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グラフの辺彩色及び f-辺彩色アルゴリズム

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グラフの辺彩色は各点に接続する辺の色が必ず異なるように辺に彩色することである．グラフの f-辺彩色は各点 v に接続している辺の高々 $f(v)$ 本しか同じ色で塗られないように辺に彩色することである．本論文は二部グラフ、平面グラフ、種数 g のグラフ、部分 k 木、s-縮退グラフ、樹化数 a のグラフ等、種々のクラスのグラフについて最少色数で辺彩色する効率の良い逐次アルゴリズム及び並列アルゴリズムを与える．

Edge-Coloring and f-Coloring for Various Classes of Graphs

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Abstract

In an ordinary edge-coloring of a graph $G = (V, E)$ each color appears at each vertex $v \in V$ at most once. An f-coloring is a generalized coloring in which each color appears at each vertex $v \in V$ at most $f(v)$ times. This paper gives efficient sequential and parallel algorithms which find ordinary edge-colorings and f-colorings for various classes of graphs such as bipartite graphs, planar graphs, graphs of fixed genus, partial k-trees, s-degenerate graphs, graphs of fixed arboricity etc.

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1 Introduction

This paper deals with a *simple* graph G which has no multiple edge or no self-loops. An *edge-coloring* of a graph G is to color all the edges of G so that no two adjacent edges are colored with the same color. The minimum number of colors needed for an edge-coloring is called the *chromatic index* of G, denoted by $\chi'(G)$. In this paper the *maximum degree* of a graph G is denoted by $\Delta(G)$ or simply by Δ. Vizing showed that $\chi'(G) = \Delta$ or $\Delta + 1$ for any simple graph G [6, 21]. The *edge-coloring problem* is to find an edge-coloring of G using $\chi'(G)$ colors. Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V$. Then an *f-coloring* of G is to color all the edges of G so that, for each vertex $v \in V$, at most $f(v)$ edges incident with v are colored with the same color. The minimum number of colors needed for an f-coloring is called the *f-chromatic index* of G and denoted by $\chi'_f(G)$. The *f-coloring problem* is to find an f-coloring of G using $\chi'_f(G)$ colors. Let $\Delta_f(G) = \max_{v \in V} \lceil d(v)/f(v) \rceil$ where $d(v)$ is the *degree* of vertex v. Hakimi and Kariv have proved that $\chi'_f(G) = \Delta_f$ or $\Delta_f + 1$ for any simple graph G [9]. An ordinary edge-coloring is a special case of an f-coloring for which $f(v) = 1$ for every vertex $v \in V$.

The f-coloring has applications to scheduling problems like the file transfer problem in a computer network [4, 15, 16]. In the model a vertex of a graph G represents a computer, and an edge does a file which one wishes to transfer between the two computers corresponding to its ends. The integer $f(v)$ is the number of communication ports available at a computer v. The edges colored with the same color represent files that can be transferred in the network simultaneously. Thus an f-coloring of G using $\chi'_f(G)$ colors corresponds to a scheduling of file transfers with the minimum finishing time.

Since the ordinary edge-coloring problem is NP-complete [11], the f-coloring problem is also NP-complete. Therefore it is very unlikely that there exists a sequential algorithm which solves the ordinary edge-coloring problem or the f-coloring problem in polynomial time. However it is known that any simple graph G can be edge-colored with $\Delta + 1$ colors in polynomial time [18, 20]. The best known algorithm for edge-coloring G with $\Delta + 1$ colors runs in time $O(\min\{n\Delta \log n, m\sqrt{n \log n}\})$ [8]. Throughout this paper n denotes the number of the vertices and m the number of the edges in G. Hakimi and Kariv's proof [9] yields a sequential algorithm which f-colors any graph using $\Delta_f + 1$ colors in time $O(mn)$. On the other hand, there are polynomial-time algorithms which find an edge-coloring using $\chi'(G)$ colors for restricted classes of graphs, as follows:

(a) an $O(m \log n)$-time sequential algorithm for bipartite graphs [5, 7];
(b) a linear-time sequential algorithm for planar graphs of $\Delta \geq 19$ [3];
(c) an $O(n \log n)$-time sequential algorithm for planar graphs of $\Delta \geq 9$ [2];
(d) a linear-time sequential algorithm for series-parallel multigraphs [24]; and
(e) a linear-time sequential algorithm for partial k-trees [22].

