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一様確率分布での例の集合から長方形を推定する方法について

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あらまし

一様分布のもとで与えられた例の集合から、目標となる長方形を推定するアルゴリズムを考えた。このアルゴリズムは、例の個数の多項式時間内に仮説となる長方形を出力し、目標の長方形に対して、この仮説が高い確率で良い近似となっているということを証明する。

和文キーワード　長方形推定問題、一様分布、ϵ 近似、凸包、凸包を包含する長方形

On the Detection of a Rectangle from a Given Set of Examples

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Abstract

In this paper, we consider the problem of detecting an unknown *target rectangle* from a given set of *examples*, i.e., a set of *randomly generated* points in the rectangle. One algorithm is presented that yields a *good* approximation of a target rectangle with *high probability* from *polynomial* number of examples.

英文 key words　rectangle detecting problem, uniform distribution, ϵ-close, convex hull, rectangle containing conconvex hull

1 Introduction

In this paper, we consider the problem of detecting an unknown *target rectangle* from a given set of *examples*, i.e., a set of *randomly generated* points in the rectangle. One algorithm is presented that yields a *good* approximation of a target rectangle with *high probability* from *polynomial* number of examples.

The above problem — Rectangle Detection Problem — has been considered [Blu89] in relation to the PAC learnability of rectangles. For example, there is an easy detecting algorithm for axes-parallel rectangles [Blu89]. That is, for any axes-parallel rectangle C, any $\epsilon > 0$, and any $\delta > 0$, if the algorithm is given more than $4/\epsilon \log(4/\delta)$ randomly chosen examples, then

$$\Pr_{\text{examples}}\{Er(H|C) < \epsilon\} > 1 - \delta,$$

where H is the output of the algorithm.

On the other hand, the problem of detecting rectangles in general has been left open. This paper presents one simple algorithm, and show that it works sufficiently well. That is, for any rectangle C, any $\epsilon > 0$, and any $\delta > 0$, if the algorithm is given more than $4\{(13/\epsilon)^2(\log 4 + \log(1/\delta))\}$ randomly chosen examples, then

$$\Pr_{\text{examples}}\{Er(H|C) < \epsilon\} > 1 - \delta,$$

where H is the output of the algorithm.

2 Preliminaries

In this paper, we consider only polygons for our object, and each polygon is denoted as an ordered tuple of points in the plain. For example, an ordered tuple of 4 points $S = \langle s_1, s_2, s_3, s_4 \rangle$ means a quadrangle with four edges $(s_1, s_2), (s_2, s_3), (s_3, s_4)$, and (s_4, s_1). Each polygon S is also considered as the set of points in S, and by $S_1 \cup S_2$, for example, we mean the *union* of polygons of S_1 and S_2. $S_1 \oplus S_2$ denotes $(S_1 - S_2) \cup (S_2 - S_1)$, and $S_1 \subset S_2$ means that S_1 is contained in S_2. For any S, area(S) denotes the area of S.

Throughout this paper, let C denote a *target rectangle*, i.e., an unknown rectangle that is a target of our detecting algorithm. Symbols using C (such as $C_\epsilon^{\mathsf{Small}}$) always denote some polygon related to C. In the algorithm (or in its analysis), X is usually regarded as *a set of examples*, i.e., a set of $x_1, x_2, ..., x_n$ that are picked up randomly from C. H usually denotes a rectangle computed by the algorithm.

3 Rectangle Detection Algorithm

Here we present our rectangle detection algorithm A_0. First we explain some functions used in the algorithm.

Let $X = \{x_1, x_2, ..., x_n\}$ be a given set of points. Define $CH(X)$ to be the convex hull $\langle s_1, s_2, ..., s_m \rangle$ of X. Note that $\{s_1, s_2, ..., s_m\} \subseteq \{x_1, x_2, ..., x_n\}$ and $m \leq n$.

For any convex polygon $S = \langle s_1, s_2, ..., s_n \rangle$ (which is $CH(X)$ in the algorithm), and any $i, 1 \leq i \leq m$, define $R(S, i)$ to be the smallest rectangle that contains S and has an edge containing (s_i, s_{i+1}).

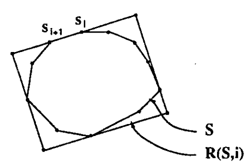

Figure 1: $R(S, i)$

Finally define $MR(S)$ to be (one of) $R(S, i)$ $(1 \leq i \leq m)$ with the smallest area.

