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# $f$-辺彩色問題の辺彩色問題への多項式時間変換可能性について

周 暁　　西関 隆夫
東北大学大学院情報科学研究科

**概 要**

グラフの辺彩色は各点に接続する辺の色が必ず異なるように辺を彩色することである．グラフの $f$-辺彩色は各点 $v$ に接続している辺の高々 $f(v)$ 本しか同じ色で塗られないように辺を彩色することである．本論文は $f$-辺彩色問題が辺彩色問題に多項式時間で変換できることを示す．

# The $f$-Coloring is Polynomial-Time Reducible to the Edge-Coloring

Xiao Zhou and Takao Nishizeki [1]
Department of System Information Sciences
Graduate School of Information Sciences
Tohoku University, Sendai 980-77, Japan

**Abstract**

In an edge-coloring of a graph $G$ each color appears at each vertex at most once. An $f$-coloring is a generalization of an edge-coloring in which each color appears at each vertex $v$ at most $f(v)$ times where $f$ is a function assigning a positive integer $f(v)$ to each vertex of $G$. In this paper we show that the $f$-coloring problem can be reduced in polynomial-time to the ordinary edge-coloring problem, that is, given a graph $G$, one can construct in polynomial-time a new graph such that an edge-coloring of the new graph using a minimum number of colors immediately induces an $f$-coloring of $G$ using a minimum number of colors.

[1] E-mail:(zhou|nishi)@ecei.tohoku.ac.jp

# 1 Introduction

This paper deals with a *simple* graph $G$ which has no multiple edge or no self-loops. An *edge-coloring* of a graph $G$ is to color all the edges of $G$ so that no two adjacent edges are colored with the same color. The minimum number of colors needed for an edge-coloring is called the *chromatic index* of $G$ and denoted by $\chi'(G)$. In the paper the *maximum degree* of a graph $G$ is denoted by $\Delta(G)$ or simply by $\Delta$. Vizing showed that $\chi'(G) = \Delta$ or $\Delta + 1$ for any simple graph $G$ [7, 22]. The *edge-coloring problem* is to find an edge-coloring of $G$ using $\chi'(G)$ colors. Let $f$ be a function which assigns a positive integer $f(v)$ to each vertex $v \in V$. Then an *$f$-coloring* of $G$ is to color all the edges of $G$ so that, for each vertex $v \in V$, at most $f(v)$ edges incident to $v$ are colored with the same color. Thus a $f$-coloring of $G$ is a partition of $G$ into spanning subgraphs whose vertex-degrees are bounded by $f$. An ordinary edge-coloring is a special case of an $f$-coloring such that $f(v) = 1$ for every vertex $v \in V$. The minimum number of colors needed for an $f$-coloring is called the *$f$-chromatic index* of $G$ and denoted by $\chi'_f(G)$. The *$f$-coloring problem* is to find an $f$-coloring of $G$ using $\chi'_f(G)$ colors. Let $\Delta_f(G) = \max_{v \in V} \lceil d(v)/f(v) \rceil$ where $d(v)$ is the *degree* of vertex $v$. Hakimi and Kariv [9] have proved that for any simple graph $G = (V, E)$

$$\Delta_f \leq \chi'_f(G) \leq \max_{v \in V} \left\lceil \frac{d(v)+1}{f(v)} \right\rceil .$$

Their result implies that $\chi'_f(G) = \Delta_f$ or $\Delta_f + 1$.

The edge-coloring and $f$-coloring have applications to scheduling problems like the file transfer problem in computer networks [5, 15, 16]. In the model a vertex of a graph $G$ represents a computer, and an edge does a file which one wishes to transfer between the two computers corresponding to its ends. The integer $f(v)$ is the number of communication ports available at a computer $v$. The edges colored with the same color represent files that can be transferred in the network simultaneously. Thus an $f$-coloring of $G$ using $\chi'_f(G)$ colors corresponds to a scheduling of file transfers with the minimum finishing time.