However no efficient algorithms have been obtained for the f-coloring problem even for restricted classes of graphs. On the other hand, NC parallel algorithms for finding an optimal edge-coloring have been obtained only for a few restricted classes of graphs such as series-parallel multigraphs [24], partial k-trees [23] and planar graphs with maximum degree $\Delta \geq 9$ or $\Delta \geq 19$ [2, 3]. However an NC parallel algorithm for finding edge-colorings with $\Delta + 1$ colors has not been known for general graphs except the case when Δ is bounded [13].

In this paper we first obtain new upper bounds on the chromatic index for three classes of graphs: graphs of genus $g \geq 1$, s-degenerate graphs, and graphs of arboricity a. (These new bounds together with known ones are listed in Table 1.) We then give efficient sequential and NC parallel algorithms which find edge-colorings for the three classes of graphs. (These new algorithms together with known ones are listed in Table 2.) We next obtain new upper bounds on the f-chromatic index for six classes of graphs: bipartite graphs, planar graphs, graphs of genus $g \geq 1$, partial k-trees, s-degenerate graphs and graphs of arboricity a. (These results are listed in Table 3.) We finally give efficient sequential and NC parallel algorithms which find f-colorings for the six classes of graphs. (These results are listed in Table 4.) In this paper the parallel computation model we use is a concurrent-read exclusive-write parallel random access machine (CREW PRAM).

Classes of graphs	Upper bounds on χ'	Refs.
Simple graph	$\Delta + 1$	[21]
Bipartite	Δ	[14]
Series-parallel	$\max\{4, \Delta\}$	[18]
Partial k-tree	$\max\{2k, \Delta\}$	[22]
Planar graph	$\max\{8, \Delta\}$	[6]
Genus $g \geq 1$	$\max\{2\lfloor(5 + \sqrt{48g+1})/2\rfloor, \Delta\}$	ours
Degeneracy s	$\max\{2s, \Delta\}$	ours
Arboricity a	$\max\{4a - 2, \Delta\}$	ours

Table 1: Upper bounds on the chromatic index.

Classes of graphs	Sequential	Parallel		Numbers of used colors	Refs.
	Time	Time	Operations		
Simple graph	$O(\min\{n\Delta \log n, m\sqrt{n \log n}\})$	open		$\Delta + 1$	[8]
Bipartite	$O(m \log n)$	open		Δ	[5, 7]
Partial k-tree	$O(n)$	$O(\log n)$	$O(n)$	χ'	[22]
Planar graph	$O(n)$	$O(\log^2 n)$	$O(n \log^2 n)$	$\max\{19, \Delta\}$	[3]
	$O(n \log n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{9, \Delta\}$	[2]
Genus $g \geq 1$	$O(n \log n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{\lceil \frac{(9+\sqrt{48g+1})^2}{8} \rceil - 1, \Delta\}$	Ours
Degeneracy s	$O(n \log n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{\lceil (s+2)^2/2 \rceil - 1, \Delta\}$	Ours
Arboricity a	$O(n \log n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{\lceil (a+2)^2/2 \rceil - 1, \Delta\}$	Ours

Table 2: Algorithms of the ordinary edge-coloring.

2 Summary Tables

In this section we summarize old results and our new ones in four tables. We first define various classes of graphs. Let s be a positive integer. A graph G is *s-degenerate* if the vertices of G can be ordered $v_1, v_2, \cdots, v_n$ such that $d(v_i, G_i) \leq s$ for each i, $1 \leq i \leq n$, where $G_i = G - \{v_1, v_2, \cdots, v_{i-1}\}$ and $d(v_i, G_i)$ is the degree of vertex v in G_i [1]. That is, G is s-degenerate if and only if G can be reduced to the trivial (or degenerate) graph K_1 by the successive removal of vertices having degree at most s. The *degeneracy* $s(G)$ of G is the minimum integer s for which G is s-degenerate. The degeneracy is also called the Szekeres-Wilf number [19]. Clearly $s(G) \leq 5$ if G is planar.

