, t^3, .., t^{d-1})$ of R^{d-1}. We consider a set $M = \{x(\tau_i) : i = 1, 2, ..n\}$ of $(d-1)n$ points on Γ. We assume $\tau_i < \tau_j$ if $i < j$. The convex hull of M is denoted by $C(M)$. It is well known that $C(M)$ has $\Omega(n^{\lfloor(d-1)/2\rfloor})$ facets. The subset M_i of M is defined by the set $\{x(\tau_j) : j \equiv i \pmod{d-1}\}$. We cluster M into $d-1$ subsets $M_1, M_2, .., M_{d-1}$.

Let us investigate the facets in detail. An index set $I = \{i_1, i_2, ..., i_{d-1}\}$ of size $d-1$ is called *special* if $I \subset \{1, 2, .., n\}$ and $i_j = i_{j-1} + 1$ if j is even. Furthermore, we set $i_{d-1} = n$ if $d-1$ is odd. We define the function $f_I(x) = \prod_{j \subset I}(x(\tau_j) - x)$. It is easy to see that this function is nonnegative on M, and zero on $x(\tau_j)$ if $j \in I$. Since the degree of $f_I(x)$ is $d-1$, similar to the argument of p.101 of [E], $f_I(x) = (u, x) - v_0$ on Γ for suitable vectors u and v_0. Hence, the hyperplane spanned by $\{x(\tau_j) : j \in I\}$ appears as a facet of $C(M)$. Then, the following claim is easily observed

Claim. *There are $\Omega(n^{\lfloor(d-1)/2\rfloor}$ facets of $C(M)$, each of which is spanned by a point set containing exactly one point of each subset M_i $(i = 1, 2, .., d-1)$.*

Let $D(M)$ be the set of dual hyperplanes of M, and let $D(C(M))$ be the dual of $C(M)$. We choose a point x in the interior of $D(C(M))$. For each hyperplane h in $D(M)$, the opposite point of x with respect to h is denoted by $x(h)$. The point set $\{x(h) : h \in D(M_i)\}$ is denoted by $\hat{M}_i$.

Let g be the d-th axis of R^d. We choose the points x_i $i = 1, 2, .., d-1$ such that the distance between each pair of these points is sufficiently large. We consider the hyperplane L_i orthogonal to g containing x_i. Now, we translate the point set $\hat{M}_i$ such that x is translated to x_i. We generate n points on g which are infinitesimally near to x_i. Let us denote V_i for the Voronoi diagram generated by these $2n$ points. V is the merged Voronoi diagram of V_i for $i = 1, 2, .., d-1$. The Voronoi region of a point on g is called a central region. Then, if we pick up a central region from each cluster, the intersection of these $d-1$ regions contributes $\Omega(n^{\lfloor(d-1)/2})$ vertices because of the claim. Thus, we obtain the theorem. □

Therefore, our upper bound is tight if $d \leq 4$. The lower bound differs from the upper bound by a factor of $O(n^{\lfloor(d-1)/2\rfloor})$ if $d \geq 5$. We conjecture that the upper bound is not tight, since the polytope $B(P)$ has a special property, that is, it has only $O(n^{\lfloor d/2\rfloor})$ faces though it has $O(n^2)$ facets.

Next, we give an upper bound for the complexity of $Pr_k(S)$. Since there are n^{k+2} faces of codimension $(k+1)$ in S, a naive upper bound is $n^{(k+2)(d-k)}$ The upper bound is generalized as follows:

Theorem 6.4 *The complexity of the projection image $Pr_k(S)$ of the $d-k-1$ skeleton of S on an affine subspace of codimension k is $O(n^{(k+1)(d-k)})$ if $k < \lfloor(d-1)/2\rfloor$ and $d-k$ is even, $O(n^{(k+1)(d-k)-1})$ if $k < \lfloor(d-1)/2\rfloor$ and $d-k$ is odd, $O(n^{(k+1)(d-k)-2})$ if $k = \lfloor(d-1)/2\rfloor$, and $O(n^{\lfloor(d-1)/2\rfloor(d-k)})$ if $k > \lfloor(d-1)/2\rfloor$.*

Proof The number of j faces of S is denoted by f_j. Let P be a (d-k+1)-face of S. The boundary $B(P)$ of $Pr_k(P)$ is a convex polytope in R^{d-k}. Let f be an arbitrary (d-k-1)-face of S. There exists a hyperplane H, which is perpendicular to the projection subspace L and contains f. It is easy to see that there exists at least a (d-k+1)-face P of S which is located in one of the half-spaces defined by H. Obviously, $Pr(f)$ is contained in $B(P)$. In fact, $Pr(f)$ is a facet of $B(P)$.

