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コオペレーティング3方向2次元有限オートマタシステム

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あらまし 本稿では、一種の制限されたコオペレーティング（4方向）2次元有限オートマタシステム（CS-2-FA）、即ち、コオペレーティング3方向2次元有限オートマタシステム（CS-TR2-FA）を導入し、主としてそのパターン認識機械としてのいくつかの性質を考察する。入力テープが正方形に限定される場合に対して、次の結果が成り立つことを示す。

(1) CS-TR2-FA は3方向2次元シンプルマルチヘッド有限オートマタと同等な受理能力を持つ。

(2) CS-2-FA は CS-TR2-FA より受理能力が真に強い。

(3) $\mathcal{L}[\text{CS-TR2-DFA}(k)^s] \subsetneq \mathcal{L}[\text{CS-TR2-NFA}(k)^s]$ 。

(4) $\cup_{1\le k<\infty}\mathcal{L}[\text{CS-TR2-DFA}(k)^s] \subsetneq \cup_{1\le k<\infty}\mathcal{L}[\text{CS-TR2-NFA}(k)^s]$ 。

(5) $\mathcal{L}[\text{CS-TR2-DFA}(k)^s]$ （$\mathcal{L}[\text{CS-TR2-NFA}(k)^s]$）$\subsetneq \mathcal{L}[\text{CS-TR2-DFA}(k+1)^s]$ （$\mathcal{L}[\text{CS-TR2-NFA}(k+1)^s]$）。

ここで、$\mathcal{L}[\text{CS-TR2-DFA}(k)^s]$ （$\mathcal{L}[\text{CS-TR2-NFA}(k)^s]$）は、k個の3方向2次元決定性（非決定性）有限オートマトンから構成される CS-TR2-FA によって受理される正方形パターン集合の族を表す。

和文キーワード 3方向2次元有限オートマトン、コオペレーティングシステム、計算複雑さ

Cooperating Systems of Three-Way Two-Dimensional Finite Automata

by

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Abstract This paper introduces a *cooperating system of three-way two-dimensional finite automata* (CS-TR2-FA) which is a restricted version of a *cooperating system of (four-way) two-dimensional finite automata* (CS-2-FA), and mainly investigates several fundamental properties of this system as a two-dimensional language acceptor whose input tapes are restricted to square ones. We show that

(1) CS-TR2-FA's are equivalent in accepting power to three-way two-dimensional simple multihead finite automata;

(2) CS-2-FA's are more powerful than CS-TR2-FA's;

(3) $\mathcal{L}[\text{CS-TR2-DFA}(k)^s] \subsetneq \mathcal{L}[\text{CS-TR2-NFA}(k)^s]$;

(4) $\cup_{1\le k<\infty}\mathcal{L}[\text{CS-TR2-DFA}(k)^s] \subsetneq \cup_{1\le k<\infty}\mathcal{L}[\text{CS-TR2-NFA}(k)^s]$; and

(5) $\mathcal{L}[\text{CS-TR2-DFA}(k)^s]$ ($\mathcal{L}[\text{CS-TR2-NFA}(k)^s]$) $\subsetneq \mathcal{L}[\text{CS-TR2-DFA}(k+1)^s]$ ($\mathcal{L}[\text{CS-TR2-NFA}(k+1)^s]$),

where $\mathcal{L}[\text{CS-TR2-DFA}(k)^s]$ ($\mathcal{L}[\text{CS-TR2-NFA}(k)^s]$) denotes the class of sets of square input tapes accepted by CS-TR2-FA's which consist of k deterministic (nondeterminstic) finite automata.

英文 key words three-way two-dimensional finite automaton, cooperating system, computational complexity

1. INTRODUCTION

A cooperating system of two-dimensional finite automata (CS-2-FA) [2,3,4] consists of a finite number of two-dimensional finite automata and a two-dimensional input tape where these finite automata work independently (in parallel). Those finite automata whose input heads scan the same cell of the input tape can communicate with each other, that is, every finite automaton is allowed to know the internal states of other finite automata on the same cell it is scanning at the moment.

In [2, 3, 4], the maze and labyrinth search problems for CS-2-FA's were studied. But there is no investigation of CS-2-FA's as the recognizers (or acceptors) of two-dimensional patterns. It is worthwhile to investigate properties of CS-2-FA's as recognizers, because CS-2-FA's may be considered as one of the simplest models of parallel two-dimensional pattern recognizers. Recently, several properties of cooperating systems of one-way finite automata and cooperating systems of one-way counter machines as one-dimensional language acceptors were investigated in [8] and [10], respectively.

In this paper, we propose a cooperating system of three-way two-dimensional finite automata (CS-TR2-FA) which is a restricted version of CS-2-FA's, and mainly investigate its several properties as two-dimensional language acceptors. The three-way two-dimensional finite automaton [7] is a two-dimensional finite automaton [1] whose input head can move left, right, or down, but not up.

The paper has seven sections in addition to this Introduction. Section 2 contains some definitions and notations. Section 3 investigates a relationship between three-way two-dimensional simple multihead finite automata (TR2-SPMHFA's) and CS-TR2-FA's. It is shown that TR2-SPMHFA's and CS-TR2-FA's are equivalent in accepting power if the input tapes are restricted to square ones. Section 4 investigates the difference between the accepting powers of CS-TR2-FA's and CS-2-FA's, and shows that CS-TR2-FA's are less powerful than CS-2-FA's. Section 5 investigates the difference between the accepting powers of deterministic and nondeterministic CS-TR2-FA's, and shows that deterministic CS-TR2-FA's are less powerful than nondeterministic CS-TR2-FA's. Section 6 shows that for classes of sets accepted by CS-TR2-FA's and CS-2-FA's, hierarchies can be obtained by varying the number of finite automata in the system. Section 7 concludes by giving some open problems. In this paper only square input tapes are considered.

2. PRELIMINARIES

Definition 2.1. Let Σ be a finite set of symbols. A *two-dimensional tape* over Σ is a two-dimensional rectangular array of elements of Σ.

The set of all two-dimensional tapes over Σ is denoted by $\Sigma^{(2)}$. Given a tape $x \in \Sigma^{(2)}$, we let $l_1(x)$ be the number of rows of x and $l_2(x)$ be the number of columns of x. If $1 \le i \le l_1(x)$ and $1 \le j \le l_2(x)$, we let $x(i,j)$ denote the symbol in x with coordinates (i,j). Furthermore, we define

$$x[(i,j),(i',j')],$$

only when $1 \le i \le i' \le l_1(x)$ and $1 \le j \le j' \le l_2(x)$, as the two-dimensional tape z satisfying the following:

(i) $l_1(z) = i' - i + 1$ and $l_2(z) = j' - j + 1$.

