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de Bruijn 及び Kautz networks の本型埋め込みについて

蓮沼 徹, 柴田幸夫
群馬大学 工学部 情報工学科

概要：d-ary de Bruijn 及び Kautz digraphs が $(d+1)$ ページに埋め込み可能であることを証明する。この結果から binary de Bruijn 及び Kautz digraphs のページナンバーが 3 に決定される。又、shuffle-exchange graph のページナンバーも 3 であることを証明する。

Embedding de Bruijn and Kautz networks in books

Toru Hasunuma and Yukio Shibata

Department of Computer Science, Gunma University
1-5-1 Tenjin-cho, Kiryu, Gunma, 376 Japan
{hasunuma, shibata}@msc.cs.gunma-u.ac.jp

Abstract. We prove that d-ary de Bruijn and Kautz digraphs can be embedded in $(d+1)$ pages. From these results, the pagenumbers of binary de Bruijn and Kautz digraphs are determined to be 3. Also we prove that the pagenumber of shuffle-exchange graph is 3.

1 Introduction

In this paper, we investigate embeddings of digraphs or graphs in structures called books. A *book* consists of a line called the *spine* and some half-planes called *pages*, sharing the spine as a common boundary.

Although we define terminologies of bookembedding only for digraphs, those for graphs are similarly defined. A *bookembedding* of a digraph G consists of the linear ordering of the vertices of G on the spine and the assignment of arcs of G to pages such that there is no crossing of arcs on each page. The *pagenumber* of G is the minimum number of pages of books in which G can be embedded. The *width* of a page in a bookembedding is the maximum number of arcs that cross any line perpendicular to the spine of the book. The *pagewidth* of a bookembedding is the maximum width of any page of the book. The *cumulative pagewidth* of a bookembedding is the sum of the widths of all pages.

The bookembedding problem has been motivated by several areas of computer science (see [2]). The most famous one is the DIOGENES approach to fault-tolerant processor arrays, proposed by Rosenberg [8]. Bookembeddings that use few pages, small pagewidth and small cumulative pagewidth correspond to more hardware-efficient DIOGENES layout. We notice the problem from this point of view. That is, we study efficient bookembeddings of networks.

Results on bookembeddings until now can be roughly divided into two categories. One is related to bookembeddings of graph class of given genus ([9],[4],[5]). The other is related to bookembeddings of special graph family. In this category, complete graph ([1]), complete bipartite graph ([6]), tree, grid, X-tree, hypercube ([2]), butterfly-like graphs ([3]), binary de Bruijn graph and shuffle-exchange graph ([7]) have been studied. The pagenumbers of these graphs except for complete bipartite graph, hypercube, binary de Bruijn graph and shuffle-exchange graph have been determined.

In this paper, we treat d-ary de Bruijn digraph, d-ary Kautz digraph and shuffle-exchange graph. In [7], it has been shown that binary de Bruijn and shuffle-exchange graphs can be embedded in 5 pages. We prove that d-ary de Bruijn and Kautz digraphs can be embedded in $(d+1)$ pages. From these results, the pagenumbers of binary de Bruijn and Kautz digraphs are determined to be 3. (Considering the undirected versions, we get similar results for de Bruijn and Kautz graphs.) Also we prove that the pagenumber of shuffle-exchange graph is 3.

Let G and H be a digraph and a graph, respectively. The vertex set and arc set of G, the vertex set and edge set of H are denoted by $V(G), A(G), V(H)$ and $E(H)$, respectively. Let $Y \subseteq A(G)$. Then $\langle Y \rangle$ stands for the subdigraph of G arc-induced by Y. For $W \subseteq E(H)$, $\langle W \rangle$ stands for the subgraph of H edge-induced by W. The underlying multi-graph of G is a graph obtained from G by replacing each arc with the corresponding edge. Moreover replacing multi-edge of the underlying multi-graph of G with single edge, the underlying graph of G is constructed. Let $v \in V(G)$. The set of arcs which are incident from v is denoted by $\Gamma_G(v)$. The set of integers $\{0, 1, \ldots, d-1\}$ is denoted by Z_d. A vertex ordering of G is a bijection from $V(G)$ to $Z_{|V(G)|}$. A vertex ordering of a graph is similarly defined.