The *genus* $g(G)$ of a graph G is the minimum number of handles which must be added to a sphere so that G can be embedded on the resulting surface. Of course, $g(G) = 0$ if and only if G is planar. We denote by $\delta(G)$ the *minimum degree* of vertices of G. It is known that $\delta(G) \leq \lfloor (5 + \sqrt{48g(G)+1})/2 \rfloor$ if $g(G) \geq 1$ [10, 12]. Any subgraph G' of G satisfies $g(G') \leq g(G)$, and hence $s(G) \leq \lfloor (5 + \sqrt{48g(G)+1})/2 \rfloor$ if $g(G) \geq 1$.

Classes of graphs	Upper bounds on χ'_f	Refs.
Simple graph	$\Delta_f + 1$	[9]
Bipartite	Δ_f	[9]
Series-parallel	$\max\{4, \Delta_f\}$	ours
Partial k-tree	$\max\{2k, \Delta_f\}$	ours
Planar graph	$\max\{8, \Delta_f\}$	ours
Genus $g \geq 1$	$\max\{2\lfloor (5 + \sqrt{48g+1})/2 \rfloor, \Delta_f\}$	ours
Degeneracy s	$\max\{2s, \Delta_f\}$	ours
Arboricity a	$\max\{4a - 2, \Delta_f\}$	ours

Table 3: Upper bounds on the f-chromatic index.

Classes of graphs	Sequential Time	Parallel Time	Parallel Operations	Numbers of used colors	Refs.
Simple graph	$O(\min\{m\Delta_f \log n, m\sqrt{m \log n}\})$	open		$\Delta_f + 1$	Ours
Bipartite	$O(m \log n)$	open		Δ_f	Ours
Planar graph	$O(n)$	$O(\log^2 n)$	$O(n \log^2 n)$	$\max\{19, \Delta_f\}$	Ours
	$O(n \log n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{9, \Delta_f\}$	Ours
Genus $g \geq 1$	$O(n \log n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{\lceil \frac{(9+\sqrt{48g+1})^2}{8} \rceil - 1, \Delta_f\}$	Ours
Partial k-tree	$O(n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{\lceil (k+2)^2/2 \rceil - 1, \Delta_f\}$	Ours
Degeneracy s	$O(n \log n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{\lceil (s+2)^2/2 \rceil - 1, \Delta_f\}$	Ours
Arboricity a	$O(n \log n)$	$O(\log^3 n)$	$O(n \log^3 n)$	$\max\{\lceil (a+2)^2/2 \rceil - 1, \Delta_f\}$	Ours

Table 4: Algorithms of the f-coloring.

A graph $G = (V, E)$ is a *k-tree* if either it is a complete graph on k vertices or it has a vertex $v \in V$ whose neighbors induce a clique of size k and $G - \{v\}$ is again a k-tree. A graph is a *partial k-tree* if and only if it is a subgraph of a k-tree. Clearly $s(G) \leq k$ for any partial k-tree G.

The *arboricity* $a(G)$ of a graph G is the minimum number of edge-disjoint forests into which G can be decomposed. Nash-Williams [17] proved that $a(G) = \max_{H \subseteq G} \lceil m(H)/(n(H) - 1) \rceil$, where H runs over all nontrivial subgraphs of G, $n(H)$ is the number of vertices and $m(H)$ the number of edges of H. His results immediately implies that $a(G) \leq s(G)$. Furthermore $a(G) \leq 3$ if G is planar, because $m(H) \leq 3n(H) - 3$ for any nontrivial subgraph H of G.

We are now ready to describe our results in detail. In Section 3 we obtain new upper bounds on the chromatic index, which are expressed in terms of Δ and one of the invariants $g(G)$, $s(G)$ and $a(G)$. The new bounds together with the known ones are listed in Table 1. The bound $\chi'(G) \leq \max\{2s(G), \Delta\}$ is a generalization of the known one that $\chi'(G) \leq \max\{2k, \Delta\}$ if G is a partial k-tree, since $s(G) \leq k$. Furthermore this bound implies that $\chi'(G) \leq \max\{2\lfloor (5 +$

$\sqrt{48g(G)+1})/2\rfloor, \Delta\}$ if $g(G) \geq 1$, and that $\chi'(G) \leq \max\{4a(G)-2, \Delta\}$ because $s(G) \leq 2a(G)-1$ as shown in Section 3.