Now the algorithm A_0 is stated as follows:

algorithm A_0;
begin
  input X; (* $X = \{x_1, x_2, .., x_n\}$ *)
  $S \leftarrow CH(X)$;
  (* $S = \langle s_1, s_2, ..., s_m \rangle, m \leq n$ *)
  $H \leftarrow MR(S)$;
halt(H)
end.

Let us now analyze the time complexity of A_0. Here we state only rough estimation (see [Kaw93] for detail).

It is known [Gra72],[Jar72] that the convex hull of given l points is computable by $\mathcal{O}(l \log l)$ basic operations. By using their algorithm, we can compute $S = CH(X)$ in $\mathcal{O}(n \log n)$ steps.

It is not hard to show that $R(S,i)$ is computable by $\mathcal{O}(m)$ basic operations for each $i, 1 \leq i \leq m$. Thus $H = MR(S)$ is $\mathcal{O}(m^2)$ step computable. Therefore, we have the following lemma.

Lemma 3.1. For a given $X = \{x_1, x_2, ..., x_n\}$, the algorithm A_0 yields some output in $\mathcal{O}(n^2)$ basic operations.

Thus, roughly speaking, A_0 runs in polynomial time w.r.t. the number of examples.

4 Analysis of Algorithm

Here we show that A_0 yields a rectangle that is *close* to the target with *high probability* when sufficiently (but still *polynomial*) number of randomly chosen examples are given.

First we introduce basic notions and notations. Recall that C, X, and H denote a target rectangle, a set of examples, and an output rectangle, respectively. For any $\epsilon > 0$, we say that H is *ϵ-close* to C if

$$Er(H|C) \stackrel{\text{def}}{=} \frac{\text{area}(H \oplus C)}{\text{area}(C)} < \epsilon,$$

where $Er(H|C)$ is called the *error* of H w.r.t. C. Our goal is to show that (if enough examples are given) H obtained by A_0 is ϵ-close to C with high probability. More Precisely, we show the following theorem.

Theorem 4.1. Some polynomial p_0 exists such that for any $\epsilon > 0$, any $\delta, 0 < \delta < 1$, if $n \geq p_0(1/\epsilon, 1/\delta)$ and $X = \{x_1, x_2, ..., x_n\} \subseteq C$ is selected randomly, then

$$\Pr_{\substack{\text{Choice}\\ \text{of } X}} \left\{ \begin{array}{l} A_0 \text{ yields } H \\ \text{that is } \epsilon\text{-close to } C \end{array} \right\} > 1 - \delta.$$

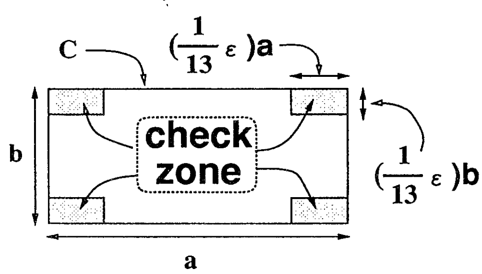

Figure 2: Check Zone

For any $\epsilon > 0$, each shadowed rectangle in the figure 2 is called a *check zone*. Let $CS_\epsilon(X)$ denote *any* one of polygons consisting of four vertices of X each of which is from each check zone. If no such polygon exists, then $CS_\epsilon(X) = \perp$.

Our analysis goes by the following two steps: (1) first show that if n is sufficiently large, then the probability that $CS_\epsilon(X) \neq \perp$ is larger than $1 - \delta$, and then (2) show that if $CS_\epsilon(X) \neq \perp$, then H (computed by A_0) is ϵ-close to C.

4.1 Analysis: step(1)

For any ϵ and δ, define $p_0(1/\epsilon, 1/\delta)$ to be

$$p_0\left(\frac{1}{\epsilon}, \frac{1}{\delta}\right) = 4\left\{\left(\frac{13}{\epsilon}\right)^2 \left(\log 4 + \log \frac{1}{\delta}\right) + 1\right\}.$$

Consider any $\epsilon > 0$ and any $\delta, 0 < \delta < 1$, and fix them in the following discussion. Here we show that if $n \geq p_0(1/\epsilon, 1/\delta)$ and $x_1, x_2, ..., x_n$ are chosen randomly from C, then probability that $CS_\epsilon(X) \neq \perp$ is greater than $1 - \delta$.

