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Topes of Oriented Matroids and Related Structures

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あらまし　有向マトロイドは超平面のarrangementの多面体領域の面の間の組合せ的性質を抽象化した概念である。有向マトロイドのトープは極大な多面体領域に対応する。本稿ではそのトープ族と関連する概念、即ちacycloid，L^1-system，median systemについて述べる。L^1-systemがconvex geometryと密接な関係があることを示す。median systemはmedian graphと同値な概念として本稿で導入する概念であり、特にmedian graphを特徴づけるために使う。acycloidとL^1-systemのperturbationについても述べる。

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Abstract

An oriented matroid can be viewed as a combinatorial abstraction of the facial incidence relations of the polyhedral cones induced by a finite arrangement of oriented hyperplanes in $\mathbf{R}^d$ through the origin. "Topes" of an oriented matroid correspond to maximal polyhedral cones. This note discusses three structures related to topes of oriented matroids, namely, acycloids, L^1 -systems and median systems. It is shown that L^1 -systems are closely related to convex geometries. Median systems are introduced as an equivalent notion of median graphs, and they are, in particular, applied to characterize median graphs. Perturbations of acycloids and L^1 -systems are studied.

1 Introduction

Let E be a finite set and let $\mathbf{A}$ be an $m \times |E|$ real matrix having $\mathbf{A}^e$ as the column vector of $\mathbf{A}$ indexed by $e \in E$. For each vector $\mathbf{x} \in \mathbf{R}^E$, $\sigma(\mathbf{x})$ denotes the signed vector of $\mathbf{x}$, that is, $\sigma(\mathbf{x}) \in \{-,0,+\}^E$ and $\sigma(\mathbf{x})_e$ is the sign of component $\mathbf{x}_e$. Let $\mathbf{V}$ be the row space of $\mathbf{A}$, i.e., $\mathbf{V} = \{\mathbf{xA} : \mathbf{x} \in \mathbf{R}^m\}$. Then the set $\sigma(\mathbf{V}) = \{\sigma(\mathbf{v}) : \mathbf{v} \in \mathbf{V}\}$ represents the partition of $\mathbf{R}^m$ by polyhedral cones induced by the subspaces $\{\mathbf{x} \in \mathbf{R}^m : \mathbf{xA}^e = 0\}$ $(e \in E)$, and also represents the facial incidence relations of the polyhedral cones. An oriented matroid is defined by a set of signed vectors satisfying certain axioms (*face axioms*) that are trivially satisfied by $\sigma(\mathbf{V})$. Besides this, an oriented matroid can be viewed as abstractions of many different concepts in linear space, see [3, 4] for the basic theory and applications.

"Topes" [9, 16] of an oriented matroid correspond to maximal polyhedral cones in the above setting. Topes can be also considered an abstraction of some properties of acyclic reorientations of loopless directed graphs, and furthermore an abstraction of combinatorial properties of partitions $\{S, E-S\}$ of a finite subset $E \subseteq \mathbf{R}^d$ such that there is a hyperplane in $\mathbf{R}^d$ separating S strictly from $E - S$. In this note, we investigate oriented matroids through their topes, and introduce and study three structures related to topes of oriented matroids, namely, acycloids [20], L^1-systems [10] and median systems.

L^1 *-systems* are defined by the reorientation property of topes of oriented matroids, and *acycloids* by the negativity closedness property in addition to the reorientation property. These two structures are useful to characterize topes and tope graphs of oriented matroids. On the other hand, they are also interesting when they are viewed from the corresponding graphs. Indeed, L^1-systems are essentially equivalent to graphs isometrically embeddable in a hypercube, and acycloids such graphs with antipodality, see [10].

Median systems are essentially equivalent to median graphs [1, 19], and they constitute a broad class of L^1-systems. Median graphs have been studied under various names or viewpoints, e.g., median algebras, median semilattices, median interval structures and maximal Helly copair hypergraphs, etc, cf. [19]. Median systems, which we introduce in this note, axiomatize median graphs by signed vectors. Since signed vectors are easy to deal with, one can give very simple proofs for propositions on median graphs.

This note consists of 4 sections. In Section 2, we review the basic notions of oriented matroids, acycloids and L^1-systems. A *convex geometry* [8] is a structure which combinatorially abstracts the notion of the convex hull of a finite set of points in Euclidean space. In Section 3, it is shown that a convex geometry is essentially equivalent to the set of positive "closed acyclons" of an acyclic L^1-system. In Section 4, several propositions on median graphs are proved or reproved by using properties of median systems. In particular, a simple characterization of median graphs, similar to Djoković's theorem [5], is obtained. In general, their proofs are shorter and easier to understand than direct proofs by the graph language or properties. Every median system is shown to be an L^1-system, and hence we should note that the results on L^1-systems hold in median systems, too.