The proofs of the upper bounds on the chromatic index immediately yield sequential algorithms which find an edge-coloring using colors no more than the bounds in $O(mn)$ time. In Section 4 we give a more efficient algorithm which may use colors more than the bounds. The algorithm edge-colors a graph G with $\max\{\lceil(a(G)+2)^2/2\rceil - 1, \Delta\}$ colors in time $O(n \log n)$. Since $a(G) \leq s(G)$ and $s(G) \leq \lfloor(5+\sqrt{48g(G)+1})/2\rfloor$, $\lceil(a(G)+2)^2/2\rceil \leq \lceil(s(G)+2)^2/2\rceil$ and $\lceil(a(G)+2)^2/2\rceil \leq \lceil(9+\sqrt{48g(G)+1})^2/8\rceil$. These consequences together with the known ones are listed in Table 2.

In Section 5 we prove that the f-coloring problem on a graph G can be reduced to the edge-coloring problem on a new graph G_f. Then, using the reduction, we derive new upper bounds on the f-chromatic index from the upper bounds on the chromatic index listed in Table 1. The new upper bounds are listed in Table 3. Furthermore, using the reduction, we derive new efficient sequential and NC parallel algorithms for the f-coloring problem from the algorithms for the edge-coloring problem listed in Table 2. The new algorithms are listed in Table 4.

3 Chromatic Index

By the classical Vizing's theorem, $\chi'(G) = \Delta$ or $\Delta + 1$ for any simple graph G [6, 21]. The main result of this section is the following theorem.

Theorem 3.1 $\chi'(G) = \Delta(G)$ *if* $\Delta(G) \geq 2s(G)$.

We observe the following upper bounds on the minimum degree $\delta(G)$ expressed in term of $a(G)$.

Lemma 3.2 $\delta(G) \leq 2a(G) - 1$.

Proof. Assume that a graph $G = (V, E)$ has no isolated vertices. Let $n = |V|$, and let n' be the number of all the vertices of degree $< 2a(G)$. Then clearly $n' + 2a(G)(n - n') \leq 2|E|$. On the other hand G can be partitioned into $a(G)$ forests, and any forest has at most $n-1$ edges. Therefore $|E| \leq a(G)(n-1)$. Thus $n' \geq 2a/(2a-1) > 1$. $\mathcal{Q.E.D.}$

If $g(G) \geq 1$, then $\delta(G) \leq \lfloor(5+\sqrt{48g(G)+1})/2\rfloor$ [10, 12]. Therefore by Lemma 3.2 we can immediately derive the following upper bounds on $s(G)$ in terms of $a(G)$ and $g(G)$. Note that $a(G') \leq a(G)$ and $g(G') \leq g(G)$ if G' is a subgraph of G.

Lemma 3.3

(a) $s(G) \leq 2a(G) - 1$.
(b) $s(G) \leq \lfloor(5+\sqrt{48g(G)+1})/2\rfloor$ *if* $g(G) \geq 1$.

By Theorem 3.1 and Lemma 3.3 we have the following corollary.

Corollary 3.4 $\chi'(G) = \Delta(G)$ *if*

(a) $\Delta(G) \geq 4a(G) - 2$; *or*
(b) $\Delta(G) \geq 2\lfloor(5+\sqrt{48g(G)+1})/2\rfloor$ *and* $g(G) \geq 1$.

4 Edge-Coloring

Theorem 3.1 and its proof yield an algorithm which edge-colors a graph G with $\max\{2s(G), \Delta\}$ colors. However the algorithm spends $O(mn)$ time, since it repeats "shifting a fan sequence" and "switching an alternating path" $O(m)$ times [23]. By Corollary 3.4 the algorithm edge-colors G with $\max\{4a(G)-2, \Delta\}$ colors in $O(mn)$ time. Since $a(G) \le 3$ for any planar graph G, the algorithm edge-colors a planar graph G with $\max\{10, \Delta\}$ colors in $O(n^2)$ time. On the other hand, an algorithm in [18, 20] edge-colors a planar graph G with $\max\{8, \Delta\}$ colors in $O(n^2)$ time. Furthermore two more efficient algorithms have been known: an algorithm in [3] edge-colors a planar graph G with $\max\{19, \Delta\}$ colors in $O(n)$ time, and an algorithm in [2] edge-colors a planar graph G with $\max\{9, \Delta\}$ colors in $O(n \log n)$ time.