First we prepare one combinatorial lemma on the probability of a certain event. Consider any probabilistic trial Φ where an event E occurs with probability p. Let $\rho_1(n, p, c)$ be the probability that E occurs at least c times in n independent trials of Φ. Then we have

$$\begin{aligned} &\rho_1(n, p, c) \\ &= \sum_{i=c}^{n} \binom{n}{i} p^i (1-p)^{n-i}. \end{aligned} \quad (1)$$

Similarly, consider any probabilistic trial Φ where each one of exclusive events $E_1, E_2, ..., E_c$ occurs with probability p. Let $\rho_2(n, p, c)$ be the probability that each $E_k, 1 \leq k \leq c$, occurs at least once in n independent trials of Φ. Then we have the following lemma.

Lemma 4.2. For any $c \geq 2$,

$$\begin{aligned} \rho_2(n, p, c) &> \rho_1(\lfloor \tfrac{n}{c} \rfloor, p, 1)^c \\ &> 1 - c(1-p)^{\lfloor \frac{n}{c} \rfloor}. \end{aligned}$$

Proof. Consider n independent trials of Φ. For each $h, 1 \leq h \leq c$, let E'_h be the event that E_k occurs at least once in $((h-1)\lfloor \frac{n}{c} \rfloor + 1)$-th $\sim h\lfloor \frac{n}{c} \rfloor$-th trials. Then clearly,

$$\begin{aligned} &\rho_2(n, p, c) \\ &> \Pr\{E'_1 \wedge E'_2 \wedge ... \wedge E'_c\} \\ &= \Pr\{E'_1\} \cdot \Pr\{E'_2\} \cdot ... \cdot \Pr\{E'_c\} \\ &\geq (\rho_1(\lfloor \frac{n}{c} \rfloor, p, 1))^c. \end{aligned} \quad (2)$$

On the other hand, from (1) we have

$$\begin{aligned} (1) &= \left\{ \sum_{i=1}^{\lfloor \frac{n}{c} \rfloor} \alpha_i \right\}^c \\ &= (1-\alpha_0)^c \ (\because \text{binomial theorem}) \\ &> 1 - c\alpha_0 \\ &= 1 - c(1-p)^{\lfloor \frac{n}{c} \rfloor}, \end{aligned}$$

where, $\alpha_i \stackrel{\text{def}}{=} \binom{\lfloor \frac{n}{c} \rfloor}{i} p^i (1-p)^{\lfloor \frac{n}{c} \rfloor - i}$. □

Now we are ready to prove our goal of step (1).

Lemma 4.3. If $n \geq p_0(1/\epsilon, 1/\delta)$, and $X = \{x_1, x_2, ..., x_n\} \subseteq C$ is chosen randomly, then the probability that $CS_\epsilon(X) \neq \perp$ is larger than $1 - \delta$.

Proof. From the definitions of $CS_\epsilon(X)$ and ρ_2, it is not so hard to see that $\Pr\{CS_\epsilon(X) \neq \perp\} = \rho_2(n, (\epsilon/13)^2, 4)$. Thus,

$$\begin{aligned} &\Pr\{CS_\epsilon(X) \neq \perp\} \\ &= \rho_2\left(n, \left(\tfrac{\epsilon}{13}\right)^2, 4\right) \\ &> 1 - 4\left\{1 - \left(\frac{\epsilon}{13}\right)^2\right\}^{\lfloor \frac{n}{4} \rfloor}. \end{aligned}$$

On the other hand, since $n \geq p_0(1/\epsilon, 1/\delta)$, we have

$$\lfloor \frac{n}{4} \rfloor > \frac{n}{4} - 1 \geq \left(\frac{13}{\epsilon}\right)^2 \log \frac{4}{\delta}. \quad (3)$$

And, by Taylor's expansion,

$$\log \frac{169}{169 - \epsilon^2} > \left(\frac{\epsilon}{13}\right)^2. \quad (4)$$

Thus, from (3) and (4),

$$\left\lfloor \frac{n}{4} \right\rfloor > \frac{\log \frac{4}{\delta}}{\log \frac{169}{169 - \epsilon^2}}.$$

Therefore,

$$1 - 4\left\{1 - \left(\frac{\epsilon}{13}\right)^2\right\}^{\lfloor \frac{n}{4} \rfloor} > 1 - \delta.$$

□

4.2 Analysis: step(2)

Let any $\epsilon > 0$ be fixed. We show that if $CS_\epsilon(X) \neq \perp$, then A_0 outputs H that is ϵ-close to C. First we define some sets. Let $C_\epsilon^{\mathsf{Small}}$ be the rectangle that has longer edges of length $(1 - \frac{2}{13}\epsilon)a$ and shorter edges of length $(1 - \frac{2}{13}\epsilon)b$ and that is located in the center of C, where