In section 2, we define the vertex ordering of de Bruijn digraph and show some properties of this ordering. In section 3, we first introduce an isomorphic decomposition of de Bruijn digraph. Then we prove that these isomorphic subdigraphs can be embedded in 2 pages under the vertex ordering defined in section 2. At the end of the section, we show that d-ary de Bruijn digraph can be embedded in $(d+1)$ pages. In section 4, we apply the bookembedding of de Bruijn digraph to bookembeddings of Kautz digraph and shuffle-exchange graph.

2 Vertex ordering of de Bruijn digraph

The D-dimensional d-ary de Bruijn digraph $B(d, D)$ is defined as follows.

$$\begin{cases} V(B(d,D)) = \{(v_0, v_1, \ldots, v_{D-1}) \mid v_i \in Z_d, 0 \le i < D\} \\ A(B(d,D)) = \{((v_0, v_1, \ldots, v_{D-1}), (v_1, \ldots, v_{D-1}, x)) \mid x \in Z_d\} \end{cases}$$

Let $v = (v_0, v_1, \ldots, v_{D-1}) \in V(B(d,D))$. If $v_i = \alpha$ for all $i \in Z_D$, then we abbreviate v to $(\alpha)^D$. We introduce some notations in order to define the vertex ordering of $B(d,D)$.

- $(v_0, v_1, \ldots, v_{D-1}) \oplus x = (v_0, v_1, \ldots, v_{D-1}, x)$ for $x \in Z_d$.
- $\rho((v_0, v_1, \ldots, v_{D-1})) = (v_0, v_1, \ldots, v_{D-2})$ for $D > 1$.
- $\sigma((v_0, v_1, \ldots, v_{D-1})) = v_{D-1}$.

We define the vertex ordering $\varphi_{d,D}$ recursively as follows.

Definition 2.1

1. $\varphi_{d,1}((i)) = i, \quad 0 \le i < d.$

2. Let $D \ge 1$. Assume that $\varphi_{d,D}$ is defined. For $v \in V(B(d,D))$, let

$$\begin{cases} O^+_{\varphi_{d,D}}(v) = \{w \in \Gamma_{B(d,D)}(v) \mid \varphi_{d,D}(v) < \varphi_{d,D}(w)\}, \\ O^-_{\varphi_{d,D}}(v) = \{w \in \Gamma_{B(d,D)}(v) \mid \varphi_{d,D}(w) < \varphi_{d,D}(v)\} \end{cases}$$

Also let $O^+_{\varphi_{d,D}}(v,i)$ be the i-th least element of $O^+_{\varphi_{d,D}}(v)$ with respect to the ordering of $\varphi_{d,D}$. Similarly, $O^-_{\varphi_{d,D}}(v,i)$ is the i-th least element of $O^-_{\varphi_{d,D}}(v)$. Let $\Phi(B(d,D)) = \{(\alpha)^D \in V(B(d,D)) \mid \alpha \in Z_d\}$. Now we define a bijection $\psi_{d,D+1}$ from $Z_{d^{D+1}}$ to $V(B(d,D+1))$ as follows.

$$\psi_{d,D+1}(d\varphi_{d,D}(v)+j) = \begin{cases} \left\{ \begin{array}{ll} v \oplus \sigma(O^-_{\varphi_{d,D}}(v, \sigma(v)-j)), & 0 \le j < \sigma(v), \\ v \oplus \sigma(v), & j = \sigma(v), \\ v \oplus \sigma(O^+_{\varphi_{d,D}}(v, d-j)), & \sigma(v) < j < d, \end{array} \right\}, & \text{if } v \in \Phi(B(d,D)), \\ v \oplus \sigma(O^-_{\varphi_{d,D}}(v, d-j)), \quad 0 \le j < d, & \text{if } v \notin \Phi(B(d,D)) \text{ and } O^+_{\varphi_{d,D}}(v) = \phi, \\ v \oplus \sigma(O^+_{\varphi_{d,D}}(v, d-j)), \quad 0 \le j < d, & \text{if } v \notin \Phi(B(d,D)) \text{ and } O^-_{\varphi_{d,D}}(v) = \phi. \end{cases}$$

Then let $\varphi_{d,D+1} = \psi^{-1}_{d,D+1}$.

