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パス幅が限られたグラフの族に対する普遍グラフ

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グラフの族 $\mathcal{F}$ に属すすべてのグラフを部分グラフとして含むグラフを $\mathcal{F}$ に対する普遍グラフという．$\mathcal{F}$ に対する枝数最小の普遍グラフは最小普遍グラフと呼ばれる．小文ではパス幅が高々 k かつ n 点上のグラフの族 $\mathcal{F}_n^k$ に対する最小普遍グラフについて考察する．まず，$\mathcal{F}_n^k$ に対する普遍グラフの枝数は少なくとも $\Omega\left(kn\log\frac{n}{k}\right)$ であることを示す．次に，$\mathcal{F}_n^k$ に対する枝数 $O\left(kn\log\frac{n}{k}\right)$ の普遍グラフを構成し，最小普遍グラフの枝数は $\Theta\left(kn\log\frac{n}{k}\right)$ であることを示す．

Universal Graphs for Graphs with Bounded Path-Width

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A graph G is said to be universal for a family $\mathcal{F}$ of graphs if G contains every graph in $\mathcal{F}$ as a subgraph. The minimum universal graph for $\mathcal{F}$ is a universal graph for $\mathcal{F}$ with the minimum number of edges. This paper considers the minimum universal graph for the family $\mathcal{F}_n^k$ of graphs on n vertices with path-width at most k. We first show that the number of edges in a universal graphs for $\mathcal{F}_n^k$ is at least $\Omega\left(kn\log\frac{n}{k}\right)$. Next, we construct a universal graph for $\mathcal{F}_n^k$ with $O\left(kn\log\frac{n}{k}\right)$ edges, and show that the number of edges in the minimum universal graph $\mathcal{F}_n^k$ is $\Theta\left(kn\log\frac{n}{k}\right)$.

1 Introduction

Given a family $\mathcal{F}$ of graphs, a graph G is said to be *universal* for $\mathcal{F}$ if G contains every graph in $\mathcal{F}$ as a subgraph. The *minimum* universal graph for $\mathcal{F}$ is a universal graph for $\mathcal{F}$ with the minimum number of edges. We denote the number of edges in a minimum universal graph for $\mathcal{F}$ by $f(\mathcal{F})$. Determining $f(\mathcal{F})$ has been known to have applications to the circuit design, data representation, and parallel computing [2, 3, 10, 12, 14]. Bhatt, Chung, Leighton, and Rosenberg showed a general upper bound for $f(\mathcal{F})$ for a family $\mathcal{F}$ of bounded-degree graphs by means of the size of separators [3]. For general families of (unbounded-degree) graphs, the following three results have been known:

(1) If $\mathcal{F}$ is the family of all planar graphs on n vertices, $f(\mathcal{F})$ is $\Omega(n \log n)$ and $O(n\sqrt{n})$ [1].

(2) If $\mathcal{F}$ is the family of all trees on n vertices, $f(\mathcal{F})$ is $\Theta(n \log n)$ [6].

(3) If $\mathcal{F}$ is the family of graphs on n vertices with proper-path-width at most 2, $f(\mathcal{F})$ is $\Theta(n \log n)$ [13].

Notice that a graph with proper-path-width at most 2 is a special kind of outerplanar graph. Notice also that $f(\mathcal{F})$ is $O(n^2)$ for any family $\mathcal{F}$ of graphs on n vertices, since K_n is trivially a universal graph for $\mathcal{F}$. This paper generalizes (3) to the family of graphs on n vertices with bounded path-width.

We consider finite undirected graphs without loops or multiple edges. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively.

Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a sequence of subsets of $V(G)$. The *width* of $\mathcal{X}$ is $\max_{1 \leq i \leq r} |X_i| - 1$. $\mathcal{X}$ is called a *path-decomposition* of G if the following conditions are satisfied: (i) For any distinct i and j, $X_i \not\subseteq X_j$; (ii) $\bigcup_{1 \leq i \leq r} X_i = V(G)$; (iii) For any edge $(u, v) \in E(G)$, there exists an i such that $u, v \in X_i$; (iv) For all a, b, and c with $1 \leq a \leq b \leq c \leq r$, $X_a \cap X_c \subseteq X_b$. The *path-width* of G, denoted by $pw(G)$, is the minimum width over all path-decompositions of G [11]. We denote the family of graphs on n vertices with path-width at most k ($k \geq 0$) by $\mathcal{F}_n^k$.