(ii) for each k, r, $[1 \le k \le l_1(z),\ 1 \le r \le l_2(z)]$,

$$z(k,r) = x(k+i-1, r+j-1).$$

We recall a *three-way two-dimensional simple k-head finite automaton* (TR2-SPk-HFA) [5,6]. A TR2-SPk-HFA M is a finite automaton with k read-only input heads operating on a two-dimensional input tape surrounded by boundary symbols #. The only one head (called the *'reading'* head) of M is capable of distinguishing the symbols in the input alphabet, and the other heads (called *'counting'* heads) of M can only detect whether they are on the boundary symbols or a symbol in the input alphabet. When an input tape x is presented to M, M determines the next state of the finite control, the next move dirction (right, left, down, or no move) of each input head, depending on the present state of the finite control, the symbol read by the reading head, and on whether or not the symbol read by each counting head is boundary symbol. We say that M accepts x if M, when started in its initial state with all its input heads on $x(1,1)$, eventually halts in an accepting state with all its heads on the bottom boundary symbols of x. As usual, we define nondeterministic and deterministic TR2-SPk-HFA's.

A *three-way two-dimensional sensing simple k-head finite automaton* (TR2-SNSPk-HFA) is the same device as a TR2-SPk-HFA except that the former can detect coincidence of the input heads.

We denote a deterministic (nondeterministic) TR2-SPk-HFA by TR2-SPk-HDFA (TR2-SPk-HNFA), and denote a deterministic (nondeterministic) TR2-SNSPk-HFA by TR2-SNSPk-HDFA (TR2-SNSPk-HNFA).

We now give a formal definition of a *cooperating system of k two-dimensional deterministic finite automata* (CS-2-DFA(k)) as an acceptor.

Definition 2.2. A CS-2-DFA(k) is a k-tuple $M = (\mathrm{FA}_1, \mathrm{FA}_2, \cdots, \mathrm{FA}_k)$, $k \ge 1$, such that for each $1 \le i \le k$,

$$\mathrm{FA}_i = (\Sigma,\ Q_i,\ X_i,\ \delta_i,\ q_{0i},\ F_i,\ \phi,\ \#),$$

where

1. Σ is a finite set of *input symbols*.

2. Q_i is a finite set of *states*.

3. $X_i = (Q_1 \cup \{\phi\}) \times \cdots \times (Q_{i-1} \cup \{\phi\}) \times (Q_{i+1} \cup \{\phi\}) \times \cdots \times (Q_k \cup \{\phi\})$, where '$\phi$' is a special state not in $(Q_1 \cup Q_2 \cup \cdots \cup Q_k)$.

4. $\delta_i : (\Sigma \cup \{\#\}) \times X_i \times Q_i \rightarrow Q_i \times \{\text{right } (= (0,+1)), \text{left } (= (0,-1)), \text{down } (= (+1,0)), \text{up } (= (-1,0)), \text{no move } (= (0,0))\}$ is the *next move function*, where '#' is the *boundary* symbol not in Σ.

5. $q_{0i} \in Q_i$ is the *initial* state of FA_i.

6. $F_i \subseteq Q_i$ is the set of *accepting* states of FA_i.

Every automaton of M independently (in parallel) works step by step on the same two-dimensional tape x over Σ surrounded by boundary symbols #. Each step is assumed to require exactly one time for its completion. For each i $(1 \le i \le k)$, let q_i be the state of FA_i at time 't'. Then each FA_i enters the next state 'p_i' at time '$t+1$' according to the function

$$\delta_i(x(\alpha,\beta), \langle q'_1, \cdots, q'_{i-1}, q'_{i+1}, \cdots, q'_k \rangle, q_i) = (p_i, (d_1, d_2)),$$

where $x(\alpha,\beta)$ is the symbol read by the input head of FA_i at time 't' and for each $j \in \{1, \cdots, i-1, i+1, \cdots, k\}$,

$$q'_j = \begin{cases} q_j \in Q_j & \text{if the input heads of } \mathrm{FA}_i \text{ and } \mathrm{FA}_j \text{ are on the same input position at the moment '}t\text{';} \\ \phi & \text{otherwise,} \end{cases}$$

and moves ist input head to $x(\alpha + d_1, \beta + d_2)$ at time '$t+1$'. We assume that the input head of FA_i never falls off the tape beyond boundary symbols.

When an input tape $x \in \Sigma^{(2)}$ is presented to M, we say that M *accepts* the tape x if each automaton of M, when started in its initial state with its input head on $x(1,1)$, eventually enters an accepting state with its input head on one of the bottom boundary symbols.

We next introduce a *cooperating system of k three-way two-dimensional deterministic finite automata* (CS-TR2-DFA(k)), with which we are mainly concerned in this paper.

Definition 2.3. A CS-TR2-DFA(k) is a CS-2-DFA(k) $M = (\mathrm{FA}_1, \mathrm{FA}_2, \cdots, \mathrm{FA}_k)$ such that the input head of each FA_i can only move left, right, or down, but not up.

To give the formal definitions of a *cooperating system of k two-dimensional nondeterministic finite automata* (CS-2-NFA(k)) and a *cooperating system of k three-way two-dimensional nondeterministic finite automata* (CS-TR2-NFA(k)) is left to the reader.

For each $X \in$ {TR2-SPk-HDFA, TR2-SPk-HNFA, TR2-SNSPk-HDFA, TR2-SNSPk-HNFA, CS-2-DFA(k), CS-2-NFA(k), CS-TR2-DFA(k), CS-TR2-NFA(k)}, by X^s we denote an X whose input tapes are restricted to square ones; by $\mathcal{L}[X]$ ($\mathcal{L}[X^s]$) we denote the class of sets of input tapes accepted by X's (X^s's). We will focuse our attention on the acceptors whose input tapes are restricted to square ones.

3. RELATIONSHIP BETWEEN TR2-SPMHFA'S AND CS-TR2-FA'S

In this section, we establish a relation between the accepting powers of three-way two-dimensional simple multihead finite automata and cooperating systems of three-way two-dimensional finite automata over square input tapes. This result will be used in the latter sections.

Lemma 3.1. For any $k \ge 1$ and any X$\in${N, D},

$$\mathcal{L}[\text{TR2-SNSP}k\text{-HXFA}^s] \subseteq \mathcal{L}[\text{CS-TR2-XFA}(2k)^s].$$

Proof. Let M be a TR2-SNSPk-HXFA^s. We will construct a CS-TR2-XFA$(2k)^s$ M' to simulate M. M' acts as follows:

1. M' simulates the moves of the reading head of M and all the left or right moves of counting heads of M by using its $(k+1)$ finite automata.