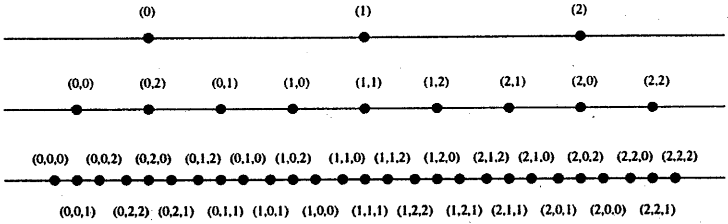

Figure 1: Configurations of the vertices of $B(3, D)$ by $\varphi_{3,D}$ for $D = 1, 2$ and 3.

Example: For $D = 1, 2$ and 3, configurations of the vertices of $B(3, D)$ on the spine by $\varphi_{3,D}$ are shown in Figure 1.

From the definition of $\varphi_{d,D}$, the following proposition is clear.

Proposition 2.2 *Let $u, v \in V(B(d, D))$, $D > 1$.*

$$\varphi_{d,D-1}(\rho(u)) < \varphi_{d,D-1}(\rho(v)) \Rightarrow \varphi_{d,D}(u) < \varphi_{d,D}(v).$$

On the following three lemmas which contain two propositions, we prove only the first proposition. Another proposition is similarly proved.

Lemma 2.3 *Let $D > 1$. Let $v \in V(B(d, D))$ such that $v \neq (\alpha)^D$ for any $\alpha \in Z_d$.*

1. $O^+_{\varphi_{d,D-1}}(\rho(v)) = \phi \Rightarrow O^+_{\varphi_{d,D}}(v) = \phi,$
2. $O^-_{\varphi_{d,D-1}}(\rho(v)) = \phi \Rightarrow O^-_{\varphi_{d,D}}(v) = \phi.$

Proof. Let $v = (v_0, v_1, \ldots, v_{D-1})$ such that $v \neq (\alpha)^D$ for any $\alpha \in Z_d$. Assume $O^+_{\varphi_{d,D-1}}(\rho(v)) = \phi$. Then $\varphi_{d,D-1}((v_1, \ldots, v_{D-2}, v_{D-1})) \leq \varphi_{d,D-1}((v_0, v_1, \ldots, v_{D-2}))$. Since $v \neq (\alpha)^D$ for any $\alpha \in Z_d$, $\varphi_{d,D-1}((v_1, \ldots, v_{D-2}, v_{D-1})) < \varphi_{d,D-1}((v_0, v_1, \ldots, v_{D-2}))$. From proposition 2.2,

$$\varphi_{d,D}((v_1, \ldots, v_{D-2}, v_{D-1}, x)) < \varphi_{d,D}((v_0, v_1, \ldots, v_{D-2}, v_{D-1})) \text{ for any } x \in Z_d.$$

That is $O^+_{\varphi_{d,D}}(v) = \phi$. □

Lemma 2.4 *Let $v \in V(B(d, D))$ and $\rho^{D-1}(v) = (\alpha)$.*

1. $\varphi_{d,D}(v) < \varphi_{d,D}((\alpha)^D) \Rightarrow O^+_{\varphi_{d,D}}(v) = \phi,$
2. $\varphi_{d,D}((\alpha)^D) < \varphi_{d,D}(v) \Rightarrow O^-_{\varphi_{d,D}}(v) = \phi.$

Proof. We use induction on D. When $D = 1$, the proposition is clear. Suppose $D > 1$. Let $v \in V(B(d, D))$ and $\rho^{D-1}(v) = (\alpha)$ such that $\varphi_{d,D}(v) < \varphi_{d,D}((\alpha)^D)$. From proposition 2.2, $\varphi_{d,D-1}(\rho(v)) \leq \varphi_{d,D-1}((\alpha)^{D-1})$. Suppose $\rho(v) \neq (\alpha)^{D-1}$. By the induction hypothesis, $O^+_{\varphi_{d,D-1}}(\rho(v)) = \phi$. Therefore from lemma 2.3, $O^+_{\varphi_{d,D}}(v) = \phi$. Then let $\rho(v) = (\alpha)^{D-1}$ and $\sigma(v) = v_{D-1} \neq \alpha$. By the construction of $\varphi_{d,D}$ from $\varphi_{d,D-1}$,

$$\varphi_{d,D-1}((\alpha)^{D-2} \oplus v_{D-1}) < \varphi_{d,D-1}((\alpha)^{D-1}).$$

From proposition 2.2,

$$\varphi_{d,D}(((\alpha)^{D-2} \oplus v_{D-1}) \oplus x) < \varphi_{d,D}((\alpha)^{D-1} \oplus v_{D-1}), \text{ for any } x \in Z_d.$$