The purpose of this paper is to prove the following:

Theorem 1 *For any integer k ($k \geq 1$) and n ($n \geq 3k$), $f(\mathcal{F}_n^k)$ is $\Theta\left(kn \log \frac{n}{k}\right)$.*

We will prove this theorem by showing that $f(\mathcal{F}_n^k)$ is $\Omega\left(kn \log \frac{n}{k}\right)$ in Section 3, and $f(\mathcal{F}_n^k)$ is $O\left(kn \log \frac{n}{k}\right)$ in Section 4. Many related results can be found in the literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14].

2 Preliminaries

k-clique of a graph G is a complete subgraph of G on k vertices. For a positive integer k, *k-trees* are defined recursively as follows: (1) The complete graph on k vertices is a k-tree; (2) Given a k-tree Q on n vertices ($n \geq k$), a graph obtained from Q by adding a new vertex adjacent to the vertices of a k-clique of Q is a k-tree on $n+1$ vertices. A k-tree Q is called a *k-path* if $|V(Q)| \leq k+1$ or Q has exactly two vertices of degree k. *k-separator* S of a k-tree G is a k-clique of G such that $G \backslash S$ has at least two connected components where $G \backslash S$ is the graph obtained from G by deleting S. For a positive integer k, *k-intercats (interior k-caterpillars)* are defined as follows: (1) A k-path is a k-intercat; (2) Given a k-intercat Q on n vertices ($n \geq k+2$), a graph obtained from Q by adding a new vertex adjacent to the vertices of a k-separator of Q is also a k-intercat on $n+1$ vertices.

A 1-path, 1-intercat, and 1-tree are an ordinary path, caterpillar, and tree, respectively. A subgraph of a k-path, k-intercat, and k-tree is called a *partial k-path, partial k-intercat,* and *partial k-tree*, respectively.

k-intercat can also be defined recursively as follows: (1) The complete graph on k vertices is a k-intercat; (2) Given a k-intercat Q on n vertices ($n \geq k$), a graph obtained from Q by adding a new vertex adjacent to the vertices of a k-clique C of Q such that $Q \backslash C$ has at most one nontrivial connected component is also a k-intercat.

A path-decomposition with width k is called a k-path-decomposition. A k-path-decomposition $(X_1, X_2, \ldots, X_r)$ is said to be *full* if $|X_i| = k+1$ $(1 \leq i \leq r)$ and $|X_j \cap X_{j+1}| = k$ $(1 \leq j \leq r-1)$.

Lemma 1 *For any graph G with $pw(G) = k$, there exists a full k-path-decomposition of G.*

Proof: Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a k-path-decomposition of G such that $\sum_{i=1}^{r}(|X_i| - k)$ is maximum. We shall show that $\mathcal{X}$ is a full k-path-decomposition of G.

Assume that $|X_i| \leq k$ for some i $(2 \leq i \leq r)$. Let $v \in X_{i-1} - X_i$. The sequence $\mathcal{X}' = (X_1, X_2, \ldots, X_{i-1}, X_i \cup \{v\}, X_{i+1}, \ldots, X_r)$ satisfies conditions (ii), (iii), and (iv) in the definition of path-decomposition. Assume that $X_j \subseteq X_i \cup \{v\}$ for some $j (\neq i)$. Since $v \notin \bigcup_{i+1 \leq p \leq r} X_p$, $j < i$. Thus $j = i-1$ since $X_j = X_j \cap (X_i \cup \{v\}) \subseteq X_{i-1}$. Therefore, $(X_1, X_2, \ldots, X_{i-2}, X_i \cup \{v\}, X_{i+1}, \ldots, X_r)$ is a k-path-decomposition of G. But this is contradicting to the choice of $\mathcal{X}$ since $|X_{i-1}| \leq k$. Thus $\mathcal{X}'$ is a k-path-decomposition of G. But again this is contradicting to the choice of $\mathcal{X}$. Thus $|X_i| = k+1$ for any i $(2 \leq i \leq r)$. Since $(X_r, \ldots, X_1)$ is also a path-decomposition of G, $|X_i| = k+1$ for any i $(1 \leq i \leq r)$.