Since the ordinary edge-coloring problem is NP-complete [11], the $f$-coloring problem is also NP-complete in general. Therefore it is very unlikely that there exists an algorithm which solves the ordinary edge-coloring problem or the $f$-coloring problem in polynomial time. Many approximation algorithms of polynomial-time complexity have been developed for the problems [2, 3, 10, 16, 17, 19, 20, 22, 24]. Any simple graph $G$ can be edge-colored with $\Delta + 1$ colors in polynomial time [18, 20]. The best known algorithm for edge-coloring $G$ with $\Delta + 1$ colors runs in time $O(\min\{n\Delta \log n, m\sqrt{n \log n}\})$ [8]. Throughout the paper $n$ denotes the number of the vertices and $m$ the number of the edges in $G$. The proof of Hakimi and Kariv's theorem [9] immediately yields an algorithm to $f$-color any graph with $\Delta_f + 1$ colors in time $O(mn)$. On the other hand, efficient exact algorithms have been developed for various classes of graphs: bipartite graphs [6, 9], planar graphs of large maximum degree [2, 3], series-parallel graphs [25], partial $k$-trees [1, 23], degenerated graphs of large maximum degree [24], etc.

In this paper we show that the $f$-coloring problem on any simple graph $G$ can be reduced in polynomial-time to the ordinary edge-coloring problem on a new graph $G^*_f$. That is, given $G$, one can construct in polynomial-time $G^*_f$ such that $\chi'_f(G) = \chi'(G^*_f)$. Thus the $f$-coloring problem is not more intractable than the ordinary edge-coloring problem although the $f$-coloring problem looks to be more difficult than the edge-coloring problem. Our construction is similar to one employed by Tutte's $f$-factor theorem [21], but is much more complicated.

# 2 Preliminaries

In this section we give some definitions. Let $G = (V, E)$ denote a graph with vertex set $V$ and edge set $E$. We often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. The paper deals with *simple* graphs without multiple edges or self-loops. An edge joining vertices $u$ and $v$ is denoted by $(u, v)$. The *degree* of vertex $v \in V(G)$ is denoted by $d(v, G)$ or simply by $d(v)$. The *maximum degree* of $G$ is denoted by $\Delta(G)$, or simply by $\Delta$.

Clearly $\chi'(G) \geq \Delta(G)$. A bipartite graph $B = (U, W, E)$ is a graph such that $v \in U$ and $w \in W$ for every edge $(v, w) \in E$. By Konig's theorem $\chi'(G) = \Delta(G)$ if $G$ is bipartite [7, 13].

Let $f$ be a function which assigns a positive integer $f(v)$ to each vertex $v \in V$. One may assume without loss of generality that $f(v) \leq d(v)$ for each vertex $v \in V(G)$. Let $d_f(v, G) = \lceil d(G, v)/f(v) \rceil$ for $v \in V$, and let $\Delta_f(G) = \max\{d_f(v) | v \in V(G)\}$. We often denote $d_f(v, G)$ simply by $d_f(v)$. Clearly $\chi'_f(G) \geq \Delta_f(G)$. It is known that $\chi'_f(G) = \Delta_f(G)$ if $G$ is bipartite [9]. For an $f$-coloring $\varphi$ of $G$, denote by $\#\varphi$ the number of colors used by $\varphi$. For a color $c$ and a vertex $v$, denote by $\#\varphi(v, c)$ the number of edges of $G$ which are incident to $v$ and colored $c$ by $\varphi$. Clearly $\#\varphi(v, c) \leq f(v)$ for each vertex $v$ and each color $c$. Let $\varphi$ use colors $c_1, c_2, \cdots, c_{\#\varphi}$. Vertex $v$ *misses* color $c_i$, $i = 1, 2, \cdots, \#\varphi$, if none of the edges incident to $v$ is colored with $c_i$ by $\varphi$.