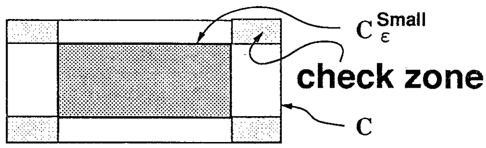

Figure 3: $C_\epsilon^{\mathsf{Small}}$

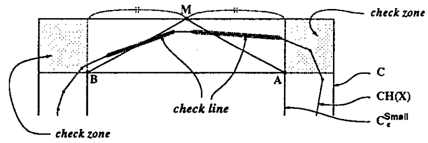

Figure 4: Check Line

a and b are length of respectively longer and shorter edges.

Our analysis goes as follows: First we show that if $CS_\epsilon(X) \neq \perp$, then $CH(X)$ contains $C_\epsilon^{\mathsf{Small}}$, that is, $CH(X)$ is not so *small* subset of C. Then we show that $MR(CH(X))$ $(= H)$ is not *large* either.

The first part of the above outline is the following lemma. (The proof is clear from definition, thus it is omitted.)

Lemma 4.4. Suppose that $CS_\epsilon(X) \neq \perp$. Then, $C_\epsilon^{\mathsf{Small}} \subset CS_\epsilon(X) \subset CH(X) \subset C$

Since H $(= MR(CH(X)))$ clearly contains $CH(X)$, we know from the above lemma that H contains a large part of C. Thus, it suffices to show that $MR(CH(X))$ is not too large compared with C. In the following, instead of $MR(CH(X))$, we investigate $R(CH(X), i_0)$ for one particular i_0, and we show that this $R(CH(X), i_0)$ is *close* to C. Recall that $MR(CH(X))$ is the smallest among $R(CH(X), i_0), 1 \leq i \leq n$; thus the fact that $(R(CH(X), i_0)$ is *close* to C (thus, not so *large*) implies that $MR(CH(X))$ is of reasonable size. $R(CH(X), i_0)$ is a rectangle that contains a *good edge* $(s_{i_0} s_{i_0+1})$ as a part of its edge.

Now let us discuss precisely. Consider one edge of C, and let M be its center. Let AB be the corresponding edge of $C_\epsilon^{\mathsf{Small}}$ (see the figure 4). Then, the edge of $CH(X)$ that intersects AM or BM is called a *check line.*

Let $s_i s_{i+1}$ be the check line of AM and $s_j s_{j+1}$ is the check line of BM. Then we have

$$\varphi(\overrightarrow{AM}) \leq \varphi(\overrightarrow{s_i s_{i+1}}) \leq \varphi(\overrightarrow{s_j s_{j+1}}) \leq \varphi(\overrightarrow{MB}),$$

where, $\varphi(\vec{v})$ means the inclination of $\vec{v}$. If furthermore, $s_i s_{s+1}$ (resp., $s_j s_{j+1}$) satisfies $\varphi(\overrightarrow{s_i s_{i+1}}) \leq \varphi(\overrightarrow{AB})$ (resp., $\varphi(\overrightarrow{AB}) \leq \varphi(\overrightarrow{s_j s_{j+1}})$), then it is called a *good* check line.

It is clear that there exists a check line on each side of C. Then now let l_A and l_B be the check line of AM and BM respectively. (Note that there may be the case that one edge crosses both AM and BM. On that case, we consider that l_A is the same as l_B.)

Lemma 4.5. Suppose that $CS_\epsilon(X) \neq \perp$. For each side of C, there exists at least one *good* check line on its side.

Proof. From the definition of a check line, we know

$$\varphi(\overrightarrow{AM}) \leq \varphi(\vec{l_A}) \leq \varphi(\vec{l_B}) \leq \varphi(\overrightarrow{MB}).$$

Suppose $\varphi(\overrightarrow{AM}) \leq \varphi(\vec{l_A}) \leq \varphi(\overrightarrow{AB})$. Then l_A is a good check line. Else then $\varphi(\overrightarrow{AB}) \leq \varphi(\vec{l_A}) \leq \varphi(\vec{l_B}) \leq \varphi(\overrightarrow{MB})$, therefore l_B is a good check line. □

We define a new type of rectangle. Let $C_\epsilon^{\mathsf{Large}}$ be the rectangle that has edges that is parallel to AM and that has apices of C on its edges (refer to Figure 5).