Assume next that $|X_i \cap X_{i+1}| \leq k-1$ for some i $(1 \leq i \leq r-1)$. Let $v \in X_i - X_{i+1}$ and $u \in X_{i+1} - X_i$. Since $v \notin \bigcup_{i+1 \leq j \leq r} X_j$ and $u \notin \bigcup_{1 \leq j \leq i} X_j$, the sequence $(X_1, \ldots, X_i, (X_{i+1} \cup \{v\}) - \{u\}, X_{i+1}, \ldots, X_r)$ is a k-path-decomposition of G contradicting the choice of $\mathcal{X}$. Thus $|X_i \cap X_{i+1}| = k$ for any i $(1 \leq i \leq r-1)$.

Thus, $\mathcal{X}$ is a full k-path-decomposition of G. □

Theorem 2 *For any graph G and an integer k $(k \geq 1)$, $pw(G) \leq k$ if and only if G is a partial k-intercat.*

Proof: Suppose that $pw(G) = h \leq k$. There exists a full h-path-decomposition $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of G by Lemma 1. If $r = 1$ then G is a subgraph of a complete graph on $h+1$ vertices, and so we conclude that G is a partial h-intercat. Thus we assume that $r \geq 2$. We construct a h-intercat H from $\mathcal{X}$ as follows:

(i) Let v_1 be a vertex in $X_1 \cap X_2$. Define that Q_1 is the complete graph on $X_1 - \{v_1\}$.

(ii) Define that Q_2 is the h-intercat obtained from Q_1 by adding v_1 and the edges connecting v_1 and the vertices in $X_1 - \{v_1\}$.

(iii) Given Q_i and the vertex $v_i \in X_i - X_{i-1}$ $(2 \leq i \leq r)$, define that Q_{i+1} is the h-intercat obtained from Q_i by adding v_i and the edges connecting v_i and the vertices in $X_i - \{v_i\}$.

(iv) Define $H = Q_{r+1}$.

From the definition of full h-path-decomposition, v_i $(2 \leq i \leq r)$ in (iii) is uniquely determined. It is easy to see that H is a h-intercat. Furthermore, we have $V(H) = V(G)$ and $E(H) \supseteq E(G)$ from the definitions of path-decomposition and Q_i. Thus G is a partial h-intercat, and so a partial k-intercat.

Conversely, suppose, without loss of generality, that G is a partial h-intercat $(h \leq k)$ with n $(n > h)$ vertices and H is a h-intercat such that $V(H) = V(G)$ and $E(H) \supseteq E(G)$. It is well-known that H can be obtained as follows:

(i) Define that $Q_1 = R_1$ is the complete graph with h vertices.

(ii) Given Q_i, R_i, and a new vertex v_i ($1 \leq i \leq n-h$), define that Q_{i+1} is the h-intercat obtained from Q_i by adding v_i and the edges connecting v_i and the vertices of R_i, and R_{i+1} is a h-clique of Q_{i+1} such that R_{i+1} contains v_i or $Q_{i+1} \backslash R_{i+1}$ has v_i as a connected component.

(iii) Define $H = Q_{n-h+1}$.

We define $X_i = V(R_i) \cup \{v_i\}$ ($1 \leq i \leq n-h$) and $\mathcal{X} = (X_1, X_2, \ldots, X_{n-h})$. It is easy to see that $|X_i| = h+1$ for any i, $\bigcup_{1 \leq i \leq n-h} X_i = V(H)$, and each vertex appears in consecutive X_i's. Thus $\mathcal{X}$ satisfies conditions (ii) and (iv) in the definition of path-decomposition, and the width of $\mathcal{X}$ is h. Since $v_i \in X_i - X_{i-1}$, and $\phi \neq V(R_{i-1}) - V(R_i) \subseteq X_{i-1} - X_i$ or $v_{i-1} = X_{i-1} - X_i$, $X_i \not\subseteq X_{i-1}$ and $X_{i-1} \not\subseteq X_i$ for any i. Thus $X_i \not\subseteq X_j$ for any distinct i and j, for otherwise $X_i = X_i \cap X_j \subseteq X_{i+1}$ ($i < j$) or $X_i = X_i \cap X_j \subseteq X_{i-1}$ ($i > j$). Hence $\mathcal{X}$ satisfies condition (i) in the definition of path-decomposition. Since each edge of H connects v_i and a vertex in $V(R_i)$ for some i or connects vertices in $V(R_1)$, both ends of each edge of H is contained in some X_i. Thus $\mathcal{X}$ satisfies condition (iii) in the definition of path-decomposition. Thus the sequence $\mathcal{X}$ is a full h-path-decomposition of H. Therefore, we have that $pw(G) \leq pw(H) \leq h \leq k$. □