There are $O(n^2)$ (d-k+1)-faces bounding P. Hence, $B(P)$ is a convex polytope with $O(n^2)$ facets in R^{d-k}. Thus, $Pr_k(S)$ is an arrangement (in R^{d-k}) of n convex polytopes, each of which has $O(n^2)$ facets.

For a vertex v of $Pr_k(S)$, let v be a vertex of the intersection of $B(P_{j(1)})$, $B(P_{j(2)})$,..., $B(P_{j(h)})$. Because of non-degeneracy, $h \leq d-k$. We denote the index set $J = \{j(1), j(2), .., j(h)\}$. The vertex v is located on the polytope $P(J) = B(P_{j(1)}) \cap B(P_{j(2)}) \cap ... \cap B(P_{j(h)})$. For a fixed J, there are at most $O(n^{2\lfloor(d-k)/2\rfloor})$ vertices on $P(J)$.

The number of $(d-k+1)$-faces in S is $f_{d-k+1} = O(n^k)$. Thus, the number of possible combinations for the index set J is $O(n^{k(d-k)})$. Thus, the number of vertices of $Pr(S)$ is $O(n^{2\lfloor(d-k)/2\rfloor}n^{k(d-k)})$, which equals to $O(n^{(k+1)(d-k)})$ if $d-k$ is even, and $O(n^{(k+1)(d-k)-1})$ if $d-k$ is odd. Hence, we have shown the theorem if $k \leq \lfloor(d-1)/2\rfloor$.

If $k > \lfloor(d-1)/2\rfloor$, $f_{(d-k-1)} = n^{\lfloor(d-1)/2\rfloor}$. Thus, the complexity is simply $n^{\lfloor(d-1)/2\rfloor(d-k)}$.

If $k = \lfloor (d-1)/2 \rfloor$, $f_{(d+1-k)} = n^k$, $f_{(d-k)} = n^{k+1}$, and $f_{(d-k-1)} = n^{k+1}$. Thus, although each $B(P)$ may contain $O(n^2)$ facets, the average of the number of facets on the polytopes in the arrangement is $O(n)$. Thus, we can save a factor of n if $d-k$ is odd, and a square factor of n if $d-k$ is even. We omit the details. □

7 Algorithmic aspect

The proof of Theorem 4.3 gives an algorithm to compute $Pr_1(S)$, which runs in $O(n^{2\lceil (d-1)/2 \rceil + d - 1})$ time. With more precise analysis, this algorithm runs in $O(n^3)$ and $O(n^4 \log n)$ time if dimension is three and four respectively. If we use randomized convex hull algorithm of Seidel [S1], the computing time is reduced to $O(n^{2\lfloor (d-1)/2 \rfloor + d - 1})$.

The output size is usually much smaller than the worst case size; thus, an output sensitive efficient algorithm is desirable. The plane sweep method solves the problem in $O(M \log n)$ time if $d = 3$ (where M is the output complexity), which is practically efficient. Further, if we use the optimal segment intersection reporting algorithm [CE], an $O(M + K \log n)$ time algorithm can be designed, where K is the number of edges in S.

In four dimensional case, the projection image is an arrangement of n convex polyhedra in the three dimensional space The total number of faces of the polyhedra in the arrangement is $O(n^2)$. Below, we give an $O(M \log n)$ method.

Let us consider the space sweep method to compute $Pr_1(S)$ We consider sweep planes $x = t$ orthogonal to x-axis, and translate it from $t = -\infty$ to $t = \infty$. The intersection $\Sigma(t)$ of $Pr_1(S)$ with the plane $x = t$ is an arrangement of convex polygons. The complexity of $\Sigma(t)$ is $O(Min(n^3, M))$. For each edge e of $\Sigma(t)$, we compute t at which the edge vanishes. For all such edges, we keep these values in a priority queue. We update this priority queue during the sweep. If the sweep comes to the abscissa of a vertex of S, more than one element of the priority queue may be updated. However, the total number of priority queue operations is $O(M)$ during the sweep. Therefore, the sweep method gives an $O(M \log n)$ time algorithm for computing $Pr(S)$.