That is $O^+_{\varphi_{d,D}}(v) = \phi$. □

We define the function $g_{\alpha,d,D}$ from $V(B(d, D))$ to Z_{D+1} as follows. If $v = (\alpha)^D$, then $g_{\alpha,d,D}(v) = 0$. If $\rho^{D-1}(v) \neq (\alpha)$, then $g_{\alpha,d,D}(v) = D$. Otherwise $g_{\alpha,d,D}(v) = i$ iff $\rho^{i-1}(v) \neq (\alpha)^{D-i+1}$ and $\rho^i(v) = (\alpha)^{D-i}$.

Lemma 2.5 *Let* $u, v \in V(B(d,D))$.

1. $\varphi_{d,D}(u) \le \varphi_{d,D}(v) \le \varphi_{d,D}((\alpha)^D) \Rightarrow g_{\alpha,d,D}(u) \ge g_{\alpha,d,D}(v)$.

2. $\varphi_{d,D}((\alpha)^D) \le \varphi_{d,D}(u) \le \varphi_{d,D}(v) \Rightarrow g_{\alpha,d,D}(u) \le g_{\alpha,d,D}(v)$.

Proof. We use induction on D. When $D = 1$, the proposition holds because $g_{\alpha,d,D}((\alpha)) = 0$ and $g_{\alpha,d,D}((x)) = 1$ for $x \neq \alpha$. Suppose $D > 1$. Let $u, v \in V(B(d,D))$ such that $\varphi_{d,D}(u) \le \varphi_{d,D}(v) \le \varphi_{d,D}((\alpha)^D)$. If $v = (\alpha)^D$, then $g_{\alpha,d,D}(v) = 0$, so the proposition clearly holds. Suppose $v \neq (\alpha)^D$. From proposition 2.2,

$$\varphi_{d,D-1}(\rho(u)) \le \varphi_{d,D-1}(\rho(v)) \le \varphi_{d,D-1}((\alpha)^{D-1}).$$

By the induction hypothesis, $g_{\alpha,d,D-1}(\rho(u)) \ge g_{\alpha,d,D-1}(\rho(v))$. Since $u, v \neq (\alpha)^D$, $g_{\alpha,d,D}(u) = g_{\alpha,d,D-1}(\rho(u)) + 1$ and $g_{\alpha,d,D}(v) = g_{\alpha,d,D-1}(\rho(v)) + 1$. Therefore $g_{\alpha,d,D}(u) \ge g_{\alpha,d,D}(v)$. □

3 Bookembedding of de Bruijn digraph

We define $T_\alpha(d,D)$ as follows.

Definition 3.1 *Let* $\alpha \in Z_d$.

$$\begin{cases} V(T_\alpha(d,D)) = V(B(d,D)), \\ A(T_\alpha(d,D)) = \{(u,v) \in A(B(d,D)) \mid \rho^{D-1}(u) = (\alpha)\} \end{cases}$$

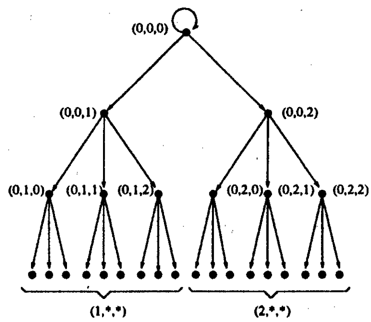

Figure 2: $T_0(3,3)$

Then $T_\alpha(d,D)$ is isomorphic to a directed tree obtained from a complete d-ary directed tree of height D by deleting a complete d-ary directed tree of height $(D-1)$ and adding a loop to the root. Clearly, $A(T_i(d,D)) \cap A(T_j(d,D)) = \phi$ for $0 \le i < j < d$ and $\cup_{0 \le i < d} A(T_i(d,D)) = A(B(d,D))$. Thus the following proposition holds.