Through this note, we assume graphs have neither loops nor multiple edges. We denote the vertex-set, the edge-set and the distance function of a graph G by $V(G), E(G)$ and d_G, respectively.

The details of this note can be found in [11], which includes the results on perturbations of acycloids and L^1-systems, and on non-matroidal acycloids.

2 Oriented matroids, acycloids and L^1-systems

Through this note, let E be a finite set. A *signed vector* X on E is an element of $\{-,0,+\}^E$, that is, X is a vector $(X_e : e \in E)$ with $X_e \in \{-,0,+\}$. The zero vector is denoted by $\mathbf{0}$. The *negative* $-X$ of a signed vector X is defined in the trivial way. For $X \in \{-,0,+\}^E$ and $S \subseteq E$, we denote the restriction of X to $E - S$ by $X \setminus S$. For $X, Y \in \{-,0,+\}^E$, we define $D(X,Y) = \{e \in E : X_e = -Y_e \neq 0\}$ and $X \circ Y = (X_e \text{ if } X_e \neq 0, \text{ and } Y_e \text{ otherwise } : e \in E)$. Here $X \circ Y$ is called the *composition* of X and Y. Several concepts are used to define oriented matroids. The following is the definition by faces [9, 16].

An *oriented matroid* (*on* E) is a pair $M = (E, \mathcal{F})$ where $\mathcal{F}$ is a set of signed vectors on E, called the *faces* of M, satisfying

- (F1) $\mathbf{0} \in \mathcal{F}$, and $X \in \mathcal{F}$ implies $-X \in \mathcal{F}$;
- (F2) if $X, Y \in \mathcal{F}$ then $X \circ Y \in \mathcal{F}$; and
- (F3) if $X, Y \in \mathcal{F}$ and $f \in D(X,Y)$, there exists $Z \in \mathcal{F}$ such that $Z_f = 0$ and $Z \setminus D(X,Y) = (X \circ Y) \setminus D(X,Y)$.

We denote by $()$ the zero vector on the empty set $\emptyset$, and define for convenience that $M = (\emptyset, \{()\})$ is an oriented matroid. A typical example is obtained from the row space $\mathbf{V}$ of a real matrix $\mathbb{A}$ as we mentioned in Section 1.

For $X, Y \in \{-,0,+\}^E$, X *conforms* to Y, $X \preceq Y$, if $Y_e = X_e$ for all e with $X_e \neq 0$. For $\mathcal{X} \subseteq \{-,0,+\}^E$, we denote by $Min\mathcal{X}$ the set of minimal elements of $\mathcal{X}$ with respect to the partial order $\preceq$. $Max\mathcal{X}$ is similarly defined. For a signed vector X on E, the set $\underline{X} \equiv \{e \in E : X_e \neq 0\}$ is called the *support* of X. We denote by ${}_{\overline{S}}X$, $S \subseteq E$, the signed vector on E obtained from X by reversing signs on S.

Let $M = (E, \mathcal{F})$ be an oriented matroid. A *tope* of M is a maximal vector of $\mathcal{F}$. We denote by $\mathcal{T}$ the set of topes of M, i.e., $\mathcal{T} = Max\mathcal{F}$. An oriented matroid is uniquely determined by its topes because the equation $\mathcal{F} = \{X : X \circ Y \in \mathcal{T} \text{ for all } Y \in \mathcal{T}\}$ holds, cf. [3]. It is well-known that the set $\mathcal{T}$ satisfies the following properties:

- (T1) $X, Y \in \mathcal{T}$ implies $\underline{X} = \underline{Y}$ (here an element which is not contained in the support of any tope is called a *loop*; the set of loops is denoted by E_0);
- (T2) $X \in \mathcal{T}$ implies $-X \in \mathcal{T}$; and
- (T3) *(reorientation property)* if $X, Y \in \mathcal{T}$ and $X \neq Y$, there exists $f \in D(X,Y)$ such that ${}_{\overline{[f]}}X \in \mathcal{T}$, where $[f]$ denotes the *parallel class* containing f, i.e. $\{e \in E :$ either $X_e = X_f$ for all $X \in \mathcal{T}$ or $X_e = -X_f$ for all $X \in \mathcal{T}\}$. Here the parallel class $[e]$ is defined only for $e \in E - E_0$.