2. M' simulates all the down moves of counting heads of M by making the right moves of input heads of its other $(k-1)$ finite automata.

3. During the simulation, if M moves its reading head down, then M' makes all of input heads of finite automata of M' move down so that all the automata of M' can keep their input heads on the same row and can communicate with each other in that row.

It is easy to see that M' can simulate M. □

Lemma 3.2. For any $k \geq 1$ and any X∈{N, D},

$$\mathcal{L}[\text{CS-TR2-XFA}(k)^s] \subseteq \mathcal{L}[\text{TR2-SNSP}(2k^2-k+1)\text{-HXFA}^s].$$

Proof. Let $M = (\text{FA}_1, \text{FA}_2, \cdots, \text{FA}_k)$ be a CS-TR2-XFA$(k)^s$. We will construct a TR2-SNSP$(2k^2-k+1)$-HXFA^s M' to simulate M. Let R denote the reading head of M', and $h_1, h_2, \cdots, h_{2k^2-k}$ denote the $2k^2-k$ counting heads of M'. M' acts as follows:

1. M' stores the internal states of $\text{FA}_1, \text{FA}_2, \cdots, \text{FA}_k$ in its finite control.
2. For each row of the input tape:
   (a) M' simulates the left or right moves of input heads of $\text{FA}_1, \text{FA}_2, \cdots, \text{FA}_k$ by using R and $h_1, h_2, \cdots, h_k$.
   (b) M' stores in its finite control the internal state of each FA_i, $1 \leq i \leq k$, when the input head of FA_i leaves the row and the order, $\langle d_1, d_2, \cdots, d_k \rangle$, in which the input heads of $\text{FA}_1, \text{FA}_2, \cdots, \text{FA}_k$ leave the row subsequently (i.e., FA_{d_1} firstly moves its input head down from the row, FA_{d_2} secondly moves its input head down from the row, and so on.),[1] and M' keeps the position where the input head of each FA_i, $1 \leq i \leq k$, leaves the row by the positions of $h_1, h_2, \cdots, h_k$.
   (c) Furthermore, for each i $(1 \leq i \leq k-1)$, the interval between the times at which FA_{d_i} and $\text{FA}_{d_{i+1}}$ move their input heads down from the row is stored by a counter with $O(n^{2k})$ space bound, which can be realized by using $h_{(2i-1)k+1}, h_{(2i-1)k+2}, \cdots, h_{(2i+1)k}$, where n is the number of rows (or columns) of the input tape.

Note that M works in $O(n^{2k})$ time, that is, if an input tape with n rows (or columns) is accepted by M, then it can be accepted by M in $O(n^{2k})$ time. Thus, it is easy to verify that M' can simulate M. □

It was shown in [5] that $\cup_{1 \leq k < \infty} \mathcal{L}[\text{TR2-SP}k\text{-HXFA}^s] = \cup_{1 \leq k < \infty} \mathcal{L}[\text{TR2-SNSP}k\text{-HXFA}^s]$ for any X∈ {N, D}. Combining this result with Lemmas 3.1 and 3.2, we have the following theorem.

Theorem 3.1. $\cup_{1 \leq k < \infty} \mathcal{L}[\text{TR2-SP}k\text{-HXFA}^s] = \cup_{1 \leq k < \infty} \mathcal{L}[\text{CS-TR2-XFA}(k)^s]$ for any X∈{N, D}.

Corollary 3.1. For any $k \geq 1$, there is no CS-TR2-NFA(k) that accepts the set of connected patterns.[2]

Proof. It is shown in [6] that the set of connected patterns is not in $\cup_{1 \leq k < \infty} \mathcal{L}[\text{TR2-SP}k\text{-HNFA}^s]$. From this result and Theorem 3.1, the corollary follows. □

Remark 3.1. It is easy to see that for each $k \geq 1$, (1) (four-way) two-dimensional sensing simple k head finite automata [5] are simulated by cooperating systems of $(k+1)$ (four-way) two-dimensional finite automata, and (2) cooperating systems of k (four-way) two-dimensional finite automata are simulated by (four-way) two-dimensional sensing simple $(k+1)$ head finite automata.

Remark 3.2. It is shown in [8] that (one-dimensional) one-way simple multihead finite automata and cooperating systems of (one-dimensional) one-way deterministic finite automata are incomparable in accepting power. From this fact, it follows that TR2-SPMHFA's and CS-TR2-DFA's are incomparable in accepting power if the input tapes are restricted to those x such that $l_1(x) > l_2(x)$. We can also show that TR2-SPMHFA's are more powerful than CS-TR2-DFA's if the input tapes are restricted to those x such that $l_1(x) < l_2(x)$.

4. THREE-WAY VERSUS FOUR-WAY

In this section, we investigate the difference between the accepting powers of CS-2-DFA$(k)^s$'s [CS-2-NFA$(k)^s$'s] and CS-TR2-DFA$(k)^s$'s [CS-TR2-NFA$(k)^s$'s].

Theorem 4.1. For each X∈{N, D}, $\mathcal{L}[\text{CS-2-DFA}(1)^s] - \cup_{1 \leq k < \infty} \mathcal{L}[\text{CS-TR2-XFA}(k)^s] \neq \emptyset$.

Proof. Let $T_1 = \{x \in \{0, 1\}^{(2)} \mid (\exists m \geq 2)\ [l_1(x) = l_2(x) = m \ \&\ x[(1,1),(1,m)] = x[(2,1),(2,m)]]\}$. Clearly, $T_1 \in \mathcal{L}$[CS-2-DFA$(1)^s$]. As shown in [5], T_1 is not in $\cup_{1 \leq k < \infty} \mathcal{L}[\text{TR2-SP}k\text{-HNFA}^s]$. From this fact and Theorem 3.1, the theorem follows. □

From Theorem 4.1, we can get the following corollary.

[1] If the input heads of $\text{FA}_{i_1}, \text{FA}_{i_2}, \cdots, \text{FA}_{i_r}$ $(1 \leq i_1 < i_2 < \cdots < i_r \leq k)$ leave the row simultaneously, we refer to the order on them as $\langle i_1, i_2, \cdots, i_r \rangle$.

[2] The definition of connected patterns can be found in [1].