Consider an edge-coloring $\varphi$ of a graph $G$. Denote by $G[c, c']$ the subgraph of $G$ induced by the edges colored with $c$ and $c'$. Clearly each connected component of $G[c, c']$ is a path or a cycle, whose edges are colored alternately with $c$ and $c'$. We call such a path (or a cycle) a *$cc'$-alternating path* (or *cycle*). A vertex $v$ is an end-vertex of such a $cc'$-alternating path if and only if $v$ misses $c$ or $c'$. For a vertex $v$, denote by $P(v; c, c')$ a $cc'$-alternating path or cycle containing $v$. *Switching path* $P(v; c, c')$ means to interchange colors $c$ and $c'$ in $P(v; c, c')$. Switching an alternating path (or cycle) yields another coloring of $G$.

# 3 Reduction

Clearly $\chi'_f(G) = 1$ if $\Delta_f(G) = 1$. By Hakim and Kariv's theorem $\chi'_f(G) = 2$ or $3$ if $\Delta_f(G) = 2$. Furthermore one can easily observe that the following lemma holds.

**Lemma 1.** *Let $G = (V, E)$ be a connected graph with $\Delta_f(G) = 2$. Then $\chi'_f(G) = 3$ if and only if the following* (i) *and* (ii) *hold:*

(i) *$d(v, G) = 2f(v)$ for every vertex $v \in V$, and*

(ii) *$|E|$ is odd.*

Thus the $f$-coloring problem can be easily solved in linear time if $\Delta_f(G) \leq 2$. Therefore, in the remaining of this section, we may assume that $\Delta_f(G) \geq 3$.

The following simple reduction is known [24]. For each vertex $v \in V$, replace $v$ with $f(v)$ copies $v_1, v_2, \cdots, v_{f(v)}$, and attach the $d(v)$ edges incident with $v$ to the copies; $\lceil d(v)/f(v) \rceil$ or $\lfloor d(v)/f(v) \rfloor$ edges to each copy $v_i$, $1 \leq i \leq f(v)$. Let $G_f$ be the resulting graph. Clearly $\Delta(G_f) = \Delta_f(G) = \max_{v \in V} \lceil d(v)/f(v) \rceil$. Since an edge-coloring of $G_f$ induces an $f$-coloring of $G$ using the same number of colors,

$$\chi'_f(G) \leq \chi'(G_f). \tag{1}$$

However, the equality in (1) does not always hold.

In this paper we show that, given a graph $G$, one can construct in polynomial-time a new graph $G^*_f$ such that

$$\chi'_f(G) = \chi'(G^*_f).$$

Given an edge-coloring of $G^*_f$ with $\chi'(G^*_f)$ colors, one can find in polynomial time an $f$-coloring of $G$ using $\chi'_f(G)$ colors. The main result of this paper is the following theorem, whose proof will be given later.

**Theorem 2.** *For any graph $G = (V, E)$ and a function $f$ such that $\Delta_f(G) \geq 3$, there is a graph $G^*_f = (V^*_f, E^*_f)$ such that $\chi'_f(G) = \chi'(G^*_f)$ and $|E^*_f|$ is polynomial in $|E|$.*

An $f$-*factor* of a graph $G = (V, E)$ is a spanning subgraph $G'$ of $G$ such that $d(v, G') = f(v)$ for every $v \in V$. For each vertex $v$ of $G$, replace $v$ with a complete bipartite graph $K_{f(v),d(v)}$ and attach to each of the $d(v)$ right vertices of $K_{f(v),d(v)}$ one of the $d(v)$ edges which were incident to $v$ in $G$. Let $G_T$ be the resulting graph. By Tutte's classical $f$-factor theorem [21], $G$ has an $f$-factor if and only if $G_T$ has a complete matching. Our Theorem 2 has a flavor of generalization of Tutte's theorem.