An *acycloid* [20] is a pair $A = (E, \mathcal{T})$ where E is a finite set and $\mathcal{T}$ is a nonempty set of signed vectors on E, called the *topes* of A, satisfying (T1) $\sim$ (T3). An acycloid is *simple* if it has no loops and every parallel class is a singleton set. The *tope graph* G_A of an acycloid $A = (E, \mathcal{T})$ is a graph such that $V(G_A) = \mathcal{T}$ and such that $X, Y \in V(G_A)$ are adjacent if and only if $D(X,Y)$ is a parallel class.

L^1-systems [10] are defined by the reorientation property of topes for simple oriented matroids: an L^1*-system* is a pair $A = (E, \mathcal{T})$, where E is a finite set and $\mathcal{T}$ is a nonempty set of elements of $\{-,+\}^E$, called the *topes* of A, satisfying

- (L1) if $X, Y \in \mathcal{T}$ and $X \neq Y$, there exists $f \in D(X,Y)$ such that ${}_{\overline{f}}X \in \mathcal{T}$; and
- (L2) for every $e \in E$, there exist $X, Y \in \mathcal{T}$ such that $X_e \neq Y_e$.

The condition (L2) is not essential but we include it for simplicity. The *tope graph* G_A of an L^1-system A is similarly defined to that of an acycloid: $V(G_A) = \mathcal{T}$ and $E(G_A) = \{[X,Y] : X,Y \in \mathcal{T}$ and $|D(X,Y)| = 1\}$. Fig.1 shows such an example.

The *hypercube* $Q(E)$ on E is the graph that has $\{-,+\}^E$ as vertex-set and $\{[X,Y] : |D(X,Y)| = 1\}$ as edge-set, cf. [14]. For two connected graphs G and G', G is *isometrically embeddable* in G' if there exists an injection $f : V(G) \to V(G')$, called an *isometric embedding* of G into G', such that $d_G(u,v) = d_{G'}(f(u), f(v))$ for all $u, v \in V(G)$.

Proposition 2.1. ([10]) *A graph G is isomorphic to the tope graph of an L^1-system if and only if G is isometrically embeddable in a hypercube.*

Note that the tope graph of an L^1-system determines the L^1-system uniquely up to reorientation, see [10, Note]. For graphs isometrically embeddable in a hypercube, Djoković's theorem is well-known. Let G be a connected graph. A subset $X \subseteq V(G)$ is *convex* in G if for all $u, v \in X$ all shortest (u,v)-paths are contained in the subgraph induced by X. For each $[a,b] \in E(G)$, define $C(a,b) = \{x \in V(G) : d_G(a,x) < d_G(b,x)\}$.

Theorem 2.2. (Djoković [5]). *A graph G is isometrically embeddable in a hypercube if and only if G is connected bipartite, and $C(a,b)$ is convex for all $[a,b] \in E(G)$.*

As some applications of acycloids and L^1-systems to oriented matroid theory, we characterized oriented matroids in terms of topes in [12, 13] (cf. [2]) and tope graphs of oriented matroids of rank at most three in [10]. The latter characterization enables us to test in a polynomial time whether a given graph is isomorphic to a graph representing adjacent relations of regions of an arrangement of pseudolines in the real projective plane $\mathbf{P}^2$.

Finally, for L^1-systems and acycloids, we will define some similar concepts to those of oriented matroids. Let $A = (E, \mathcal{T})$ be an L^1-system or an acycloid. The sets of *faces*, *acyclons* and *circuits* of A are defined by $\mathcal{F} = \{X : X \circ Y \in \mathcal{T}$ for all $Y \in \mathcal{T}\}$, $\mathcal{A} = \{X : X \preceq Y$ for some $Y \in \mathcal{T}\}$, and $\mathcal{C} = Min\{X : X \not\preceq Y$ for all $Y \in \mathcal{T}\}$, respectively. By these definitions, we immediately obtain $\mathcal{T} = Max\mathcal{A} = Max\mathcal{F}$, $\mathcal{A} = \{X : X \not\succeq Y$ for all $Y \in \mathcal{C}\}$, and $\mathcal{C} = Min(\{-,0,+\}^E - \mathcal{A})$.