Corollary 4.1. For each $k \geq 1$ and each X$\in$\{N, D\}, (1) $\mathcal{L}$[CS-TR2-XFA$(k)^s$] $\subsetneq$ $\mathcal{L}$[CS-2-XFA$(k)^s$], and (2) $\cup_{1 \leq k < \infty}\mathcal{L}$[CS-TR2-XFA$(k)^s$] $\subsetneq$ $\cup_{1 \leq k < \infty}\mathcal{L}$[CS-2-XFA$(k)^s$].

5. NONDETERMINISM VERSUS DETERMINISM

In this section, we investigate the difference between the accepting powers of CS-TR2-NFA$(k)^s$'s and CS-TR2-DFA$(k)^s$'s.

Theorem 5.1. $\mathcal{L}$[CS-TR2-NFA$(1)^s$] $-$ $\cup_{1 \leq k < \infty}\mathcal{L}$[CS-TR2-DFA$(k)^s$] $\neq \emptyset$.

Proof. Let $T_2 = \{x \in \{0,\ 1\}^{(2)} \mid (\exists m \geq 2)\ [l_1(x) = l_2(x) = m]\ \&\ \exists i(1 \leq i \leq m)\ [x(1,i) = x(2,i) = 1]\}$. Clearly, $T_2 \in \mathcal{L}$[CS-TR2-NFA$(1)^s$]. As shown in [5], T_2 is not in $\cup_{1 \leq k < \infty}\mathcal{L}$[TR2-SP$k$-HDFA^s]. From this fact and Theorem 3.1, the theorem follows. □

From Theorem 5.1, we can get the following corollary.

Corollary 5.1. For each $k \geq 1$, (1) $\mathcal{L}$[CS-TR2-DFA$(k)^s$] $\subsetneq$ $\mathcal{L}$[CS-TR2-NFA$(k)^s$], and (2) $\cup_{1 \leq k < \infty}\mathcal{L}$[CS-TR2-DFA$(k)^s$] $\subsetneq$ $\cup_{1 \leq k < \infty}\mathcal{L}$[CS-TR2-NFA$(k)^s$].

6. HIERARCHIES BASED ON THE NUMBER OF AUTOMATA

6.1. Four-Way Case

We first investigate how the number of automata of CS-2-FA^s's affects the accepting power.

Theorem 6.1.1. For each $k \geq 1$ and each X$\in$ \{N, D\}, $\mathcal{L}$[CS-2-XFA$_{\{0\}}(k)^s$]$\subsetneq$ $\mathcal{L}$[CS-2-XFA$_{\{0\}}(k+2)^s$], where $\mathcal{L}$[CS-2-XFA$_{\{0\}}(k)^s$] denote the class of sets of square tapes over a one-letter alphabet accepted by CS-2-XFA(k)'s.

Proof. It is easy to prove that every CS-2-DFA(k) [CS-2-NFA(k)] can be simulated by a (four-way) two-dimensional sensing deterministic [nondeterministic] k-head finite automaton, and every (four-way) two-dimensional sensing deterministic [nondeterministic] k-head finite automaton can be simulated by a CS-2-DFA$(k+1)$ [CS-2-NFA$(k+1)$]. As shown in [9], for sets of square tapes over a one-letter alphabet, (four-way) two-dimensional sensing deterministic [nondeterministic] $(k+1)$-head finite automata are more powerful tnan the cooresponding k-head finite automata. From these facts, the theorem follows. □

Unfortunately, it is unknown whether $\mathcal{L}$[CS-2-XFA$_{\{0\}}(k)^s$]$\subsetneq\mathcal{L}$[CS-2-XFA$_{\{0\}}(k+1)^s$] for any $k \geq 1$ and for any X$\in$\{D,N\}. It is also unknown whether $\mathcal{L}$[CS-2-XFA$(k)^s$]$\subsetneq\mathcal{L}$[CS-2-XFA$(k+1)^s$] for any $k \geq 2$ and for any X$\in$\{D,N\}. (It is easy to show that $\mathcal{L}$[CS-2-XFA$(1)^s$]$\subsetneq\mathcal{L}$[CS-2-XFA$(2)^s$].)

6.2. Three-Way Case

We next investigate how the number of automata of CS-TR2-FA^s's affects the accepting power.

For each $n \geq 1$, let $T(n) = \{x \in \{0,\ 1\}^{(2)} \mid (\exists m \geq n)\ [l_1(x) = l_2(x) = m\ \&\ x[(1,1),(1,m)] = x[(2,1),(2,m)] \in R_n(m)\ \&\ x[(3,1),(m,m)] \in \{0\}^{(2)}]\}$, where $R_n(m) = \{x \in \{0,\ 1\}^{(2)} \mid l_1(x) = 1,\ l_2(x) = m\ \&\ (x \text{ has exactly } n \text{ 1's})\}$ for each $m \geq n$. It is obvious that for any fixed positive integer n, $T(n)$ can be accepted by a CS-TR2-DFA(n).

We first consider the following problem: given a fixed positive integer n, find a CS-TR2-FA which accepts $T(n)$ and uses the minimum number of automata. Unfortunately, we cannot generally solve the problem in the present paper, but we give the lower and upper bounds. Let $f(n)$ denote the minimum number of automata required for deterministic CS-TR2-FA's to accept $T(n)$, and $g(n)$ denote the minimum number of automata required for nondeterministic CS-TR2-FA's to accept $T(n)$. Clearly, $g(n) \leq f(n)$ for any $n \geq 1$.

Theorem 6.2.1. For each $k \geq 1$, (1) $f(k^2+k-1) \leq 2k-1$, (2) $f(k^2+2k) \leq 2k$, and (3) $f(k(k-1)/2+1) \geq k$.

Befor giving the proof of Theorem 6.2.1, we will give an example for showing how some CS-TR2-DFA(2) accepts $T(3)$, which will be used as a basis step in the proof of (1) (or (2)) of Theorem 6.2.1 below.

Example 6.2.1. $T(3) \in \mathcal{L}$[CS-TR2-DFA$(2)^s$].

Proof. Let $T'(3) = \{x[(1,1),(2,l_2(x))] \mid x \in T(3)\}$. We actually show that there exists a CS-TR2-DFA(2) $M(2)$ =(FA$_1$, FA$_2$) accepting $T'(3)$, since one can easily make $M(2)$ accept $T(3)$. Let $h_i(t)$ denote the position of input head h_i of FA$_i$ at time t for each $i \in \{1,2\}$.

If the automaton (FA$_1$ or FA$_2$) moves its input head one cell every n steps, we say that the *speed* of its input head is $1/n$.