Proposition 3.2 *$B(d,D)$ is decomposed to $T_0(d,D), T_1(d,D), \ldots, T_{d-1}(d,D)$.*

Let $V_i(T_\alpha(d,D)) = \{v \in V(T_\alpha(d,D)) \mid g_{\alpha,d,D}(v) = i\}$. Then $V_i(T_\alpha(d,D))$ is the set of vertices whose distance from the root are i. Also let

$$\begin{cases} V_i^-(T_\alpha(d,D)) = \{v \in V_i(T_\alpha(d,D)) \mid \varphi_{d,D}(v) < \varphi_{d,D}((\alpha)^D)\}, \\ V_i^+(T_\alpha(d,D)) = \{v \in V_i(T_\alpha(d,D)) \mid \varphi_{d,D}((\alpha)^D) < \varphi_{d,D}(v)\} \quad \text{for } 1 \le i \le D. \end{cases}$$

Lemma 3.3

$$\begin{cases} |V_i^-(T_\alpha(d,D))| = d^{i-1}\alpha, \\ |V_i^+(T_\alpha(d,D))| = d^{i-1}(d-\alpha-1). \end{cases}$$

Proof. By lemma 2.4, it is sufficient to show the case $i = 1$. From the definition of $\varphi_{d,D}$,

$$\begin{cases} O^-_{\varphi_{d,D}}((\alpha)^D) = \{(\alpha)^{D-1} \oplus x \mid x \in Z_\alpha\}, \\ O^+_{\varphi_{d,D}}((\alpha)^D) = \{(\alpha)^{D-1} \oplus x \mid x \in \{\alpha+1, \alpha+2, \dots, d-1\}\}. \end{cases}$$

Since $V_1^-(T_\alpha(d,D)) = O^-_{\varphi_{d,D}}((\alpha)^D)$ and $V_1^+(T_\alpha(d,D)) = O^+_{\varphi_{d,D}}((\alpha)^D)$, the proposition holds in the case $i = 1$. □

Let $A_i(T_\alpha(d,D)) = \{(u,v) \in A(T_\alpha(d,D)) \mid v \in V_i(T_\alpha(d,D))\}$.

Also let

$$\begin{cases} A_i^-(T_\alpha(d,D)) = \{(u,v) \in A_i(T_\alpha(d,D)) \mid v \in V_i^-(T_\alpha(d,D))\}, \\ A_i^+(T_\alpha(d,D)) = \{(u,v) \in A_i(T_\alpha(d,D)) \mid v \in V_i^+(T_\alpha(d,D))\} \quad \text{for } 1 \le i \le D. \end{cases}$$

Clearly, $|A_i^-(T_\alpha(d,D))| = |V_i^-(T_\alpha(d,D))|$ and $|A_i^+(T_\alpha(d,D))| = |V_i^+(T_\alpha(d,D))|$.

Lemma 3.4 *Regarding $\varphi_{d,D}$ as a vertex ordering of $T_\alpha(d,D)$ and assigning elements of $A_i(T_\alpha(d,D))$, $i = 0, 1, \dots, D$ to two pages according to the parity of i, 2-page bookembedding of $T_\alpha(d,D)$ is constructed. In this bookembedding, the widths of pages are $d^{D-1}\beta$ and $d^{D-2}\beta$, where $\beta = \max(\alpha, d-\alpha-1)$.*

Proof. We use induction on D. When $D = 1$, the proposition is clear. Suppose $D > 1$. Now assume that the proposition is not true. Thus there exist $(u,v), (x,y) \in A(T_\alpha(d,D))$ such that (u,v) and (x,y) cross on a page. Without loss of generality, we can set $\varphi_{d,D}(u) < \varphi_{d,D}(x)$. Suppose $\rho(v) = \rho(y)$. Then $(u,y) \in A(T_\alpha(d,D))$. But this is impossible because any vertex of $T_\alpha(d,D)$ has in-degree of 1. Therefore $\rho(v) \neq \rho(y)$.

Case 1. $\rho(u) \neq \rho(x)$:

If $\rho(v) = \rho(u)$, then there is no crossing of (u,v) and (x,y). Suppose $\rho(v) = \rho(x)$. Then $(u,x) \in A(T_\alpha(d,D))$. This contradicts the fact that (u,v) and (x,y) are on the same page. Thus $\rho(v) \neq \rho(u)$ and $\rho(v) \neq \rho(x)$. Similarly, $\rho(y) \neq \rho(u)$ and $\rho(y) \neq \rho(x)$. Therefore $\rho(u), \rho(v), \rho(x)$ and $\rho(y)$ are all distinct. Then the fact that (u,v) and (x,y) cross on a page implies that $(\rho(u), \rho(v))$ and $(\rho(x), \rho(y))$ cross on a page in the 2-page bookembedding of $T_\alpha(d, D-1)$. This contradicts the induction hypothesis.