3 Lower Bound

Let $d_G(v)$ be the degree of a vertex v in G. Let $D(G) = (\delta_G^1, \delta_G^2, \ldots, \delta_G^n)$ be the degree sequence for a graph G with n vertices, where $\delta_G^1 \geq \delta_G^2 \geq \cdots \geq \delta_G^n$. For graphs G and H with m and n vertices, respectively, we define $D(G) \geq D(H)$ if and only if $m \geq n$ and $\delta_G^i \geq \delta_H^i$ for any i ($1 \leq i \leq n$).

Lemma 2 *If a graph G is a universal graph for a family $\mathcal{F}$ of graphs, $D(G) \geq D(H)$ for any graph H in $\mathcal{F}$.*

Proof: For otherwise, G can not contain H as a subgraph. □

Lemma 3 *For any integer k ($k \geq 1$) and i ($1 \leq i \leq \left\lfloor \frac{n-2k}{k} \right\rfloor$), there exists a k-intercat $R(k,i)$ on n vertices such that $\delta_{R(k,i)}^{ki} \geq \left\lfloor \frac{n-2k}{i} \right\rfloor + k$.*

Proof: Let $r = \left\lfloor \frac{n-2k}{i} \right\rfloor$. $R(k,i)$ can be constructed as follows:

1. Define that $Q(k,k+1)$ is the complete graph on the vertices $V(Q(k,k+1)) = \{v_1, v_2, \ldots, v_{k+1}\}$.
2. Given $Q(k,j)$ ($k+1 \leq j \leq 2k-1$), define that $Q(k,j+1)$ is the k-intercat obtained from $Q(k,j)$ by adding a vertex v_{j+1} and k edges (v_{j+1}, v_{j-m}) ($0 \leq m \leq k-1$).
3. Given $Q(k,j)$ ($2k \leq j \leq (i-1)r + 2k - 1$), define that $Q(k,j+1)$ is the k-intercat obtained from $Q(k,j)$ by adding a vertex v_{j+1} and k edges $\left(v_{j+1}, v_{\left\lfloor \frac{j-2k}{r} \right\rfloor r+k+h}\right)$ where $h = m$ if $m \geq j - \left\{\left(\left\lfloor \frac{j-2k}{r} \right\rfloor + 1\right) r + k\right\}$, $h = r + m$ ($1 \leq m \leq k$) otherwise.
4. Given $Q(k,j)$ ($(i-1)r + 2k \leq j \leq n-1$), define that $Q(k,j+1)$ is the k-intercat obtained from $Q(k,j)$ by adding a vertex v_{j+1} and k edges $(v_{j+1}, v_{(i-1)r+k+m})$ ($1 \leq m \leq k$).
5. Define $R(k,i) = Q(k,n)$.

It is easy to see that $d_{R(k,i)}(v_{sr+k+m}) = r + k$ ($0 \leq s \leq i-2, 1 \leq m \leq k$), and $d_{R(k,i)}(v_{(i-1)r+k+m}) \geq r + k$ ($1 \leq m \leq k$). Thus we have $\delta_{R(k,i)}^{ki} \geq r + k$. □

Theorem 3 *For any integer k ($k \geq 1$) and n ($n \geq 3k$), $f(\mathcal{F}_n^k)$ is $\Omega\left(kn \log \frac{n}{k}\right)$.*

Proof: Let G be a universal graph for $\mathcal{F}_n^k$. By Lemmas 2, 3, and Theorem 2,