We first construct a new graph $G^*$ from $G$ in the following way: for each vertex $v \in V$ add $\Delta_f(G)f(v) - d(v, G)$ new vertices to $G$ and join $v$ with these new vertices. (See Figure 1.) Let $G^* = (V^*, E^*)$ be the resulting graph. We denote the sets of new vertices and edges by $V'$ and $E'$, respectively. Thus $V^* = V \cup V'$ and $E^* = E \cup E'$. Extend the function $f$ as follows: $f(v) = 1$ for each vertex $v \in V'$. Clearly $d(v, G^*) = \Delta_f(G)f(v)$ for each vertex $v \in V$, and $d(v, G^*) = 1$ for each vertex $v \in V'$. Hence $\Delta_f(G^*) = \Delta_f(G)$, One can easily observe that the following lemma holds.

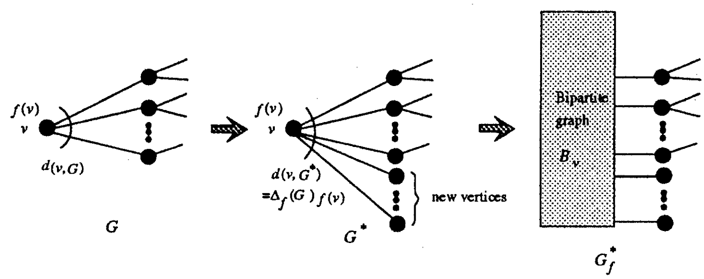


Figure 1: Transformations from $G$ to $G^*$ and $G^*_f$.

**Lemma 3.** $\chi'_f(G) = \chi'_f(G^*)$.

We next construct a graph $G^*_f$ from $G^*$ as follows. For positive integers $\alpha$ and $\beta$, let $B(\alpha, \beta) = (U, W_a \cup W_b, E_B)$ be a connected bipartite graph such that

- $d(v) = \alpha$ for each vertex $v \in U \cup W_a$,
- $d(v) = \alpha - 1$ for each vertex $v \in W_b$, and
- $|W_b| = \alpha\beta$.

We have the following lemma on bipartite graphs $B(\alpha, \beta)$, whose proof will be given later.

**Lemma 4.** *For any $\alpha \geq 3$ and $\beta \geq 1$ there is a bipartite graph $B(\alpha, \beta)$ such that*

(i) $|E(B)| = O(\alpha^3\beta \log_{\alpha-1} \alpha\beta)$; *and*

(ii) *for any partition of $W_b$ into $\alpha$ subsets $W_1, W_2, \cdots, W_\alpha$ such that $|W_i| = \beta$, there is an edge-coloring $\varphi$ of $B(\alpha, \beta)$ with $\alpha$ colors $c_1, c_2, \cdots, c_\alpha$ such that each of the $\beta$ vertices in $W_i$ misses color $c_i$ for every $i = 1, 2, \cdots, \alpha$.*

For each vertex $v \in V(G^*)$ such that $d(v, G^*) \geq 2$, replace $v$ with a copy $B_v = (U, W_a \cup W_b, E_B)$ of $B(\alpha, \beta_v)$ such that $\alpha = \Delta_f(G)$ and $\beta_v = f(v)$. and attach to each of the vertices in

$W_b$ one of the edges which were incident to $v$ in $G^*$ (the set of these edges is denoted by $E_v$). (See Figure 1.) Let $G_f^*$ be the resulting graph. Clearly

$$E(G^*) = \bigcup_{v \in V} E_v,$$

$d(v, G_f^*) = 1$ or $\Delta_f(G)$ for each vertex $v \in V(G_f^*)$, and hence $\Delta(G_f^*) = \Delta_f(G) = \Delta_f(G^*)$.

We are now ready to prove Theorem 2.

**Proof of Theorem 2.**

(a) $\chi'_f(G) \le \chi'(G_f^*)$.