Case 2. $\rho(u) = \rho(x)$:

As shown in the case 1, $\rho(v) \neq \rho(x)$ and $\rho(y) \neq \rho(u)$. By proposition 2.2 it is sufficient to consider the following three subcases. That is, (u,v) and (x,y) can not cross in the other three subcases.

$$(a) \qquad \varphi_{d,D-1}(\rho(y)) < \varphi_{d,D-1}(\rho(u)) < \varphi_{d,D-1}(\rho(v)).$$

$$(b) \qquad \varphi_{d,D-1}(\rho(u)) < \varphi_{d,D-1}(\rho(v)) < \varphi_{d,D-1}(\rho(y)).$$

$$(c) \qquad \varphi_{d,D-1}(\rho(v)) < \varphi_{d,D-1}(\rho(y)) < \varphi_{d,D-1}(\rho(u)).$$

The subcase (a) implies $\varphi_{d,D}(y) < \varphi_{d,D}(u) < \varphi_{d,D}(x) < \varphi_{d,D}(v)$. But this contradicts lemma 2.4. Then we consider the subcase (b). Now $(\rho(u), \rho(v)), (\rho(u), \rho(y)) \in A(T_\alpha(d, D-1))$. By the construction of $\varphi_{d,D}$ from $\varphi_{d,D-1}$,

$$\varphi_{d,D}(\rho(u) \oplus \sigma(\rho(y))) < \varphi_{d,D}(\rho(u) \oplus \sigma(\rho(v))).$$

But $\rho(u) \oplus \sigma(\rho(y)) = x$ and $\rho(u) \oplus \sigma(\rho(v)) = u$. This contradicts the assumption that $\varphi_{d,D}(u) < \varphi_{d,D}(x)$. Similarly, the subcase (c) contradicts the assumption. Therefore $T_\alpha(d,D)$ can be embedded in 2 pages.

From lemma 2.4 and lemma 2.5, it is easily checked that the pagewidths are

$$\max_{i:even} \max(|V_i^-(T_\alpha(d,D))|, |V_i^+(T_\alpha(d,D))|) \text{ and } \max_{i:odd} \max(|V_i^-(T_\alpha(d,D))|, |V_i^+(T_\alpha(d,D))|).$$

By lemma 3.3, the widths of pages are determined. □

In Figure 3, we show the 2-page bookembeddings of $T_\alpha(3,3)$ for $\alpha = 0, 1$ and 3, where the loops are omitted.

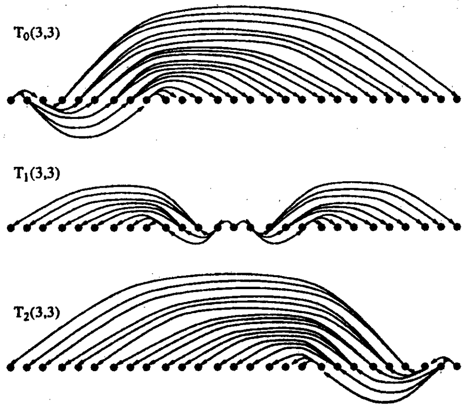

Figure 3: The 2-page bookembeddings of $T_\alpha(3,3)$ for $\alpha = 0, 1$ and 2.

Theorem 3.5 *$B(d,D)$ can be embedded in $(d+1)$ pages where the pagewidth and cumulative pagewidth are $(d-1)d^{D-1}$ and $\frac{1}{4}d^{D-2}(3d^3 - 2d^2 + 4d - d(d \bmod 2) - 4)$, respectively.*

Proof. In the 2-page bookembedding of $T_\alpha(d,D)$, let $Y_1(T_\alpha(d,D))$ and $Y_2(T_\alpha(d,D))$ be the sets of arcs assigned to each page. Also let $A_D(T_\alpha(d,D)) \subseteq Y_1(T_\alpha(d,D))$. Then $\rho^{D-1}(v) = (\alpha)$ for any vertex $v \in V(\langle Y_2(T_\alpha(d,D))\rangle)$. Now let $U_i = \{v \in V(B(d,D)) \mid \rho^{D-1}(v) = i\}$. From the construction of $\varphi_{d,D}$, the vertices of U_i are continuous on the spine with respect to $\varphi_{d,D}$. Thus elements of $Y_2(T_\alpha(d,D)), \alpha = 0, 1, \ldots, d-1$, can be assigned on one page without crossing. Therefore $B(d,D)$ can be embedded in $(d+1)$ pages. The pagewidth follows lemma 3.4. The cumulative pagewidth is obtained by computing $\sum_{0 \le \alpha < d} \max(\alpha, d-\alpha-1)d^{D-1} + (d-1)d^{D-2}$. □