3 L^1-systems and convex geometries

A function $\phi : \mathcal{P}(E) \to \mathcal{P}(E)$, where E is a finite set, is called a *closure (operator)* if it satisfies

(1) $S \subseteq \phi(S) = \phi(\phi(S))$; and
(2) $R \subseteq S$ implies $\phi(R) \subseteq \phi(S)$,

for all $R, S \subseteq E$. A subset S of E is said to be *closed* if $\phi(S) = S$. For the purpose of simplicity, the empty set is assumed to be closed. A closure ϕ is said to be *anti-exchange* [6] if ϕ satisfies

(3) given a closed set S and two distinct elements e, f of $E - S$,
then $e \in \phi(S \cup f)$ implies $f \notin \phi(S \cup e)$.

The anti-exchange closure is a generalization of the order ideals of a poset and it has many natural examples, such as the convex hull on finite points in $\mathbf{R}^n$, the transitive closure on the edges of an acyclic directed graph, the tree closure on the edges of a tree, etc, see [6].

A *convex geometry* [8] is a pair $(E, \mathbf{G})$ where E is a finite set and $\mathbf{G} \subseteq \mathcal{P}(E)$ satisfying

(G1) $\emptyset, E \in \mathbf{G}$;
(G2) $\mathbf{G}$ is closed under intersection; and
(G3) if $S \in \mathbf{G}$ and $S \neq E$, then there exists $f \in E - S$ such that $S \cup f \in \mathbf{G}$.

Anti-exchange closures ϕ on E and convex geometries $(E, \mathbf{G})$ are equivalent under the following correspondences:

$$\mathbf{G} = \text{the collection of closed sets of } \phi;$$
$$\phi(S) = \cap\{R \in \mathbf{G} : S \subseteq R\} \ (S \subseteq E).$$

In this section, we show that convex geometries are closely related to to L^1-systems. For $\mathcal{X} \subseteq \{-,0,+\}^E$ and $X \in \{-,0,+\}^E$, define $\mathcal{X}(X) = \{Y \in \mathcal{X} : X \preceq Y\}$. Given two signed vectors $X, Y \in \{-,0,+\}^E$, define their *intersection* by $X \cap Y = (X_e \text{ if } X_e = Y_e, \text{ and } 0 \text{ otherwise } : e \in E)$.

Let $A = (E, \mathcal{T})$ be an L^1-system with acyclons $\mathcal{A}$. For an acyclon $X \in \mathcal{A}$, we define $cl(X) = \cap\mathcal{T}(X)$ and call it the *closure* of X. We say an acyclon $X \in \mathcal{A}$ is *closed* if $cl(X) = X$, and we denote by $\mathcal{D}$ the set of closed acyclons of A, i.e., $\mathcal{D} = \{X \in \mathcal{A} : cl(X) = X\}$. It is easily checked that $X^1, X^2 \in \mathcal{D}$ implies $X^1 \cap X^2 \in \mathcal{D}$. Hence for $X \in \mathcal{A}$, $cl(X)$ is the smallest closed acyclon to which X conforms, and we can describe $cl(X) = \cap\mathcal{D}(X)$. Note that $\mathcal{T} = Max\mathcal{D}$ and $\mathcal{F} \subseteq \mathcal{D}$ hold.

Now we consider the poset $L(\mathcal{D}) = (\mathcal{D} \cup \{1\}, \preceq)$, where 1 is an imaginary greatest element, i.e., an element such that $X \preceq 1$ for all $X \in \mathcal{D}$. This poset $L(\mathcal{D})$ forms a lattice as in the following theorem. We show in Fig.2 the lattice $L(\mathcal{D})$ of the L^1-system in Fig.1.

Theorem 3.1. *Let $A = (E, \mathcal{T})$ be an L^1-system with closed acyclons $\mathcal{D}$. Then the poset $L(\mathcal{D})$ forms a coatomic lattice, in which the meet $X \wedge Y$ and the join $X \vee Y$ are defined by*

$$X \wedge Y = X \cap Y,$$
$$X \vee Y = \cap\{Z \in \mathcal{D} \cup \{1\} : X \preceq Z \text{ and } Y \preceq Z\}$$

for $X, Y \in \mathcal{D} \cup \{1\}$, where consider $X \wedge 1 = X$ for $X \in \mathcal{D} \cup \{1\}$. Moreover $L(\mathcal{D})$ has the J-D chain property and the height function h is given by $h(X) = |\underline{X}|$ for $X \in \mathcal{D}$.

In the case where A is a simple acycloid, the above theorem was proved by Tomizawa [20]. In simple acycloids, moreover, the set of atoms of $L(\mathcal{D})$ is given by $Min(\mathcal{D} - \{\mathbf{0}\}) = \{X \in \{-,0,+\}^E : |\underline{X}| = 1\}$ and $L(\mathcal{D})$ is also atomic [20].