$$\begin{aligned}
2|E(G)| = \sum_{v \in V(G)} d_G(v) \geq \sum_{i=1}^{n} \delta_G^i > \sum_{i=1}^{\lfloor \frac{n-2k}{k} \rfloor k} \delta_G^i \geq k \sum_{i=1}^{\lfloor \frac{n-2k}{k} \rfloor} \delta_G^{ki} \\
&\geq k \sum_{i=1}^{\lfloor \frac{n-2k}{k} \rfloor} \left(\left\lfloor \frac{n-2k}{i} \right\rfloor + k \right) \\
&> k \sum_{i=1}^{\lfloor \frac{n-2k}{k} \rfloor} \left(\frac{n-2k}{i} + k - 1 \right) \\
&> k \left\{ (n-2k) \log_e \left(\left\lfloor \frac{n-2k}{k} \right\rfloor + 1 \right) + (k-1) \left\lfloor \frac{n-2k}{k} \right\rfloor \right\} \\
&> k \left\{ (n-2k) \log_e \left(\frac{n-2k}{k} \right) + (k-1) \left(\frac{n-2k}{k} - 1 \right) \right\} \\
&= k(n-2k) \log_e \left(\frac{n-2k}{k} \right) + (k-1)(n-3k).
\end{aligned}$$

Thus $|E(G)|$ is $\Omega\left(kn \log \frac{n}{k}\right)$. □

4 Upper Bound

We show an upper bound by constructing the graph G_n^k with n vertices and $O\left(kn \log \frac{n}{k}\right)$ edges, and proving that G_n^k is a universal graph for $\mathcal{F}_n^k$.

Let G_n^k ($k \geq 1, n \geq 1$) be the graph obtained by the following construction procedure:

1. Let $v_1, v_2, \ldots, v_n$ be the vertices of G_n^k.
2. Let $k^* = 2^{\lceil \log k \rceil}$. For any integer i with $1 \leq i \leq n$, let b_i be the maximum integer such that $2^{b_i} | i$. For every i ($1 \leq i \leq n$), join v_i by an edge to v_j such that $1 \leq j \leq n$ and $1 \leq |i - j| \leq 3k^* 2^{b_i} + k - 1$, if v_i is not adjacent to v_j.

Theorem 4 *For any integer k ($k \geq 1$) and n ($n \geq 1$), $|E(G_n^k)| = O\left(kn \log \frac{n}{k}\right)$.*

Proof: For any integer i with $1 \leq i \leq n$, let b_i be the maximum integer such that $2^{b_i} | i$. Note that $|\{i | b_i = h, 1 \leq i \leq n\}| = \left\lfloor \frac{n+2^h}{2^{h+1}} \right\rfloor$ and $|\{i | b_i \geq h, 1 \leq i \leq n\}| = \left\lfloor \frac{n}{2^h} \right\rfloor$ for any h ($h \geq 0$). Since $2\left(3k^* 2^{\log \frac{n}{6k^*}} + k - 1\right) > n$, the total number of edges added in Step 2 is at most

$$\begin{aligned}
&\sum_{h=0}^{\lfloor \log \frac{n}{6k^*} \rfloor} 2(3k^* 2^h + k - 1) \left\lfloor \frac{n + 2^h}{2^{h+1}} \right\rfloor + n \left\lfloor \frac{n}{2^{\lfloor \log \frac{n}{6k^*} \rfloor + 1}} \right\rfloor \\
&< \sum_{h=0}^{\lfloor \log \frac{n}{6k^*} \rfloor} (3k^* 2^h + k - 1) \left(\frac{n}{2^h} + 1 \right) + \frac{n^2}{2^{\log \frac{n}{6k^*}}} \\
&= \sum_{h=0}^{\lfloor \log \frac{n}{6k^*} \rfloor} \left\{ (3k^* n + k - 1) + \frac{(k-1)n}{2^h} + 3k^* 2^h \right\} + \frac{n^2}{\frac{n}{6k^*}} \\
&\leq (3k^* n + k - 1) \left(\log \frac{n}{6k^*} + 1 \right) + (2k - 1)(n - 3k^*) + 6k^* n
\end{aligned}$$

$$< \quad (6kn+k-1)\left(\log\frac{n}{6k}+1\right)+(2k-1)(n-3k)+12kn$$
$$= \quad (6kn+k-1)\log\frac{n}{6k}+(20k-1)n-(6k^2-4k+1).$$

Thus $|E(G_n^k)| < (6kn+k-1)\log\frac{n}{6k}+(20k-1)n-(6k^2-4k+1)$, and $|E(G_n^k)| = O\left(kn\log\frac{n}{k}\right)$. □

Theorem 5 *For any integer k ($k \geq 1$) and n ($n \geq 1$), G_n^k is a universal graph for $\mathcal{F}_n^k$.*

Proof: It is sufficient to show that any k-intercat is a subgraph of G_n^k by Theorem 2. Let R be a k-intercat in $\mathcal{F}_n^k$. We shall show that R is a subgraph of G_n^k. If $n \leq 8k-1$, R is a subgraph of G_n^k since G_n^k is the complete graph on n vertices. Thus we assume that $n \geq 8k$.