In higher dimensional case, an output sensitive algorithm to compute a convex hull in $O(n^2 + h \log n)$ time is developed by Seidel [S2], where h is the number of faces on the convex hull. Let k_i be the number of vertices of $Pr(S)$, which lie on the projected images of a $(d-2-i)$-dimensional faces of S. If we apply Seidel's output sensitive convex hull algorithm, we obtain a slightly output sensitive algorithm. The time complexity is at most $O(n^{d+1}) + \sum_{i=0}^{d-2} n^i k_i \log n$. It is easy to see that $k_i = O(n^{2\lfloor ((d-1)/2) \rfloor + d - 1 - i})$, and practically much smaller.

8 Applications

Point location problems

Let us consider the point location structure for a convex subdivision S consisting of n regions. Although good data structures are known for the point location in an arrangement, data structure of poly-logarithmic query point location in other convex subdivision usually have huge space complexity.

A naive senario is, given a convex subdivision S, we project it to the space of codimension 1, and make a point location structure for $Pr_1(S)$. Then, given a point p, first locate it in $Pr_1(S)$, and next locate it in the fiber. By using similar list search [Co] or fractional cascading [CG], the space complexity is as same as that for the point location structure of $Pr_1(S)$. Thus, it is important to analyze the complexity of $Pr_1(S)$, although this method is space-expensive if d is large. Note that this method need not the convexity of S in principle.

Also, another possibility is we first make a point location structure of $Pr_k(S)$ and its fibers for $k \geq 2$. The topological structure of the fiber is stable provided that it is a fiber of a point in a given cell of $Pr_k(S)$. If $k = 2$, we equip the point location structure of Edelsbrunner-Lee-Preparata [LP, EGS] for the fibers, and we have a point location structure of S (note that the *monotoneness* of the fiber is also stable in a region of $Pr_2(S)$). If we can permit $O(log^2)$ time query, by applying the Preparata and Tamassia's technique [PT], the space complexity is $O(s(Pr_2(S))log^2 n)$, where $s(Pr_2(S))$ is the space complexity for the point location in $Pr_2(S)$.

It is necessary to analyze the complexity of the projected image for such point location structures. Moreover, it is often important to describe the relation of the complexity and n.

For instance, suppose we construct the convex hull P of n points in the $(d+1)$-dimensional space. P contains at most n vertices and $O(n^{\lfloor (d+1)/2 \rfloor})$ faces. We would like to construct a data structure, such that given a hyperplane H which does not intersect P, query the nearest point of P to H efficiently. In the dual space, this problem is the point location problem in a convex subdivision S of the d-dimensional space into n convex regions. For each vertex $v = v_1, .., v_{d+1}$ of P, we dualize it to a hyperplane $H(v)$ defined by $x_{d+1} = v_1 x_1 + .. + v_1 x_d - v_{d+1}$ in $d+1$-dimensional space. We construct the upper (resp.lower) envelope of the arrangement associated with these hyperplanes, and project it to the hyperplane $x_d = 0$ to obtain a convex subdivision S (resp.S') of the d-dimensional space. Given a hyperplane $H : c_1 x_1 + c_2 x_2 + .. + c_{d+1} x_{d+1} = c_{d+2}$, we locate the point $(c_1, ..., c_d)$ in S and S'. The reported hyperplanes are the dual plane of the nearest point to H and that of the farthest point to H. In above application, it is desirable to estimate the complexity of the point location structure by using n.

Further, in above case, S is a projected image of a lower envelope of an arrangement of n hyperplanes in $(d+1)$-dimensional space. In such a case, the following sampling method can reduce the space complexity: First, we choose r_1 hyperplanes randomly, and consider the projection S_1 of the lower envelope of the arrangement constructed from the sampled hyperplanes. We prepare the point location structure of S_1. Next, we choose $r_2 > r_1$ hyperplanes, and consider the projection S_2 of the lower envelope of the associated arrangement. We prepare the point location structure of the clipped portion of S_2 to each region of S_1. If we continue the process until $r_k = n$, we obtain a hierarchical point location structure of S.