Now we estimate the error of $C_\epsilon^{\mathsf{Small}}$ and $C_\epsilon^{\mathsf{Large}}$.

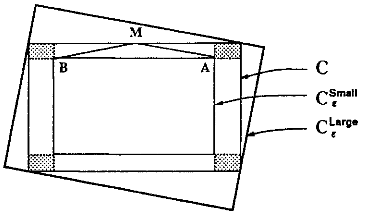

Figure 5: $C_\epsilon^{\mathsf{Large}}$

Lemma 4.6.

$$\begin{cases} (1) & Er(C_\epsilon^{\mathsf{Small}}) < \frac{4}{13}\epsilon \\ (2) & Er(C_\epsilon^{\mathsf{Large}}) < \frac{4}{11}\epsilon \end{cases}$$

Proof. First we prove (1). From the definition of $C_\epsilon^{\mathsf{Small}}$, we know

$$\text{area}(C_\epsilon^{\mathsf{Small}}) = (1 - \frac{2}{13}\epsilon)^2 \cdot \text{area}(C).$$

Then,

$$\begin{aligned} Er(C_\epsilon^{\mathsf{Small}}) &= \frac{\text{area}(C_\epsilon^{\mathsf{Small}}) \oplus \text{area}(C)}{\text{area}(C)} \\ &= 1 - (1 - \tfrac{2}{13}\epsilon)^2 \\ &= \tfrac{4}{13}\epsilon(1 - \tfrac{1}{13}\epsilon) \\ &< \frac{4}{13}\epsilon \end{aligned}$$

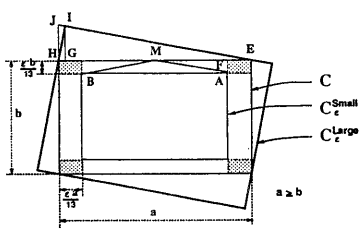

Figure 6: $Er(C_\epsilon^{\mathsf{Large}})$

Next we prove (2). Let us give a name to points as figure 6. We know $\triangle MAF$ is similar to $\triangle EJH$. Then,

$$AF : FM = JH : HE$$

$$\frac{1}{13}\epsilon b : (\frac{1}{2} - \frac{1}{13}\epsilon)a = JH : HE$$

$$\begin{aligned} \therefore \text{area}(\triangle EJH) &= \frac{1}{2} JH \cdot HE \\ &= \frac{1}{2} \cdot \frac{\frac{1}{13}\epsilon}{(\frac{1}{2} - \frac{1}{13}\epsilon)} \text{area}(C) \\ &< \frac{1}{2} \cdot \frac{\frac{1}{13}\epsilon}{(\frac{1}{2} - \frac{1}{13})} \text{area}(C) \quad (\because 0 < \epsilon < 1) \\ &= \frac{1}{11}\epsilon \\ \therefore Er(C_\epsilon^{\mathsf{Large}}) &< \frac{4 \cdot \text{area}(\triangle EJH)}{\text{area}(C)} \\ &< \frac{4}{11}\epsilon \end{aligned}$$

□

Let us introduce another error ratio.

$$Er_\epsilon(H|C) \stackrel{\text{def}}{=} \frac{\text{area}(H \oplus C_\epsilon^{\mathsf{Small}})}{\text{area}(C)}$$

This ratio is called *pseudo error* of H. When C is clear from the context, $Er(H|C)$ and $Er_\epsilon(H|C)$ may be abbreviated as $Er(H)$ and $Er_\epsilon(H)$.

Here we first analyze the pseudo error of H $(= MR(CH(X)))$.

Lemma 4.7. Suppose that $CS_\epsilon(X) \neq \perp$. Then, $Er_\epsilon(MR(CH(X))) < Er_\epsilon(C_\epsilon^{\mathsf{Large}})$.

Proof. First we prove the following upper bound.(5):

$$Er_\epsilon(MR(CH(X))) \leq Er_\epsilon(R(CH(X), i_0))$$

Here, i_0 is a good check line of a longer edge of C. Suppose that $CS_\epsilon(X) \neq \perp$. Then we know $C_\epsilon^{\mathsf{Small}} \subset MR(CH(X))$ and $C_\epsilon^{\mathsf{Small}} \subset R(CH(X), i_0)$. Recall that the area of $MR(CH(X))$ is the minimum among all the area of $R(CH(X), i)$ $(1 \leq i \leq m)$. Thus,

$$\text{area}(MR(CH(X))) \leq \text{area}(R(CH(X), i_0)).$$

Hence, we have (5).