First of all, we give labels to the vertices of R as follows:

1. Let R' be a graph obtained from R by deleting all vertices of degree k in R, and $w_1 \in V(R)-V(R')$ be a vertex adjacent to w in R such that $d_{R'}(w)=k$. Let $w_2, w_3, \ldots, w_{k+1}$ be the vertices adjacent to w_1 in R. Give labels "1", "2", ..., "$k+1$" to $w_1, w_2, \ldots, w_{k+1}$, respectively. Set $i=k+2$.
2. Give the label "i" to the unlabeled vertex of R such that: (i) adjacent to the k labeled vertices; (ii) the degree in R is as small as possible subject to (i).
3. If $i=n$, halt. Otherwise, set $i=i+1$ and return to Step 2.

It should be noted that if the vertex given the label "i" in Step 2 is not uniquely determined, then degrees of these vertices in R are k. We denote the vertex with label "i" by u_i. Define $l_i = \max\{d|(u_i, u_{i+d}) \in E(R) \cup (u_i, u_i)\}$ for any i ($1 \leq i \leq n$). Let $l_i^* = 2^{\lceil \log l_i \rceil}$ if $l_i \geq 1$, otherwise, $l_i^* = 1$.

For the labeling above, we have the following three lemmas. Lemmas 4 and 5 are trivial, so we omit the proof.

Lemma 4 *If $(u_x, u_z) \in E(R)$ then $(u_x, u_y) \in E(R)$ for any distinct x, y, and z ($1 \leq x < y < z \leq n$).*

Lemma 5 *For any vertex u_i ($1 \leq i \leq n$), $|\{u_j|(u_i, u_j) \in E(R), j < i\}| = \min\{k, i-1\}$.*

Lemma 6 *For any vertex u_i ($1 \leq i \leq n-1$), $l_i = 0$ if and only if $|\{u_j|(u_{i+1}, u_j) \in E(R), j < i\}| = k$.*

Proof: For $1 \leq i \leq k$, $l_i > 0$ since $(u_{k+1}, u_i) \in E(R)$, and $|\{u_j|(u_{i+1}, u_j) \in E(R), j < i\}| = i-1 < k$ by Lemma 5. Thus assume that $k+1 \leq i \leq n-1$. Suppose that $|\{u_j|(u_{i+1}, u_j) \in E(R), j < i\}| = k$. By Lemma 5, $(u_{i+1}, u_i) \notin E(R)$. Thus $l_i = 0$ by Lemma 4. Conversely, suppose that $l_i = 0$ ($k+1 \leq i \leq n-1$). By the definition of l_i, $(u_{i+1}, u_i) \notin E(R)$. From Lemma 5, u_{i+1} has k edges connecting u_j such that $j < i$. □

Now we define mapping $\phi: V(R) \rightarrow V(G_n^k)$ as follows:

1. Let $k^* = 2^{\lceil \log k \rceil}$, $U = V(G_n^k)$, and $i = 1$.
2. Let $m_i^* = \left\lceil \frac{l_i^*}{2k^*} \right\rceil$. Let s_i be the minimum j such that $v_j \in U$ and $m_i^* | j$. Define that $\phi(u_i) = v_{s_i}$. Let $U = U - \{v_{s_i}\}$.
3. If $i = n$, halt. Otherwise, set $i = i+1$, and return Step 2.

Lemma 7 *ϕ is a $1-1$ mapping satisfying*

$$(*) \quad -k \le s_i - i \le \left\lceil \frac{l_i^*}{2} \right\rceil - 1$$

and

$$(\star) \quad s_i - i \le l_i - k - 1 \text{ if } m_i^* \ge 2$$

for any i where $\phi(u_i) = v_{s_i}$ $(1 \le i \le n)$.

Proof: We show the lemma by induction on i. Notice that $k \le k^* < 2k$, and $l_i \le l_i^* < 2l_i$ if $l_i \ge 1$.