In this section we give an $O(n \log n)$ algorithm for edge-coloring a graph G of fixed $a(G)$. The number of used colors may exceed $\max\{4a(G)-2, \Delta\}$ but does not exceed $\max\{c_a(G), \Delta\}$ where

$$c_a(G) = \left\lceil \frac{(a(G)+2)^2}{2} \right\rceil - 1.$$

If G is planar, then $a(G) \le 3$ and hence $c_a(G) = 12$. Thus our algorithm has a flavor of generalization of the two algorithms above in [2] and [3]. Furthermore we give a NC parallel algorithm. The main result of this section is the following.

Theorem 4.1 *A graph G of fixed arboricity $a(G)$ can be edge-colored by at most* $\max\{c_a(G), \Delta(G)\}$ *colors in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations.*

Since $a(G) \le s(G)$ and $s(G) \le \lfloor (5 + \sqrt{48g(G)+1})/2 \rfloor$ if $g(G) \ge 1$, we have the following corollary.

Corollary 4.2

(a) *A graph G of fixed $s(G)$ can be edge-colored by at most* $\max\{\lceil (s(G)+2)^2/2 \rceil - 1, \Delta\}$ *colors in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations.*

(b) *A graph G of fixed $g(G) \ge 1$ can be edge-colored by at most* $\max\{\lceil (9+\sqrt{48g(G)+1})^2/8 \rceil - 1, \Delta\}$ *colors in $O(n \log n)$ sequential time or in $O(\log^3 n)$ parallel time with $O(n \log^3 n)$ operations.*

5 f-Coloring

In this section we give efficient sequential and NC parallel algorithms for the f-coloring problem on various classes of graphs. We first show that the f-coloring problem on a graph G can be reduced to the edge-coloring problem on a new graph G_f defined below. We may assume without loss of generality that $f(v) \le d(v)$ for each $v \in V$. For each vertex $v \in V$ with $f(v) \ge 2$, replace v with $f(v)$ copies $v_1, v_2, \cdots, v_{f(v)}$, and attach the $d(v)$ edges incident with v to the copies; $\lceil d(v)/f(v) \rceil$ edges to each copy v_i, $1 \le i \le f(v)-1$, and the remaining edges to the last copy $v_{f(v)}$. Let G_f be the resulting graph. Clearly $\Delta(G_f) = \Delta_f(G) = \max_{v \in V} \lceil d(v)/f(v) \rceil$, and the number of edges in G_f is equal to that of G. If G is a simple graph, G_f is also a simple graph. Clearly an ordinary edge-coloring of G_f induces an f-coloring of G. One can easily observe that the following lemmas hold.

Lemma 5.1 *For a graph G there exists G_f such that*

(a) G_f *is bipartite if* G *is bipartite;*
(b) G_f *is planar if* G *is planar;*
(c) $g(G_f) \leq g(G)$*;*
(d) $s(G_f) \leq s(G)$*; and*
(e) $a(G_f) \leq a(G)$.

Lemma 5.2 *Let* $\mathcal{G}$ *be a class of graphs such that* $G_f \in \mathcal{G}$ *if* $G \in \mathcal{G}$*, and let* α*,* β *and* γ *be real numbers.*

(a) *If there exists a sequential algorithm which edge-colors any graph* G *in* $\mathcal{G}$ *with* $\max\{\alpha, \beta\Delta(G) + \gamma\}$ *colors in* $T(m)$ *time, then there exists a sequential algorithm which* f*-colors any graph* G *in* $\mathcal{G}$ *with* $\max\{\alpha, \beta\Delta_f(G) + \gamma\}$ *colors in* $O(T(m))$ *time.*

(b) *If there exists a parallel algorithm which edge-colors any graph* G *in* $\mathcal{G}$ *with* $\max\{\alpha, \beta\Delta(G)+\gamma\}$ *colors in* $T(m)$ *parallel time with* $P(m)$ *operations, then there exists a parallel algorithm which* f*-colors any graph* G *in* $\mathcal{G}$ *with* $\max\{\alpha, \beta\Delta_f(G) + \gamma\}$ *colors in* $O(T(m))$ *parallel time with* $O(P(m))$ *operations.*

Proof. (a) Let $G = (V, E)$ be a graph in $\mathcal{G}$. One can construct G_f from G in linear time. Using the assumed algorithm, one can find an edge-coloring of G_f using $\max\{\alpha, \beta\Delta_f(G) + \gamma\}$ colors in $O(T(m))$ time. The edge-coloring of G_f immediately induces an f-coloring of G using $\max\{\alpha, \beta\Delta_f(G) + \gamma\}$ colors. Thus one can find a desired f-coloring of G in $O(T(m))$ time in total.