Consider the case when an input tape x with 2 rows and m columns such that $x[(1,1),(1,m)]$, $x[(2,1),(2,m)] \in R_3(m)$ is presented to $M(2)$ which acts as follows. (Input tapes in the form different from the above can easily be rejected by $M(2)$.) What we have to show is that how $M(2)$ checks whether the symbol on $p'_2(i)$ is 1 for each $i \in \{1,2,3\}$, where $p'_2(i)$ denotes the position just under the position, $p_1(i)$, of the i-th 1 in the first row (counting from left to right).

1. FA_1 and FA_2 move h_1 and h_2 simultaneously to the position $p_1(1)$ (at some time t_0^x). (Thus, $h_1(t_0^x) = h_2(t_0^x) = p_1(1)$.) Then
2. (a) FA_1 moves h_1 down one cell at speed 1 (thus, $h_1(t_1^x) = p'_2(1)$, $h_2(t_1^x - 1) = p_1(1)$, where $t_1^x = t_0^x + 1$), and then checks whether the symbol on $p'_2(1)$ is 1. If this is the case, then FA_1 moves h_1 to the right at speed 1 from $p'_2(1)$ to the position of the next symbol 1 (denoted by $p_2(2)$) if it really exists, and then moves h_1 to the right at speed 1/2 from $p_2(2)$ to the position of next symbol 1 (denoted by $p_2(3)$) if it really exists. Otherwise FA_1 halts forever (on the right boundary symbol attached to the second row of x), that is, $M(2)$ rejects x.
   (b) FA_2 moves h_2 to the right from $p_1(1)$ to $p_1(2)$ at speed 1 and from $p_1(2)$ to $p_1(3)$ at spedd 1/2, and moves h_2 down one cell at speed 1.
3. The time at which h_2 reaches $p'_2(3)$ is denoted by t_2^x. If h_1 and h_2 simultaneously reach $p_2(3)$ at time t_2^x (i.e., $p'_2(3) = p_2(3)$), then goto 4. Otherwise $M(2)$ rejects x. Note that h_1 and h_2 simultaneously reach $p_2(3)$ at time t_2^x if and only if
   (a) $l_1 + l_2 = l'_1 + l'_2$, and
   (b) $l_1 + 2l_2 = l'_1 + 2l'_2$

   (where, for each $i \in \{1,2\}$, l_i denotes the distance from $p_1(i)$ to $p_1(i+1)$, and l'_1 and l'_2 denote the distance from $p'_2(1)$ to $p_2(2)$ and from $p_2(2)$ to $p_2(3)$, respectively), thus, $l_1 = l'_1$ and $l_2 = l'_2$.
4. $M(2)$ accepts x if the number of 1's in the second row is exactly 3. Otherwise $M(2)$ rejects x.

It will be obvious that $M(2)$ accepts $T'(3)$. □

Proof of Theorem 6.2.1: The proofs of (1) and (2) are similar. We only give the proof of (2) here. To prove (2) is equivalent to proving that: for each $k \geq 1$, $T(k^2+2k) \in \mathcal{L}$[CS-TR2-DFA$(2k)^s$].

For each $n \geq 1$, let $T'(n) = \{x[(1,1),(2,l_2(x))] \mid x \in T(n)\}$. For convenience, we prove by induction on k that $T'(k^2+2k) \in \mathcal{L}$[CS-TR2-DFA$(2k)$]. It will be obvious that (2) follows from this fact.

Basis (k=1): There exists a CS-TR2-DFA(2) $M(2)$ =(FA_1, FA_2) accepting $T'(3)$ which satisfies the property that for each $x \in T'(3)$, there are three times $t_0^x > t_1^x > t_2^x$ during the accepting computation of $M(2)$ on x such that

0) $h_1(t_0^x) = h_2(t_0^x) = p_1(1)$,

1) $h_1(t_1^x) = p'_2(1)$, $h_2(t_1^x - 1) = p_1(1)$, and

2) $h_1(t_2^x) = h_2(t_2^x) = p'_2(3)$,

where $h_i(t)$ denotes the position of input head h_i of FA_i at time t, $p_1(i)$ denotes the position of the i-th 1 in the first row (counting from left to right), and $p'_2(i)$ denotes the position just under $p_1(i)$. This is shown in Example 6.2.1.

Inductive Hypothesis: Suppose that for each $1 \leq j \leq k$, there exists a CS-TR2-DFA$(2j)$ $M(2j)$ =(FA_1, FA_2, $\cdots$, FA_{2j}) accepting $T'(j^2+2j)$ which satisfies the property that for each $x \in T'(j^2+2j)$, there are $2j+1$ times $t_0^x > t_1^x > \cdots > t_{2j}^x$ during the accepting computation of $M(2j)$ on x such that

0) $h_1(t_0^x) = h_2(t_0^x) \cdots = h_{2j}(t_0^x) = p_1(d_j[1])$,

1) $h_1(t_1^x) = p'_2(d_j[1])$, $h_2(t_1^x - 1) = h_3(t_1^x - 1) = \cdots = h_{2j}(t_1^x - 1) = p_1(d_j[1])$,

...

i) $h_1(t_i^x) = h_2(t_i^x) = \cdots = h_i(t_i^x) = p'_2(d_j[i])$,
$h_{i+1}(t_i^x - 1) = h_{i+2}(t_i^x - 1) = \cdots = h_{2j}(t_i^x - 1) = p_1(d_j[i])$,

...

2j) $h_1(t_{2j}^x) = h_2(t_{2j}^x) = \cdots = h_{2j}(t_{2j}^x) = p'_2(d_j[2j])$,

where

$$d_r[i] = \begin{cases} i(i+1)/2 & \text{for } 1 \leq i \leq r+1, \\ i(4r-i+3)/2 - r(r+1) & \text{for } r+1 < i \leq 2r. \end{cases}$$

(See Fig.1.)

Inductive Step: We show that there exists a CS-TR2-DFA$(2(k+1))$ $M(2(k+1))$ =(FA_1, FA_2, $\cdots$, FA_{2k+2}) accepting $T'(k^2+4k+3)$ which satisfies the property that for each $x \in T'(k^2+4k+3)$, there are $2k+3$ times $t_0^x > t_1^x > \cdots > t_{2k+2}^x$ during the accepting computation of $M(2(k+1))$ on x such that

0) $h_1(t_0^x) = h_2(t_0^x) \cdots = h_{2k+2}(t_0^x) = p_1(d_{k+1}[1])$,

1) $h_1(t_1^x) = p_2'(d_{k+1}[1])$, $h_2(t_1^x - 1) = h_3(t_1^x - 1) = \cdots = h_{2k+2}(t_1^x - 1) = p_1(d_{k+1}[1])$,

...

i) $h_1(t_i^x) = h_2(t_i^x) = \cdots = h_i(t_i^x) = p_2'(d_{k+1}[i])$,
$h_{i+1}(t_i^x - 1) = h_{i+2}(t_i^x - 1) = \cdots = h_{2k+2}(t_i^x - 1) = p_1(d_{k+1}[i])$,

...