Vizing's theorem [7, 22] implies that $\chi'(G_f^*) = \Delta(G_f^*)$ or $\Delta(G_f^*)+1$, and Hakimi and Kariv's theorem [9] implies that $\chi'_f(G^*) = \Delta_f(G^*)$ or $\Delta_f(G^*)+1$. If $\chi'(G_f^*) = \Delta(G_f^*) + 1$, then

$$\chi'(G_f^*) = \Delta(G_f^*) + 1 = \Delta_f(G^*) + 1 \ge \chi'_f(G^*) = \chi'_f(G).$$

Thus we may assume that $\chi'(G_f^*) = \Delta(G_f^*)$: there is an edge-coloring $\varphi$ of $G_f^*$ using $\Delta(G_f^*)$ ($= \Delta_f(G^*)$) colors. Then all $\Delta(G_f^*)$ colors appear around every vertex of maximum degree $\Delta(G_f^*)$. Let $\psi$ be a restriction of $\varphi$ to $E(G^*)$: $\psi(e) = \varphi(e)$ for each edge $e \in E(G^*)$. Note that $E(G^*) \subseteq E(G_f^*)$. We claim that $\psi$ is an $f$-coloring of $G^*$ and hence $\chi'(G_f^*) = \Delta(G_f^*) \ge \chi'_f(G^*) = \chi'_f(G)$. It suffices to show that $\#\psi(v,c) = f(v)$ for each vertex $v \in V$ and each color $c$, that is, exactly $f(v)$ edges in $E_v$ are colored with the same color $c$.

For any two colors $c_1$ and $c_2$, every $c_1c_2$-alternating path (not a cycle) in $G_f^*$ ends only at a vertex of degree 1 and does not end at any vertex in a bipartite graph $B_v$ for any vertex $v \in V$ in $G_f^*$. Therefore, the edges in $E_v$ appearing in a $c_1c_2$-alternating path (or cycle) $P$, are colored alternately with $c_1$ and $c_2$ in $P$, and exactly one half of them are colored with $c_1$ and the other half with $c_2$. Furthermore every edge in $E_v$ colored with $c_1$ or $c_2$ is contained in exactly one of the $c_1c_2$-alternating paths. Thus the same number of edges in $E_v$ are colored with $c_1$ and $c_2$, respectively. Therefore exactly $f(v)$ edges among the $f(v)\Delta(G_f^*)$ edges in $E_v$ are colored with each of the $\Delta(G_f^*)$ colors.

(b) $\chi'_f(G) \ge \chi(G_f^*)$.

If $\chi'_f(G^*) = \Delta_f(G^*) + 1$, then

$$\chi'_f(G) = \chi'_f(G^*) = \Delta_f(G^*) + 1 = \Delta(G_f^*) + 1 \ge \chi'(G_f^*).$$

Thus we may assume that $\chi'_f(G^*) = \Delta_f(G^*)$. Then there is an $f$-coloring $\psi$ of $G^*$ using $\alpha$ colors $c_1, c_2, \cdots, c_\alpha$ where $\alpha = \Delta_f(G^*) = \Delta(G_f^*)$. We now claim that there is an edge-coloring $\varphi$ of $G_f^*$ using the $\alpha$ colors. Let $\varphi(e) = \psi(e)$ for each edge $e \in E(G^*) = \cup_{v\in V} E_v$. For each vertex $v \in V$ and for each color $c_i$, $i = 1, 2, \cdots, \alpha$, let $W_i(v)$ be the set of vertices $v \in W_b$ in $B_v$ which is an end of edge $e \in E_v$ such that $\psi(e) = c_i$. Note that $|W_i(v)| = \beta_v = f(v)$. By Lemma 4 (ii) there is an edge-coloring $\eta_v$ of $B_v = B(\alpha, \beta)$ using the $\alpha$ colors $c_1, c_2, \cdots, c_\alpha$ such that each vertex $w \in W_i(v)$ misses color $c_i$ for every $i$, $1 \le i \le d(v)$. Let $\varphi(e) = \eta_v(e)$ for each edge $e \in E(B_v)$ and each vertex $v \in V$. Then $\varphi$ is a correct edge-coloring of $G_f^*$ using $\Delta(G_f^*)$ colors. Therefore

$$\chi'(G_f^*) \le \#\varphi = \#\psi = \chi'_f(G^*) = \chi'_f(G).$$

(c) $|E(G_f^*)|$ is polynomial in $|E|$.