For $X \in \{-,0,+\}^E$, we define $X^+ = \{e \in E : X_e = +\}$ and similarly define X^-. Also we define, for $\mathcal{X} \subseteq \{-,0,+\}^E$, $\mathcal{X}^+ = \{X \in \mathcal{X} : X^- = \emptyset\}$ and $\underline{\mathcal{X}} = \{\underline{X} : X \in \mathcal{X}\}$. An L^1-system A is *acyclic* if it has the positive tope $(+ + \cdots +)$, or equivalently it has no positive circuits.

Proposition 3.2. *Let A be an acyclic L^1-system with closed acyclons $\mathcal{D}$. Then the pair $(E, \underline{\mathcal{D}^+})$ is a convex geometry.*

Next, we show that every convex geometry is obtained this way. By the definition of convex geometries and by [6, Lemma 3.2], we have the following lemma.

Lemma 3.3. *Let $(E, \mathbb{G})$ be a convex geometry. Then the poset $L = (\mathbb{G}, \subseteq)$ is a lattice with the J-D chain property and the height function h satisfies $h(S) = |S|$ for all $S \in \mathbb{G}$.*

Proposition 3.4. *Let $(E, \mathbb{G})$ be a convex geometry and put $\mathcal{G} = \{X \in \{-,+\}^E : X^+ \in \mathbb{G}\}$. Then the pair $A = (E, \mathcal{G})$ is an acyclic L^1-system with $\mathbb{G} = \underline{\mathcal{D}^+}$, where $\mathcal{D}$ is the set of closed acyclons of A.*

Denote by $\mathcal{K}$ the set of all convex geometries, and by $\mathcal{K}_{L1}$ the set of convex geometries obtained from L^1-systems as in Proposition 3.2. As an immediate consequence of Propositions 3.2 and 3.4, we have

Theorem 3.5. $\mathcal{K}_{L1} = \mathcal{K}$.

For an acyclic oriented matroid M with circuits $\mathcal{C}$, Las Vergnas [15] defined the following closure, called the *convex hull* in M;

$$Conv_M(S) = S \cup \{e \in E - S : \exists X \in \mathcal{C} \text{ such that } X^- = \{e\} \text{ and } X^+ \subseteq S\} \ (S \subseteq E).$$

This closure is a generalization of the notion of convex hull in $\mathbf{R}^n$ and the closed sets are called the *convex sets* of M. Edelman showed in [7] that if M is simple, this closure is anti-exchange, and hence the convex sets of M forms a convex geometry. The following proposition shows that the related anti-exchange closure of the convex geometry obtained from an L^1-system is a natural extension of the convex hull $Conv_M$.

Proposition 3.6. *Let A be an acyclic L^1-system with closed acyclons $\mathcal{D}$ and circuits $\mathcal{C}$ and let $\underline{cl}$ be the anti-exchange closure associated with the convex geometry $(E, \underline{\mathcal{D}^+})$. Then, for all $S \subseteq E$, we have $\underline{cl}(S) = S \cup \{e \in E - S : \exists X \in \mathcal{C}$ such that $X^- = \{e\}$ and $X^+ \subseteq S\}$.*

What convex geometries arise from the convex sets of some acyclic simple oriented matroid [7, 8]? This open problem can be now restated as follows: characterize the set $\mathcal{K}_{om}$ of convex geometries obtained from oriented matroids as in Proposition 3.2.

Since the lattice $L(\mathcal{D})$ of closed acyclons of a simple acycloid is atomic, in particular, we know that if a convex geometry $(E, \mathbb{G})$ is an element of $\mathcal{K}_{om}$, then the lattice $(\mathbb{G}, \subseteq)$ is atomic. Hence, we know that the interval $[0, (++++)]$ of the lattice in Fig.2 belongs to $\mathcal{K}_{L1} - \mathcal{K}_{om}$ $(= \mathcal{K} - \mathcal{K}_{om})$. An atomic example belonging to $\mathcal{K} - \mathcal{K}_{om}$ is given in [6]. The above mentioned open problem is still open.

4 Median systems and median graphs

A graph G is *median* [19] if G is connected, and for any three vertices x, y, z there exists a unique vertex u such that u lies on a shortest (x,y)-path, a shortest (y,z)-path, and a shortest (z,x)-path. This vertex u is denoted by $m(x,y,z)$ and called the *median* of x, y and z. All trees, and all undirected Hasse diagrams of distributive lattices are median. Median systems, introduced in this section, axiomatize median graphs by signed vectors.