2k+2) $h_1(t_{2k+2}^x) = h_2(t_{2k+2}^x) = \cdots = h_{2k+2}(t_{2k+2}^x) = p_2'(d_{k+1}[2k+2])$.

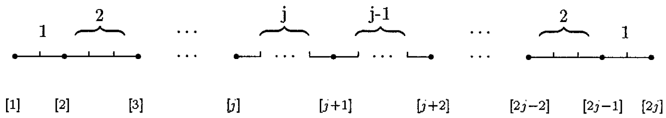

Fig. 1. '•' and '↾' denote $j(j+2)$ 1's in the first row of x, and

'[i]' denotes the position $p_1(d_j[i])$ for each $1 \leq i \leq 2j$.

Consider the case when an input tape x with 2 rows and m columns such that $x[(1,1),(1,m)]$, $x[(2,1),(2,m)] \in R_{(k^2+4k+3)}(m)$ is presented to $M(2(k+1))$ which acts as follows. (Input tapes in the form different from the above can easily be rejected by $M(2(k+1))$.)

1. $M(2(k+1))$ first verifies in a similar way as $M(2k)$ that the symbol on $p_2'(i)$ is 1 for each $i \in \{1, 2, \cdots, d_{k+1}\ [k+1]\}$. That is, by using the corresponding $2k$ automata (FA_1, FA_2, $\cdots$, FA_{2k}), $M(2(k+1))$ simulates the action of $M(2k)$ for checking whether the symbol just under the i-th 1 in the first row is 1 when an input tape y with 2 rows and m columns such that $y[(1,1),(1,m)]$, $y[(2,1),(2,m)] \in R_{(k^2+2k)}(m)$ is presented to $M(2k)$, while FA_{2k+1} and FA_{2k+2} are idle, and move in the same way as FA_{2k}. If this verification is successful, then goto 2. Otherwise $M(2(k+1))$ rejects x.

2. (a) Inductive hypothesis implies that h_1, h_2, $\cdots$, h_{k+1} simultaneously reach $p_2'(d_{k+1}\ [k+1])$ at some time t_{k+1}^x, and h_{k+2}, h_{k+3}, $\cdots$, h_{2k+2} reach $p_1(d_{k+1}[k+1])$ at time $t_{k+1}^x - 1$. Then FA_{k+1} moves h_{k+1} to the right from $p_2'(d_{k+1}[k+1])$ to the position (denoted by $p_2(d_{k+1}[k+1]+1)$) of the next symbol 1 at speed $1/(n^k + n^{k-1} + \cdots + n + 1)$, from $p_2(d_{k+1}[k+1]+1)$ to the position (denoted by $p_2(d_{k+1}[k+1]+2)$) of the next symbol 1 at speed $1/(n^{k-1} + n^{k-2} + \cdots + n + 1)$, $\cdots$, from $p_2(d_{k+1}[k+1]+k)$ to the position (denoted by $p_2(d_{k+1}[k+1]+k+1)$) of the next symbol 1 at speed 1, and from $p_2(d_{k+1}[k+1]+k+1)$ to the position (denoted by $p_2(d_{k+1}[k+1]+k+2) = p_2(d_{k+1}[k+2])$) of the next symbol 1 at speed 1/2 if there really exist at least $k+2$ 1's to the right hand of $p_2'(d_{k+1}[k+1])$, where $n = 2(m+2)$. Moreover, h_1, h_2, $\cdots$, h_{k+1} reach $p_2(d_{k+1}[k+2])$ at the same time if it realy exists. For otherwise, $M(2(k+1))$ rejects x. FA_{k+1} may do this with FA_1, FA_2, $\cdots$, FA_k.
Note that FA_{k+1} can move h_{k+1} at speed $1/(n^i + n^{i-1} + \cdots + n + 1)$ for any $i \leq k$ by means of FA_1, FA_2, $\cdots$, FA_k. For example, suppose that h_1 and h_{k+1} reach some cell $x(2,j)$ at the same time, and then FA_{k+1} (FA_1) moves h_{k+1} (h_1) to the neighbor cell just when h_1 meets h_{k+1} again after FA_1 moved h_1 one cycle of the second row ($2(m+2)$ cells) at speed 1. This makes h_{k+1} move at speed $1/(n+1)$ on $x(2,j)$.
(b) FA_{k+2} moves h_{k+2} to the right from $p_1(d_{k+1}[k+1])$ to $p_1(d_{k+1}[k+1]+1)$ at speed $1/(n^k + n^{k-1} + \cdots + n + 1)$, from $p_1(d_{k+1}[k+1]+1)$ to $p_1(d_{k+1}[k+1]+2)$ at speed $1/(n^{k-1} + n^{k-2} + \cdots + n + 1)$, $\cdots$, from $p_1(d_{k+1}[k+1]+k)$ to $p_1(d_{k+1}[k+1]+k+1)$ at speed 1, and from $p_1(d_{k+1}[k+1]+k+1)$ to $p_1(d_{k+1}[k+1]+k+2) = p_1(d_{k+1}[k+2])$ at speed 1/2. Moreover, h_{k+2}, h_{k+3}, $\cdots$, h_{2k+2} reach $p_1(d_{k+1}[k+2])$ simultaneously. This can be done by means of FA_{k+3}, FA_{k+4}, $\cdots$, FA_{2k+2}. Then FA_{k+2} moves h_{k+2} down one cell at speed 1.