Suppose that $\phi(u_j) = v_{s_j}$ $(1 \le j \le i-1, 1 \le i \le n-k-1)$ are determined by the algorithm in such a way that: conditions $(*)$ and $(\star)$ hold for any j $(1 \le j \le i-1)$, $v_j \notin U$ for any j $(1 \le j \le i-h-1)$, and $v_{i-h} \in U$ $(0 \le h \le k, h < i)$.

First, assume that $0 \le h \le k-1$. We show that the conditions $(*)$ and $(\star)$ hold for i. We have

$$-h \le s_i - i \le -h + (h+1)m_i^* - 1 \le (h+1)\left(\left\lceil \frac{l_i^*}{2k^*} \right\rceil - 1\right) < \frac{(h+1)l_i^*}{2k} \le \frac{l_i^*}{2} \le \left\lceil \frac{l_i^*}{2} \right\rceil .$$

If $m_i \ge 2$ then

$$s_i - i \le (h+1)\left(\frac{l^*}{2k^*} - 1\right) \le (h+1)\left(\frac{l_i - 1}{k} - 1\right) \le \frac{(h+1)(l_i - k - 1)}{k} \le l_i - k - 1.$$

It should be noted that $s_i < i + \left\lceil \frac{l_i^*}{2} \right\rceil \le i + l_i \le n$ if $l_i \ge 1$, $s_i < i + \left\lceil \frac{l_i^*}{2} \right\rceil \le i + 1 < n$ otherwise. Next, assume that $h = k$. We will show that $m_i^* = 1$ and $s_i - i = -k$. Since $v_{i-k} \in U$, $m_j^* \ge 2$ for any vertex u_j with $s_j \ge i-k+1$. Since $s_j - j \le l_j - k - 1$ for such u_j by the induction hypothesis, $(u_j, u_{s_j+k+1}) \in E(Q)$. Since $s_j + k + 1 > i + 1 > j$, $(u_j, u_{i+1}) \in E(Q)$ by Lemma 4. By the assumption that $v_{i-k} \notin U$, there are k vertices with $s_j \ge i-k+1$. Thus $m_i^* = 1$ by Lemma 6 and $s_i - i = -k$. In either case, induction hypothesis is satisfied.

Suppose that $\phi(u_j) = v_{s_j}$ $(1 \le j \le i = n-k-1)$ are determined by the algorithm in such a way that: conditions $(*)$ and $(\star)$ hold for any j $(1 \le j \le i)$, $v_j \notin U$ for any j $(1 \le j \le i-h-1)$, and $v_{i-h} \in U$ $(0 \le h \le k)$. Since $l_j \le n-j \le k$ for $j \ge n-k$, $m_j^* = 1$. We have $-k \le s_j - j \le 0$ for $n-k \le j \le n$.

Thus ϕ is $1-1$ mapping satisfying $(*)$ and $(\star)$ for any i. □

Lemma 8 *If $\phi(u_i) = v_{s_i}$, then $(v_{s_i}, v_j) \in E(G_n^k)$ for any v_j such that $1 \le j \le n$ and $1 \le |s_i - j| \le \left\lceil \frac{3}{2} l_i^* \right\rceil + k - 1$.*

Proof: Since $\left\lceil \frac{l_i^*}{2k^*} \right\rceil | s_i$, $(v_{s_i}, v_j) \in E(G_n^k)$ for any v_j such that $1 \le j \le n$ and $|s_i - j| \le 3k^* \left\lceil \frac{l_i^*}{2k^*} \right\rceil + k - 1$. If $l_i^* \ge 2k^*$ then $3k^* \left\lceil \frac{l_i^*}{2k^*} \right\rceil + k - 1 = \frac{3l_i^*}{2} + k - 1 = \left\lceil \frac{3l_i^*}{2} \right\rceil + k - 1$. If $l_i^* < 2k^*$ then $3k^* \left\lceil \frac{l_i^*}{2k^*} \right\rceil + k - 1 = 3k^* + k - 1 \ge \left\lceil \frac{3l_i^*}{2} \right\rceil + k - 1$. □

Lemma 9 *If $(u_i, u_j) \in E(R)$ then $(\phi(u_i), \phi(u_j)) \in E(G_n^k)$.*

Proof: Without loss of generality, we assume that $i < j$, $\phi(u_i) = v_{s_i}$, and $\phi(u_j) = v_{s_j}$. Notice that $1 \le j - i \le l_i \le l_i^*$. From Lemma 7, we have $-k \le s_i - i \le \left\lceil \frac{l_i^*}{2} \right\rceil - 1$ and $-k \le s_j - j \le \left\lceil \frac{l_j^*}{2} \right\rceil - 1$. Thus $-(\left\lceil \frac{l_i^*}{2} \right\rceil + k - 2) \le s_j - s_i \le l_i^* + \left\lceil \frac{l_j^*}{2} \right\rceil + k - 1$.