By Lemma 4 (i) we have $|E(B_v)| = O(\alpha^3\beta_v \log_{\alpha-1} \alpha\beta_v)$ for each vertex $v \in V$ where $\alpha = \Delta_f(G)$ and $\beta_v = f(v)$. Noting $\Delta_f(G) \le \Delta(G)$, $f(v) \le \Delta(G)$, and $\sum_{v\in V} f(v) \le$

$\sum_{v\in V} d(v,G) \le 2|E|$, we have

$$\begin{aligned}|E(G_f^*)| &= O\left(|E| + \sum_{v\in V}|E(B_v)|\right)\\ &= O\left(|E| + (\Delta_f(G))^3\sum_{v\in V} f(v)\log_{\alpha-1}\Delta(G)\right)\\ &\le O\left(|E|(\Delta_f(G))^3\log_{\alpha-1}\Delta(G)\right).\end{aligned}$$

*Q.E.D.*

In the remaining of this section we prove Lemma 4. We first consider an edge-coloring $\varphi$ of a complete bipartite graph $K_{\alpha-1,\alpha-1} = (U_b^1, W_b^1, E_B^1)$ using $\alpha$ colors $c_1, c_2, \cdots, c_\alpha$. Clearly each vertex $v \in U_b^1 \cup W_b^1$ misses exactly one color. Let $U_b^1 = \{u_1, u_2, \cdots, u_{\alpha-1}\}$ and $W_b^1 = \{w_1, w_2, \cdots, w_{\alpha-1}\}$. Let $S = \{s_1, s_2, \cdots, s_{\alpha-1}\}$ where $s_i \in \{c_1, c_2, \cdots, c_\alpha\}$ for each $i$, $1 \le i \le \alpha-1$. Let $T = \{t_1, t_2, \cdots, t_{\alpha-1}\}$ be any permutation of $S$.
Then we have the following lemma.

**Lemma 5.** *There is an edge-coloring $\varphi$ of $K_{\alpha-1,\alpha-1} = (U_b^1, W_b^1, E_B^1)$ using $\alpha$ colors $c_1, c_2, \cdots,$ $c_\alpha$ such that vertex $u_i$ misses color $s_i$ and $w_i$ misses $t_i$ for all $i$, $1 \le i \le \alpha-1$.*

For a positive integer $p$, let $B^p(\alpha) = (U_a^p \cup U_b^p, W_a^p \cup W_b^p, E_B^p)$ be a bipartite graph such that

- $d(v, B^p) = \alpha$ for each vertex $v \in U_a^p \cup W_a^p$,
- $d(v, B^p) = \alpha - 1$ for each vertex $v \in U_b^p \cup W_b^p$, and
- $|U_b^p| = |W_b^p| = (\alpha-1)^p$.