3. The time at which h_{k+2} reaches $p_2'(d_{k+1}[k+2])$ is denoted by t_{k+2}^x. $M(2(k+1))$ continues the computation on x if h_1, h_2, $\cdots$, h_{k+2} reach $p_2(d_{k+1}[k+2])$ at time t_{k+2}^x (i.e., $p_2'(d_{k+1}[k+2]) = p_2(d_{k+1}[k+2])$), that is, if $h_1(t_{k+2}^x) = h_2(t_{k+2}^x) = \cdots = h_{k+2}(t_{k+2}^x) = p_2'(d_{k+1}[k+2])$, and $h_{k+3}(t_{k+2}^x - 1) = h_{k+4}(t_{k+2}^x - 1) = \cdots = h_{2k+2}(t_{k+2}^x - 1) = p_1(d_{k+1}[k+2])$. Otherwise $M(2)$ rejects x. Note that h_1, h_2, $\cdots$, h_{k+2} reach $p_2(d_{k+1}[k+2])$ at time t_{k+2}^x if and only if

   (a) $l_{d_{k+1}[k+1]} + l_{d_{k+1}[k+1]+1} + \cdots + l_{d_{k+1}[k+1]+k+1}$
   $= l'_{d_{k+1}[k+1]} + l'_{d_{k+1}[k+1]+1} + \cdots + l'_{d_{k+1}[k+1]+k+1}$, and

   (b) $l_{d_{k+1}[k+1]} \cdot n^k +$
   $(l_{d_{k+1}[k+1]} + l_{d_{k+1}[k+1]+1}) \cdot n^{k-1} + \cdots +$
   $(l_{d_{k+1}[k+1]} + l_{d_{k+1}[k+1]+1} + \cdots + l_{d_{k+1}[k+1]+k-1}) \cdot n +$
   $l_{d_{k+1}[k+1]} + l_{d_{k+1}[k+1]+1} + \cdots + l_{d_{k+1}[k+1]+k} + 2l_{d_{k+1}[k+1]+k+1}$

$= l'_{d_{k+1}[k+1]} \cdot n^k +$
$(l'_{d_{k+1}[k+1]} + l'_{d_{k+1}[k+1]+1}) \cdot n^{k-1} + \cdots +$
$(l'_{d_{k+1}[k+1]} + l'_{d_{k+1}[k+1]+1} + \cdots + l'_{d_{k+1}[k+1]+k-1}) \cdot n +$
$l'_{d_{k+1}[k+1]} + l'_{d_{k+1}[k+1]+1} + \cdots + l'_{d_{k+1}[k+1]+k} + 2l'_{d_{k+1}[k+1]+k+1}$

(where l_i denotes the distance from $p_1(i)$ to $p_1(i+1)$, $l'_{d_{k+1}[k+1]}$ denotes the distance from $p'_2(d_{k+1}[k+1])$ to the $p_2(d_{k+1}[k+1]+1)$, and $l'_{d_{k+1}[k+1]+i}$ denotes the distance from $p_2(d_{k+1}[k+1]+i)$ to the $p_2(d_{k+1}[k+1]+i+1)$), thus

$$l_{d_{k+1}[k+1]} = l'_{d_{k+1}[k+1]}, \quad l_{d_{k+1}[k+1]+1} = l'_{d_{k+1}[k+1]+1}, \quad \cdots, \quad l_{d_{k+1}[k+1]+k+1} = l'_{d_{k+1}[k+1]+k+1}.$$

(This means that the symbol on $p'_2(i)$ is 1 for each $i \in \{d_{k+1}[k+1]+1, d_{k+1}[k+1]+2, \cdots, d_{k+1}[k+2]\}$).

4. $M(2(k+1))$ finally verifies in a similar way as $M(2k)$ that the symbol on $p'_2(i)$ is 1 for each $i \in \{d_{k+1}[k+2]+1, d_{k+1}[k+2]+2, \cdots, d_{k+1}[2k+2]\}$. That is, by using FA_1, FA_2, $\cdots$, FA_k, FA_{k+3}, FA_{k+4}, $\cdots$, FA_{2k+2} corresponding to the $2k$ automata of $M(2k)$, $M(2(k+1))$ simulates the action of $M(2k)$ for checking whether the symbol under the j-th 1 in the first row is 1 for each $j \in \{d_k[k]+1, d_k[k]+2, \cdots, d_k[2k]\}$ when an input tape y with 2 rows and m columns such that $y[(1,1),(1,m)]$, $y[(2,1),(2,m)] \in R_{(k^2+2k)}(m)$ is presented to $M(2k)$, while FA_{k+1} and FA_{k+2} are idle, and move in the same way as FA_k. If this verification is successful, then goto 5. Otherwise $M(2(k+1))$ rejects x.

5. $M(2(k+1))$ accepts x if the number of 1's in the second row is exactly k^2+4k+3. Otherwise $M(2(k+1))$ rejects x.

It is obvious that by inductive hypothesis the action of $M(2(k+1))$ described above satisfies the required property. This completes the proof of (2).

We now prove (3). Suppose that there is a CS-TR2-DFA$(k-1)^s$ $M(k-1) = (\text{FA}_1, \text{FA}_2, \cdots, \text{FA}_{k-1})$ accepting $T(k(k-1)/2+1)$. Let h_i denote the input head of FA_i for each $i \in \{1,2,\cdots,k-1\}$.

For each $m \geq k(k-1)/2+1$, let $V(m) = \{x \in T(k(k-1)/2+1) \mid l_1(x) = l_2(x) = m\}$, and for each permutation $\sigma : \{1,2,\cdots,k-1\} \to \{1,2,\cdots,k-1\}$, let $W_\sigma(m)$ be the set of all input tapes $x \in V(m)$ such that during the accepting computation of $M(k-1)$ on x, input heads $h_{\sigma(1)}$, $h_{\sigma(2)}$, $\cdots$, $h_{\sigma(k-1)}$ leave the first row of x in this order.[3] Then there must exist some permutation τ such that

$$|W_\tau(m)| \geq |V(m)|/(k-1)! = \Omega(m^{k(k-1)/2+1}).^{4}$$

For each $x \in W_\tau(m)$ and each $1 \leq i \leq k-1$, let $q_{\tau(i)}(x)$, $p_{\tau(i)}(x)$ and $t_{\tau(i)}(x)$ denote the internal state of $\text{FA}_{\tau(i)}$, the position of $h_{\tau(i)}$ and the time, respectively, when $h_{\tau(i)}$ leaves the first row during the accepting computation of $M(k-1)$ on x.