Next we define a new type of rectangle. Let $ER(CH(X), i_0)$ be the rectangle that has apices of C on its edges, and each of that edge is parallel to the edge of $R(CH(X), i_0)$. Clearly

$ER(CH(X), i_0)$ includes $R(CH(X), i_0)$. Then we have (6):

$$Er_\epsilon(R(CH(X), i_0)) < Er_\epsilon(ER(CH(X), i_0)).$$

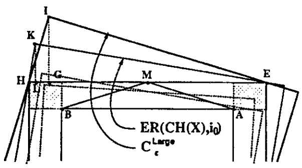

Figure 7: $ER(CH(X), i_0)$ and $C_\epsilon^{\mathsf{Large}}$

Now note the inclinations of $ER(CH(X), i_0)$ and $C_\epsilon^{\mathsf{Large}}$. As figure 7, EI, an edge of $C_\epsilon^{\mathsf{Large}}$, is parallel to AM. On the other hand, EK, an edge of $ER(CH(X), i_0)$, is parallel to a good check line $s_{i_0}s_{i_0+1}$. From the definition of a good check line, we know $\varphi(\overrightarrow{AM}) \leq \varphi(\overrightarrow{s_{i_0}s_{i_0+1}})$. Then $\varphi(\overrightarrow{EI}) \leq \varphi(\overrightarrow{EK})$. Therefore the area of $\triangle EIH$ is larger than (or equal to) the area of $\triangle EKH$. Thus,

$$Er_\epsilon(ER(CH(X), i_0)) \leq Er_\epsilon(C_\epsilon^{\mathsf{Large}}).$$

From (5),(6), and the above, we have the lemma.
□

Now we can prove our goal of step (2).

Lemma 4.8. Suppose that $CS_\epsilon(X) \neq \perp$. Then, $Er(MR(CH(X))) < \epsilon$.

Proof.

$$\begin{aligned}
& Er(MR(CH(X))) \\
&= \frac{\text{area}(MR(CH(X)) \oplus C)}{\text{area}(C)} \\
&= \frac{\text{area}(C - MR(CH(X)))}{\text{area}(C)} + \frac{\text{area}(MR(CH(X)) - C)}{\text{area}(C)}
\end{aligned}$$

$$\begin{aligned}
& \frac{\text{area}(C - MR(CH(X)))}{\text{area}(C)} \\
&< \frac{\text{area}(C - C_\epsilon^{\mathsf{Small}})}{\text{area}(C)} \\
&= Er(C_\epsilon^{\mathsf{Small}}) \\
&< \tfrac{4}{13}\epsilon \ (\because \text{lemma 4.6})
\end{aligned}$$

$$\begin{aligned}
& \frac{\text{area}(MR(CH(X)) - C)}{\text{area}(C)} \\
&< \frac{\text{area}(MR(CH(X)) - C_\epsilon^{\mathsf{Small}})}{\text{area}(C)} \\
&= Er_\epsilon(MR(CH(X))) \\
&< Er_\epsilon(C_\epsilon^{\mathsf{Large}}) \ (\because \text{lemma 4.7}) \\
&= Er(C_\epsilon^{\mathsf{Large}}) + Er(C_\epsilon^{\mathsf{Small}}) \\
&< (\tfrac{4}{11} + \tfrac{4}{13})\epsilon \ (\because \text{lemma 4.6})
\end{aligned}$$

$$\begin{aligned}
\therefore\ & Er(MR(CH(X))) \\
&< \tfrac{4}{13}\epsilon + (\tfrac{4}{11} + \tfrac{4}{13})\epsilon \\
&= \tfrac{140}{143}\epsilon < \epsilon
\end{aligned}$$

□

Clearly Theorem 4.1 is derived from Lemma 4.3 and lemma 4.8.

5 Conclusion

In this paper, we present an algorithm A_0 to detect a general rectangle. That is, for any $\epsilon > 0$, any $\delta, 0 < \delta < 1$, and any rectangle C, if n, the number of examples to the algorithm, is larger than $4\{(13/\epsilon)^2(\log 4 + \log(1/\delta)) + 1\}$, then the following probability is guaranteed.

$$\Pr_{\text{Sample Data}}\{Er(H|C) < \epsilon\} > 1 - \delta,$$

where H is the output of the algorithm A_0.

This idea will be applied to a PAC learning algorithm of general rectangles under the uniform distribution. (see [Kaw93] for detail).

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