Clearly a complete bipartite graph $K_{\alpha-1,\alpha-1}$ is a bipartite graph $B^1(\alpha)$ with $U_a^1 = W_a^1 = \phi$. One can recursively construct a bipartite graph $B^p(\alpha)$ from $2(\alpha-1)^{p-1}$ copies of $B^1(\alpha)$ and $\alpha-1$ copies of $B^{p-1}(\alpha)$ as in Figure 2. The construction is similar as that of a well-known Clos permutation network [4, 12, 14]. $B^p(\alpha)$ consists of three stages connected in cascade: the first stage consists of $(\alpha-1)^{p-1}$ copies of $B^1(\alpha)$, the second $\alpha-1$ copies of $B^{p-1}(\alpha)$, and the third $(\alpha-1)^{p-1}$ copies of $B^1(\alpha)$. Vertex $w_i$ in the $j$th copy of $B^1(\alpha)$ on the first stage is joined to the $j$th vertex in $U_b^{p-1}$ of the $i$th copy of $B^{p-1}(\alpha)$ on the second stage for each $i$, $1 \le i \le \alpha-1$, and each $j$, $1 \le j \le (\alpha-1)^{p-1}$. Similarly right vertices on the second stage are joined to left vertices on the third stages. The sets $U_a^p, U_b^p, W_a^p, W_b^p$ of vertices in $B^p(\alpha)$ are defined as follows:

- $U_a^p$ is the set of left vertices in the second and third stages;
- $W_a^p$ is the set of right vertices in the first and second stages;
- $U_b^p$ is the set of leftmost vertices in $B^p(\alpha)$; and
- $W_b^p$ is the set of righttmost vertices in $B^p(\alpha)$.

Let $U_b^p = \{u_1, u_2, \cdots, u_{(\alpha-1)^p}\}$ and $W_b^p = \{w_1, w_2, \cdots, w_{(\alpha-1)^p}\}$, then we have the following lemma which is a generalization of Lemma 5.

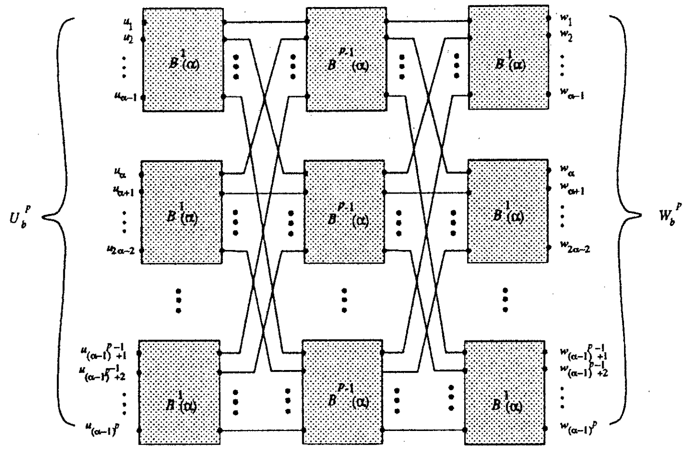


Figure 2: Construction of the bipartite graph $B^p(\alpha)$.

**Lemma 6.** *Let $\alpha$ and $p$ be two integers such that $\alpha \geq 3$ and $p \geq 1$. Let $S = \{s_1, s_2, \cdots, s_{(\alpha-1)^p}\}$ where $s_i \in \{c_1, c_2, \cdots, c_\alpha\}$ for each $i$, $1 \leq i \leq (\alpha-1)^p$. Let $T = \{t_1, t_2, \cdots, t_{(\alpha-1)^p}\}$ be any permutation of $S$. Then $B^p(\alpha) = (U_a^p \cup U_b^p, W_a^p \cup W_b^p, E_B^p)$ can be edge-colored with $\alpha$ colors $c_1, c_2, \cdots, c_\alpha$ so that $u_i$ misses $s_i$ and $w_i$ misses $t_i$ for all $i$, $1 \leq i \leq (\alpha-1)^p$.*

**Proof.** Similar to that for the Clos permutation network [4, 12]. $\mathcal{Q.E.D.}$

We are now ready to prove Lemma 4.