Theoretically, if we set $r_i = c^i$ for a (large) constant c, the space complexity can be reduced to $O(n^{\lfloor (d+1)/2 \rfloor + \epsilon})$ if we sacrifice the query time by a factor of $O(\log n)$. There is a space-time tradeoff caused by the size of the samples. In order to tune up the numbers of samples, we must analyze $Pr_1(S)$ (or $Pr_2(S)$) by using n, if we use the projection method for the local structures.

Below, we study the efficiency of the projection method for the dimension $3, 4$, and 5.

In three dimensional case, the projection method to a codimension 1 space is given by [THI]. From Theorem 2.1 and Theorem 2.2, the space complexity is $\theta K n$ for a convex subdivision S of n regions and k edges. The projection method to codimension 2 space, which is in fact a space-sweep method, is given by Preparata and Tamassia [PT]. The space complexity is reduced to $O(K \log^2 n)$, although the query time is increased to $O(\log^2 n)$. Therefore, the former method is space-expensive by a factor of $\frac{n}{\log^2 n}$ compared to the latter method if we choose the worst projection direction, although it may be practically comparable if we can find a good projection direction.

Further, let us consider the following vertical penetrating query problem in the arrangement $A(H)$ of n hyperplanes in the four-dimensional space. For a given point p in the space, we query the facet which is shot by the ray-shooting from p with respect to the direction vector $(0, 0, 0, 1)$. This problem is the dual problem of a multi(four)-dimensional list query problem; that is, we preprocess a point set M such that we can efficiently query the point which is nearest to a given hyperplane.

It is known that the cell containing q can be queried in $O(\log n)$ time with $O(n^4)$ space [CF]. In each cell, the ray shooting problem is essentially the point location problem in the projection of the cap of the cell. Let $f_3(C)$ and $f_2(C)$ be the number of 3-dimensional faces and that of 2-dimensional faces of a cell C of the arrangement. Then, from Theorem 2.1, the ray shooting query can be answered in $O(\log n)$ time if we use $\sum_{C \in A(H)} f_2(C) f_3(C)$ space. From the famous zone theorem [E], $\sum_{C \in A(H)} f_2(C) f_3(C) = O(n^4)$.

Theorem 8.1 *The ray shooting query in the arrangement of four dimensional space is done in $O(\log n)$ time using $O(n^4)$ space.*

We remark that the best known bound for $\sum_{C \in A(H)} \{f_2(C)\}^2$ is $O(n^4 \log n)$ [AMS]; thus we save a factor of $\log n$ by using Theorem 2.1..

For a four dimensional subdivision S, the complexity of $Pr_1(S)$ is $\Theta(n^4)$. Applying Preparata-Tamassia's approach for the point location in

$Pr_1(S)$, and make a similar list search structure for the fibers, we have a $O(n^4 \log^2 n)$ space $O(\log^2 n)$ query time structure for the point location in S.

The projection method to a codimension 2 space also needs $O(\log^2 n)$ time $O(n^4 \log^2 n)$ space, since the complexity of $Pr_2(S)$ is $O(n^4)$. However, in more presice analysis, the complexity of $Pr_2(S)$ is $O(K^2)$, while $Pr_1(S)$ is $O(Kn^2)$, where K is the number of edges in S. Thus, the projection method to a codimension 2 space is better.

In five dimensional case, we only deal with the projection method to a codimension 2 space, since the other method is too expensive. By Theorem 4.5, the complexity of $Pr_2(S)$ is $O(n^7)$. Thus, we obtain an $O(log^2 n)$ query time and $O(n^7 \log^2 n)$ space structure. The space complexity is often less than cube of the input size, since $K^3 = \Omega(n^9)$.

Transparent Graphics

Projection complexity is related to the ray tracing structure of a transparent object. Let l be a ray vertical to the projection plane in 3-dimensional space, and l path trough the convex subdivision S. Suppose a particle (e.g. a photon) is shot along l, and affected (e.g. change the wave length) in each region of S. Then the property (e.g. color) of the image on S depends on the region containing it in $Pr(S)$. Although we have dealt with the projection on a plane so far, we can easily obtain similar results for the central projection on a sphere. The result on the number of topological change caused by the rotation has potential application to the visibility problem from a moving viewpoint.

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