Before introducing median systems, we need a theorem by Mulder [18, Thm.1;Lemma 2]. We will present below a simple proof to it using Djoković's theorem, see [11] for the proof.

Theorem 4.1. (Mulder). *A graph G is median if and only if G is isometrically embeddable in some hypercube Q such that for any three vertices of G their median in Q is also a vertex of G.*

Note that every hypercube $Q(E)$ is median, and that the median of $X, Y, Z \in V(Q(E)) = \{-,+\}^E$ is the signed vector U such that, for all $e \in E, U_e = i$ if and only if at least two of X_e, Y_e and Z_e are i. We will denote this signed vector U by $< X, Y, Z >$.

A *median system* is a pair $A = (E, \mathcal{T})$ where E is a finite set and $\mathcal{T}$ is a nonempty set of elements of $\{-,+\}^E$, called the *topes* of A, satisfying

- (M1) $X, Y, Z \in \mathcal{T}$ implies $< X, Y, Z > \in \mathcal{T}$; and
- (M2) A is simple, i.e., for every $e \in E$, there exist $X, Y \in \mathcal{T}$ s.t. $X_e \neq Y_e$, and for every distinct $e, f \in E$, there exist $X, Y \in \mathcal{T}$ such that $X_e = X_f$ and $Y_e = -Y_f$.

We define for convenience that $A = (\emptyset, \{()\})$ is a median system. The pair $(E, \mathcal{T} = \{-,+\}^E)$ is the only acyclid (oriented matroid) on E which is a median system on E, see [11].

Let $A = (E, \mathcal{T})$ be a median system. Then for a subset $S \subseteq E$, the pair $A - S = (E - S, \{X \backslash S : X \in \mathcal{T}\})$ is also a median system, called the *deletion* of A *by* S. The *tope graph* G_A of A is a graph with $V(G_A) = \mathcal{T}$ and $E(G_A) = \{[X, Y] : X, Y \in \mathcal{T}$ and $|D(X, Y)| = 1\}$. An example of such a graph is given in Fig.3.

Proposition 4.2. *Every median system $A = (E, \mathcal{T})$ is an L^1 -system.*

Median systems are equivalent to median graphs by the following proposition.

Proposition 4.3. *A graph G is isomorphic to the tope graph of a median system if and only if G is median.*

Now let $A = (E, \mathcal{T})$ be an L^1 -system. Let $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$ be such that $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}$ and $\mathcal{T}_1 \cap \mathcal{T}_2 \neq \emptyset$, and such that for any $X \in \mathcal{T}_1 - \mathcal{T}_2$ there exists no $e \in E$ with ${}_{\bar{e}}X \in \mathcal{T}_2 - \mathcal{T}_1$. Let $p \notin E$, and put $\mathcal{T}' = \{X + p^+ : X \in \mathcal{T}_1\} \cup \{X + p^- : X \in \mathcal{T}_2\}$, where $X + p^i$ $(i \in \{-,+\})$ denotes the signed vector Z on $E \cup \{p\}$ with $Z_p = i$ and $Z_e = X_e$ for other element e. Then we call the pair $A' = (E \cup \{p\}, \mathcal{T}')$ the *expansion* of A with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. The expansion A' is called *convex* if there exist closed acyclons X^1 and X^2 of A such that $\mathcal{T}_1 = \mathcal{T}(X^1)$ and $\mathcal{T}_2 = \mathcal{T}(X^2)$.

Proposition 4.4. *A pair $A = (E, \mathcal{T})$ of a finite set E and $\emptyset \neq \mathcal{T} \subseteq \{-,+\}^E$ is a median system if and only if A can be obtained from the smallest median system $(\emptyset, \{()\})$ by a sequence of convex expansions.*

Let G be a graph. For $X, Y \subseteq V(G), [X, Y]$ denotes the set of edges with one endpoint in X and the other in Y. Now let $V_1, V_2 \subseteq V(G)$ satisfy $V_1 \cup V_2 = V(G), V_1 \cap V_2 \neq \emptyset$ and $[V_1 - V_2, V_2 - V_1] = \emptyset$. The *expansion* [17] of G with respect to V_1 and V_2 is the graph G' constructed as follows:

(i) replace each vertex $v \in V_1 \cap V_2$ by two vertices u_v, u'_v, which are joined by an edge;
(ii) join u_v to the neighbours of v in $V_1 - V_2$ and u'_v to those in $V_2 - V_1$;
(iii) if $v, w \in V_1 \cap V_2$ and $[v, w] \in E(G)$, then join u_v to u_w and u'_v to u'_w.