(b) G_f can be easily obtained in $O(\log m)$ parallel time with $O(m)$ operations. $\mathcal{Q.E.D.}$

It is known that $\chi'(G) = \Delta(G)$ for bipartite graphs [14], $\chi'(G) \leq \max\{8, \Delta(G)\}$ for planar graphs [6], and $\chi'(G) \leq \max\{2k, \Delta(G)\}$ for partial k-trees [22]. Therefore, by Theorem 3.1, Corollary 3.4 and Lemmas 5.1, 5.2, we have the following upper bounds on χ'_f.

Theorem 5.3

(a) $\chi'_f(G) = \Delta_f(G)$ *if* G *is bipartite* [9]*;*
(b) $\chi'_f(G) \leq \max\{8, \Delta_f(G)\}$ *if* G *is planar;*
(c) $\chi'_f(G) \leq \max\{2\lfloor(5 + \sqrt{48g(G) + 1})/2\rfloor, \Delta_f(G)\}$ *if* $g(G) \geq 1$*;*
(d) $\chi'_f(G) \leq \max\{2s(G), \Delta_f\}$*;*
(e) $\chi'_f(G) \leq \max\{2k, \Delta_f(G)\}$ *if* G *is a partial* k*-tree; and*
(f) $\chi'_f(G) \leq \max\{4a(G) - 2, \Delta_f(G)\}$.

If G is a partial k-tree, then G_f is not always a partial k-tree, but $s(G_f) \leq s(G) \leq k$. therefore (e) above is an immediate consequence of (d). Using the previous results listed in Table 2 and ours in Theorem 4.1, Lemmas 5.1 and 5.2, we have the following results.

Theorem 5.4

(a) *Graphs* G *can be* f*-colored by at most* $\Delta_f(G)+1$ *colors in* $O(\min\{m\Delta_f \log n, m\sqrt{m \log n}\})$ *sequential time.*

(b) *Bipartite graphs* G *can be* f*-colored by* $\chi'_f(G)$ *colors in* $O(m \log n)$ *sequential time.*

(c) *Planar graphs* G *can be* f*-colored by* $\max\{9, \Delta_f(G)\}$ *colors in* $O(n \log n)$ *sequential time or in* $O(\log^2 n)$ *parallel time with* $O(n \log^2 n)$ *operations.*

(d) *Planar graphs* G *can be* f*-colored by* $\max\{19, \Delta_f(G)\}$ *colors in* $O(n)$ *sequential time or in* $O(\log^3 n)$ *parallel time with* $O(n \log^3 n)$ *operations.*

(e) *Graphs G of fixed* $g(G) \geq 1$ *can be f-colored by* $\max\{\lceil (9+\sqrt{48g(G)+1})^2/8 \rceil - 1, \Delta_f(G)\}$ *colors in* $O(n \log n)$ *sequential time or in* $O(\log^3 n)$ *parallel time with* $O(n \log^3 n)$ *operations.*

(f) *Graphs G of fixed* $s(G)$ *can be f-colored by* $\max\{\lceil (s(G)+2)^2/2 \rceil - 1, \Delta_f(G)\}$ *colors in* $O(n \log n)$ *sequential time or in* $O(\log^3 n)$ *parallel time with* $O(n \log^3 n)$ *operations.*

(g) *Partial k-trees G can be f-colored by* $\max\{\lceil (k+2)^2/2 \rceil - 1, \Delta_f(G)\}$ *colors for fixed k in* $O(n \log n)$ *sequential time or in* $O(\log^3 n)$ *parallel time with* $O(n \log^3 n)$ *operations.*

(h) *Graphs G of fixed* $a(G)$ *can be f-colored by* $\max\{\lceil (a(G)+2)^2/2 \rceil - 1, \Delta_f(G)\}$ *colors in* $O(n \log n)$ *sequential time or in* $O(\log^3 n)$ *parallel time with* $O(n \log^3 n)$ *operations.*

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