For each $x \in W_\tau(m)$, let

$$t(x) = (t_{\tau(2)}(x) - t_{\tau(1)}(x), t_{\tau(3)}(x) - t_{\tau(2)}(x), \cdots, t_{\tau(k-1)}(x) - t_{\tau(k-2)}(x)),$$

and

$$u(x) = \langle (q_{\tau(1)}(x), p_{\tau(1)}(x)), \cdots, (q_{\tau(k-1)}(x), p_{\tau(k-1)}(x)), t(x) \rangle.$$

Clearly, for each $2 \leq i \leq k-1$, $t_{\tau(i)}(x) - t_{\tau(i-1)}(x) = O(m^{k-i})$, because otherwise $\text{FA}_{\tau(i)}$, $\cdots$, $\text{FA}_{\tau(k-1)}$ would enter a loop on the first row, and thus $M(k-1)$ would never accept x. So $| \{u(x) \mid x \in W_\tau(m)\} | = O(m^{k(k-1)/2})$. Therefore, it follows that for large m

$$| W_\tau(m) | > | \{u(x) \mid x \in W_\tau(m)\} |,$$

and so there exist two diffirent input tapes $x, y \in W_\tau(m)$ such that $u(x) = u(y)$. Let z be the tape obtained from x by replacing the second row of x with the second row of y. It follows that z is also accepted by $M(k-1)$. This is a contradiction, because z is not in $T(k(k-1)/2+1)$. This completes the proof of (3). □

Theorem 6.2.2. $g(2k^2-5k+4) \geq k$, for $k \geq 1$.

Proof. The proof is very similar to that of (3) of Theorem 6.2.1. Suppose, to the contrary, that there is a CS-TR2-NFA$(k-1)^s$ $M(k-1) = (\text{FA}_1, \text{FA}_2, \cdots, \text{FA}_{k-1})$ accepting $T(2k^2-5k+4)$. Let h_i denote the input head of FA_i for each $i \in \{1,2,\cdots,k-1\}$.

For each $m \geq 2k^2-5k+4$, let $V(m) = \{x \in T(2k^2-5k+4) \mid l_1(x) = l_2(x) = m\}$. With each $x \in V(m)$, We associate one fixed accepting computation, $c(x)$, of $M(k-1)$ on x in which $M(k-1)$ operates in $O(m^{2(k-1)})$ time. Furthermore, for each permutation $\sigma : \{1,2,\cdots,k-1\} \to \{1,2,\cdots,k-1\}$, let $W_\sigma(m)$ be the set of all input tapes $x \in V(m)$ such that during $c(x)$, input heads $h_{\sigma(1)}$, $h_{\sigma(2)}$, $\cdots$, $h_{\sigma(k-1)}$ leave the first row of x in this order.[5] Then there must exist some permution τ such that

$$|W_\tau(m)| \geq |V(m)|/(k-1)! = \Omega(m^{2k^2-5k+4}).$$

[3] See footnote 1.
[4] For any finite set A, $| A |$ denotes the number of elements of A.
[5] See footnote 1.

For each $x \in W_\tau(m)$ and each $1 \leq i \leq k-1$, let $q_{\tau(i)}(x)$, $p_{\tau(i)}(x)$ and $t_{\tau(i)}(x)$ denote the internal state of $\text{FA}_{\tau(i)}$, the position of $h_{\tau(i)}$ and the time, respectively, when $h_{\tau(i)}$ leaves the first row of x during $c(x)$.

For each $x \in W_\tau(m)$, let

$$t(x) = (t_{\tau(2)}(x) - t_{\tau(1)}(x), t_{\tau(3)}(x) - t_{\tau(2)}(x), \cdots, t_{\tau(k-1)}(x) - t_{\tau(k-2)}(x)),$$

and

$$u(x) = \langle (q_{\tau(1)}(x), p_{\tau(1)}(x)), \cdots, (q_{\tau(k-1)}(x), p_{\tau(k-1)}(x)), t(x) \rangle.$$

Clearly, for each $2 \leq i \leq k-1$, $t_{\tau(i)}(x) - t_{\tau(i-1)}(x) = O(m^{2(k-1)})$. So $| \{u(x) \mid x \in W_\tau(m)\} | = O(m^{2k^2-5k+3})$. Therefore, it follows that for large m

$$| W_\tau(m) | > | \{u(x) \mid x \in W_\tau(m)\} |,$$

and so there exist two diffirent input tapes $x, y \in W_\tau(m)$ such that $u(x) = u(y)$. Let z be the tape obtained from x by replacing the second row of x with the second row of y. Clearly, from $c(x)$ and $c(y)$, we can construct an accepting computation of $M(k-1)$ on z. This is a contradiction, because z is not in $T(2k^2 - 5k + 4)$. This completes the proof of the theorem. □

From Theorems 6.2.1 and 6.2.2, we can get the following theorem.

Theorem 6.2.3. For each $k \geq 1$ and each X∈{D,N}, $\mathcal{L}$[CS-TR2-XFA$(k)^s$]$\subsetneq\mathcal{L}$[CS-TR2-XFA$(k+1)^s$].

Proof. For each $k \geq 1$, let $D(k) = \max\{n \mid f(n) = k\}$ and $N(k) = \max\{n \mid g(n) = k\}$. From Theorem 6.2.1 (3) and Theorem 6.2.2, we have

$$D(k) \leq k(k+1)/2 \quad \text{and} \quad N(k) \leq 2k^2 - k,$$

respectively.

For each X∈{D,N}, let M be a CS-TR2-XFA$(k)^s$ accepting $T(X(k))$. From M, we can easily construct a CS-TR2-XFA$(k+1)^s$ M' which accepts $T(X(k)+1)$. Thus $T(X(k)+1) \in\mathcal{L}$[CS-TR2-XFA$(k+1)^s$]. From this and the fact that $T(X(k)+1) \notin\mathcal{L}$[CS-TR2-XFA$(k)^s$], it follows that $T(X(k)+1) \in\mathcal{L}$[CS-TR2-XFA$(k+1)^s$]$-\mathcal{L}$[CS-TR2-XFA$(k)^s$]. □

7. CONCLUSION

We conclude this paper by giving several open problems except the open problems stated in the previous section.

1. For each $k \geq 2$,

   $$\mathcal{L}[\text{CS-2-DFA}(k)^s]\subsetneq\mathcal{L}[\text{CS-2-NFA}(k)^s]?$$

   Note that $\mathcal{L}$[CS-2-DFA$(1)^s$]$\subsetneq\mathcal{L}$[CS-2-NFA$(1)^s$]. (See [1]).

2. For each $k \geq 1$, and each X∈ {D,N},

   $$\mathcal{L}[\text{CS-TR2-XFA}_{\{0\}}(k)^s]\subsetneq\mathcal{L}[\text{CS-TR2-XFA}_{\{0\}}(k+1)^s],$$

   where $\mathcal{L}$[CS-TR2-XFA$_{\{0\}}(k)^s$] denote the class of sets of square tapes over a one-letter alphabet accepted by CS-TR2-XFA(k)'s?

3. For $n \geq 4$, $g(n) < f(n)$? (It is easy to show that for $1 \leq n \leq 3$, $g(n) = f(n)$.)

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