The expansion G' is called *convex* if V_1 and V_2 are convex subsets of $V(G)$.

Lemma 4.5. *In the tope graph G_A of an L^1-system $A = (E, \mathcal{T})$, a set $\mathcal{X}$ of vertices (topes) is convex if and only if $\mathcal{X} = \mathcal{T}(X)$ for some closed acyclon X of A.*

By this lemma and Proposition 4.4, we immediately obtain

Theorem 4.6. (Mulder [17]). *A graph G is median if and only if G can be obtained from a one-vertex graph K_1 by a sequence of convex expansions.*

Let $A = (E, \mathcal{T})$ be a pair with a finite set E and $\emptyset \neq \mathcal{T} \subseteq \{-,+\}^E$. For $X, Y \in \mathcal{T}$, we define $I_A(X,Y) = \{Z \in \mathcal{T} : D(X,Z) \subseteq D(X,Y)\}$, called the *interval* between X and Y. The index A of $I_A(X,Y)$ is often omitted when it is clear from the context. Note that if $< X, Y, Z >$ is an element of $\mathcal{T}$ then it is the unique element of $I_A(X,Y) \cap I_A(Y,Z) \cap I_A(Z,X)$.

Proposition 4.7. *An L^1-system $A = (E, \mathcal{T})$ is median if and only if A satisfies*

(M3) *if $e \in E$ and $X, Y \in \mathcal{T}$ satisfy $X_e = Y_e$ and ${}_{\bar{e}}X$, ${}_{\bar{e}}Y \in \mathcal{T}$, then $Z \in I_A(X,Y)$ implies ${}_{\bar{e}}Z \in \mathcal{T}$.*

The next theorem is similar to Djoković's theorem which characterizes the graphs isometrically embeddable in a hypercube.

Theorem 4.8. *A graph G is median if and only if G is connected bipartite, and $U(a,b)$ is convex for all $[a,b] \in E(G)$, where $U(a,b) = \{x \in C(a,b) : \exists y \in C(b,a)$ s.t. $[x,y] \in E(G)\}$.*

We proved Theorems 4.6 and 4.8 by using median systems. Such proofs are indeed shorter and easier to understand than direct proofs by the graph language or properties. In the following, we will show another example like this.

Lemma 4.9. *Let $A = (E, \mathcal{T})$ be an L^1 -system with faces $\mathcal{F}$ and closed acyclons $\mathcal{D}$. Then A is median if and only if $\mathcal{F} = \mathcal{D}$ holds.*

Now, we extend the definition of $< X, Y, Z >$, $X, Y, Z \in \{-,+\}^E$, to the case of any odd numbered elements of $\{-,+\}^E$. That is, for $X^1, X^2, ..., X^{2k+1} \in \{-,+\}^E$, $k \geq 0$, $< X^1, X^2, ..., X^{2k+1} >$ denotes the signed vector U such that, for all $e \in E$, $U_e = i$ if and only if at least $k+1$ of $X_e^1, X_e^2, ..., X_e^{2k+1}$ are i. Then we have

Proposition 4.10. *Let $A = (E, \mathcal{T})$ be a median system and let $X^1, X^2, ..., X^{2k+1} \in \mathcal{T}$. Then $< X^1, X^2, ..., X^{2k+1} > \in \mathcal{T}$.*

For vertices $v_1, v_2, ..., v_p$ of a connected graph G, any vertex m that minimize the sum $\sum_{i=1}^{p} d(x, v_i)$, $x \in V(G)$, is called a *median* of $v_1, v_2, ..., v_p$. Note that a median graph is a connected one in which any three vetices admit a unique median. It is clear that Proposition 4.10 proves the following result by Bandelt and Barthélemy [1]: a connected graph G is median if and only if each odd numbered family of vertices in G admits a unique median.

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References

[1] Bandelt H.-J. and Barthélemy J.P. Medians in median graphs. *Discrete Appl. Math.*, 8:131–142, 1984.

[2] Bienia W. and Cordovil R. An axiomatic of non-Radon partitions of oriented matroids. *Europ. J. Combin.*, 8:1–4, 1987.

[3] Björner A., Las Vergnas M., Sturmfels B., White N., and Ziegler G. M. Oriented matroids, in Combinatorial Geometries: Advanced Theory. to appear.

[4] Bland R. and Las Vergnas M. Orientability of matroids. *J.Combin. Theory, Ser. B*, 24:94–123, 1978.