**Proof of Lemma 4.** To construct the bipartite graph $B^p(\alpha) = (U_a^p \cup U_b^p, W_a^p \cup W_b^p, E_B^p)$, we use $2(\alpha-1)^{p-1}$ copies of $B^1(\alpha)$ and $\alpha - 1$ copies of $B^{p-1}(\alpha)$. The number of new edges joining the first and second stages and joining the second and third stages is $2(\alpha-1)^p$ in total. Clearly $|E(B^1)| = (\alpha-1)^2$. Therefore the size of the graph $B^p(\alpha)$ is

$$\begin{aligned} |E(B^p)| &= (\alpha-1)|E(B^{p-1})| + 2(\alpha-1)^{p-1}|E(B^1)| + 2(\alpha-1)^p \\ &= (\alpha-1)|E(B^{p-1})| + 2\alpha(\alpha-1)^p. \end{aligned}$$

Solving the recursive equation above, we have

$$|E(B^p)| = (2p\alpha - \alpha - 1)(\alpha-1)^p < 2p\alpha(\alpha-1)^p.$$

We now construct a required bipartite graph $B(\alpha, \beta) = (U, W_a \cup W_b, E_B)$ from $B^p(\alpha)$. We choose vertices $w_1, w_2, \cdots, w_{\alpha\beta}$ as those of $W_b$. Therefore $p$ must satisfy $(\alpha-1)^p \geq \alpha\beta$. Thus we choose $p = \lceil \log_{\alpha-1} \alpha\beta \rceil$. Note that $\alpha \geq 3$ and $(\alpha-1)^p \leq (\alpha-1)\alpha\beta$. Then we have $|E(B^p)| = O(\alpha^3 \beta \log_{\alpha-1} \alpha\beta)$.

Let $Q = \{u_{\alpha\beta+1}, \cdots, u_{(\alpha-1)^p}\}$ and $R = \{w_{\alpha\beta+1}, \cdots, w_{(\alpha-1)^p}\}$. Join each vertex $u_i \in Q$ with $w_i \in R$ for each $i$, $\alpha\beta + 1 \leq i \leq (\alpha-1)^p$. Add $\beta$ new vertices $x_1, x_2, \cdots, x_\beta$, and join $x_i$ with $\alpha$ vertices $u_{(i-1)\alpha+1}, u_{(i-1)\alpha+2}, \cdots, u_{i\alpha} \in U_b^p$ for each $i = 1, 2, \cdots, \beta$. Let $B(\alpha, \beta) = (U, W_a \cup W_b, E_B)$ be the resulting bipartite graph, where

$$U \;=\; U_a^p \cup U_b^p,$$

$$W_a = W_a^p \cup R \cup \{x_1, x_2, \cdots, x_p\}, \text{ and}$$
$$W_b = \{w_1, w_2, \cdots, w_{\alpha\beta}\}.$$

Clearly $|E_B| = O(\alpha^3\beta \log_{\alpha-1} \alpha\beta)$. Thus we have proved (i).

We next prove (ii). By Lemma 6 there is an edge-coloring $\varphi$ of $B^p(\alpha)$ using $\alpha$ colors $c_1, c_2, \cdots, c_\alpha$ such that

- each $u_i$, $i = 1, 2, \cdots, \alpha\beta$, misses color $c_{i \bmod \alpha}$,
- every vertex $w \in W_i$, $i = 1, 2, \cdots, \alpha$, misses $c_i$, and
- every vertex $v \in Q \cup R$ misses $c_\alpha$

where we let $c_0 = c_\alpha$. We can extend $\varphi$ to a required edge-coloring $\varphi'$ of $B(\alpha, \beta)$ as follows:

$$\varphi'(e) = \begin{cases} \varphi(e) & \text{if } e \in E(B^p(\alpha)); \\ c_\alpha & \text{if } e = (u, w) \text{ with } u \in Q \text{ and } w \in R; \\ c_j & \text{if } e = (x_j, u_{(j-1)\alpha+i}) \text{ for each } i \text{ and } j,\ i = 1, 2, \cdots, \alpha \text{ and } j = 1, 2, \cdots, \beta. \end{cases}$$

$\mathcal{Q.E.D.}$

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