If $l_j^* > l_i^*$ then $|s_j - s_i| \leq l_i^* + \left\lceil \frac{l_j^*}{2} \right\rceil + k - 1 < l_j^* + \left\lceil \frac{l_j^*}{2} \right\rceil + k - 1 = \left\lceil \frac{3}{2} l_j^* \right\rceil + k - 1$. From Lemma 8, we have $(v_{s_i}, v_{s_j}) \in E(G_n^k)$. If $l_j^* \leq l_i^*$ then $|s_j - s_i| \leq l_i^* + \left\lceil \frac{l_j^*}{2} \right\rceil + k - 1 \leq l_i^* + \left\lceil \frac{l_i^*}{2} \right\rceil + k - 1 = \left\lceil \frac{3}{2} l_i^* \right\rceil + k - 1$. From Lemma 8, we have $(v_{s_i}, v_{s_j}) \in E(G_n^k)$. □

By Lemma 9, we conclude that R is a subgraph of G_n^k. This completes the proof of Theorem 5. □

Theorem 1 follows from Theorems 2, 3, and 4.
Minimum universal graphs for k-trees ($k \geq 2$) are open.

References

[1] L. Babai, F. R. K. Chung, P. Erdös, R. L. Graham, and J. Spencer, On graphs which contain all sparse graphs, *Ann. Discrete Math.*, 12, pp. 21–26, 1982.

[2] S. N. Bhatt, F. R. K. Chung, F. T. Leighton, and A. L. Rosenberg, Optimal sumulations of tree machines, in *Proceedings of 27th FOCS*, pp. 274–282, 1986.

[3] S. N. Bhatt, F. R. K. Chung, F. T. Leighton, and A. L. Rosenberg, Universal graphs for bounded-degree trees and planar graphs, *SIAM J. Disc. Math.*, 2, pp. 145–155, 1989.

[4] F. R. K. Chung, Universal graphs and induced-universal graphs, *J. Graph Theory*, 14, pp. 443–454, 1990.

[5] F. R. K. Chung and R. L. Graham, On graphs which contain all small trees, *J. Combin. Theory*, Series B(24), pp. 14–23, 1978.

[6] F. R. K. Chung and R. L. Graham, On universal graphs for spanning trees, *J. London Math. Soc.*, 27, pp. 203–211, 1983.

[7] F. R. K. Chung, R. L. Graham, and D. Coppersmith, On trees which contain all small trees, in *The Theory and Applications of Graphs*, pp. 265–272, John Wiley & Sons, 1981.

[8] F. R. K. Chung, R. L. Graham, and N. Pippenger, On graphs which contain all small trees II, in *Proc. 1976 Hungarian Colloquium on Combinatorics*, pp. 213–223, 1978.

[9] F. R. K. Chung, R. L. Graham, and J. Shearer, Universal caterpillars, *J. Combin. Theory*, Series B(31), pp. 348–355, 1981.

[10] F. R. K. Chung, A. L. Rosenberg, and L. Snyder, Perfect storage representations for families of data structures, *SIAM J. Algebraic Disc. Methods*, 4, pp. 548–565, 1983.

[11] N. Robertson and P. D. Seymour, Graph minors. I. Excluding a forest, *Journal of Combinatorial Theory*, Series B(35), pp. 39–61, 1983.

[12] A. R. Rosenberg, L. J. Stockmeyer, and L. Snyder, Uniform data encodings, *Theoret. Comput. Sci.*, 2, pp. 145–165, 1980.

[13] S. Ueno, T. Kitakaku, and Y. Kajitani, Universal graphs for 2-paths, in *Proc. 1991 IEEE ISCAS*, pp. 1147–1151, 1991.

[14] L. G. Valliant, Universality considerations in VLSI circuits, *IEEE Trans. Comput.*, C-30, pp. 135–140, 1981.