[5] Djoković D. Ž. Distance-preserving subgraphs of hypercubes. *J.Combin. Theory, Ser. B*, 14:263–267, 1973.

[6] Edelman P.H. Meet-distributive lattices and the anti-exchange closure. *Algebra Universalis*, 10:290–299, 1980.

[7] Edelman P.H. The lattice of convex sets of an oriented matroid. *J.Combin. Theory, Ser. B*, 33:239–244, 1982.

[8] Edelman P.H. and Jamison R.E. The theory of convex geometries. *Geom. Dedicata*, 19:247–270, 1985.

[9] Fukuda K. *Oriented matroid programming.* PhD thesis, University of Waterloo, 1982.

[10] Fukuda K. and Handa K. Antipodal graphs and oriented matroids. *Discrete Math.*, to appear.

[11] Handa K. Topes of oriented matroids and related structures. submitted.

[12] Handa K. The faces and coboundaries of an acycloid. In *Topology and Computer Science*, pages 535–545, Tokyo, 1987. Kinokuniya Co., Ltd.

[13] Handa K. A characterization of oriented matroids in terms of topes. *Europ. J. Combin.*, 11:41–45, 1990.

[14] Harary F., Hayes J.P., and Wu H.-J. A survey of the theory of hypercube graphs. *Comput. Math. Applic.*, 15:277–289, 1988.

[15] Las Vergnas M. Convexity in oriented matroids. *J.Combin. Theory, Ser. B*, 29:231–243, 1980.

[16] Mandel A. *Topology of oriented matroids.* PhD thesis, University of Waterloo, 1982.

[17] Mulder H. M. The structure of median graphs. *Discrete Math.*, 24:197–204, 1978.

[18] Mulder H. M. n-cubes and median graphs. *J. Graph Theory*, 4:107–110, 1980.

[19] Mulder H. M. *The interval function of a graph*, volume 132 of *Math. Centre Tracts.* Mathematisch Centrum, Amsterdam, 1980.

[20] Tomizawa N. Theory of acycloid and holometry. *RIMS kokyuroku*, 534 (Graph Theory and Applications):91–138, February 1984. (in Japanese).

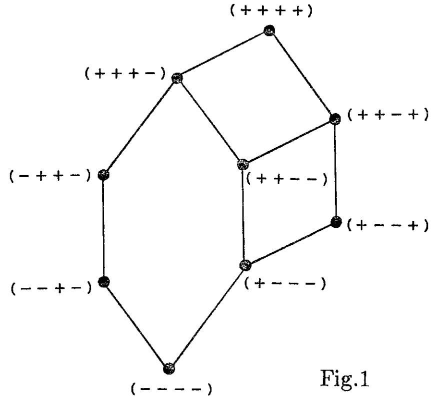

Fig.1

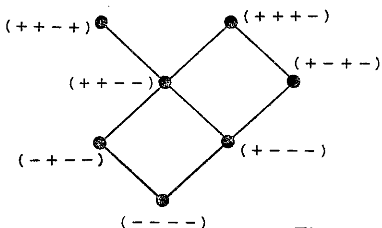

Fig.3

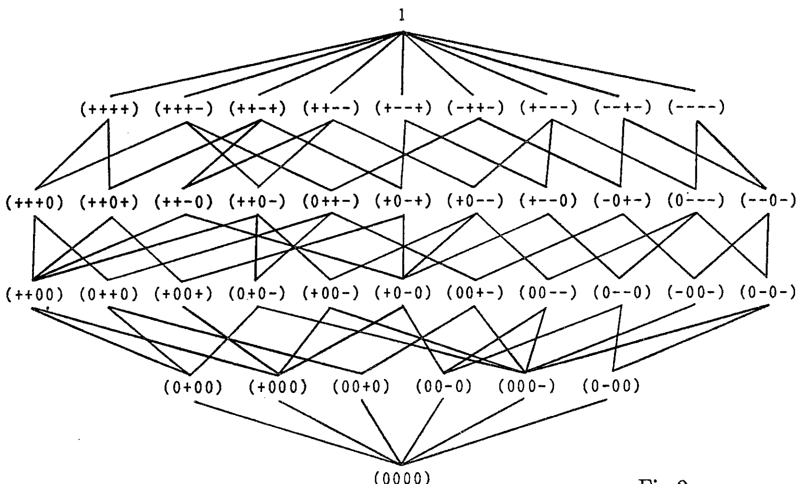

Fig.2