Corollary 3.6 *$B(2,D)$ can be embedded in 3 pages where the pagewidth and cumulative pagewidth are 2^{D-1} and $5 \cdot 2^{D-2}$, respectively.*

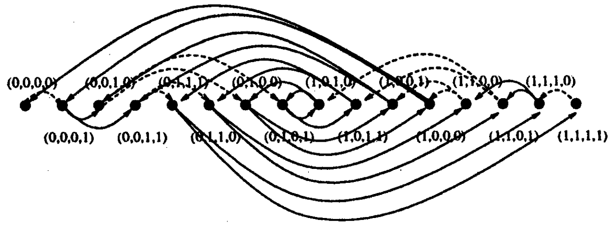

Figure 4: The 3-page bookembedding of $B(2,4)$.

4 Bookembeddings of Kautz digraph and shuffle-exchange graph

4.1 Kautz digraph

The Kautz digraph $K(d,D)$ is defined as follows.

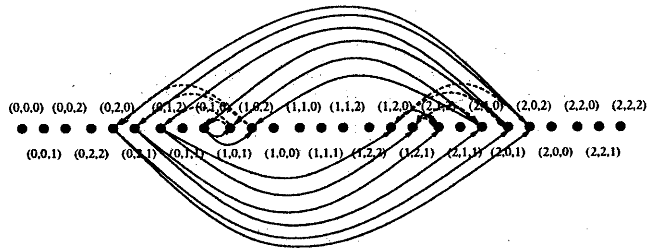

Figure 5: The 3-page bookembedding of $K(2,3)$.

$$\begin{cases} V(K(d,D)) = \{(v_0, v_1, \ldots, v_{D-1}) \mid v_i \in Z_{d+1}, 0 \le i < D, v_j \ne v_{j+1}, 0 \le j < D-1\} \\ A(K(d,D)) = \{((v_0, v_1, \ldots, v_{D-1}), (v_1, \ldots, v_{D-1}, x)) \mid x \in Z_{d+1},\ x \ne v_{D-1}\} \end{cases}$$

Theorem 4.1 *$K(d,D)$ can be embedded in $(d+1)$ pages where the pagewidth and cumulative pagewidth are d^D and $\frac{1}{4}d^{D-1}(3d^2+4d+(d \bmod 2))$, respectively.*

Proof. Now $K(d,D)$ is a subdigraph of $B(d+1,D)$ and $A(K(d,D)) \subseteq \cup_{0\le\alpha\le d} A_D(T_\alpha(d+1,D))$. From the $(d+2)$-page bookembedding of $B(d+1,D)$, it is realized that $K(d,D)$ can be embedded in $(d+1)$ pages. In this bookembedding of $K(d,D)$, the pagewidth are $\max_{0\le\alpha\le d}(|V(K(d,D)) \cap V_D^-(T_\alpha(d+1,D))|, |V(K(d,D)) \cap V_D^+(T_\alpha(d+1,D))|)$. Here

$$\begin{cases} V_D^-(T_\alpha(d+1,D)) = \{(x, v_1, \ldots, v_{D-1}) \mid x \in Z_\alpha,\ v_i \in Z_{d+1},\ 1 \le i < D\}, \\ V_D^+(T_\alpha(d+1,D)) = \{(y, v_1, \ldots, v_{D-1}) \mid y \in \{\alpha+1, \ldots, d\},\ v_i \in Z_{d+1},\ 1 \le i < D\}. \end{cases}$$

Thus $|V(K(d,D)) \cap V_D^-(T_\alpha(d+1,D))| = d^{D-1}\alpha$ and $|V(K(d,D)) \cap V_D^+(T_\alpha(d+1,D))| = d^{D-1}(d-\alpha)$. Therefore the pagewidth is d^D. The cumulative pagewidth is obtained by computing $\sum_{0\le\alpha\le d} \max(\alpha, d-\alpha)d^{D-1}$. □

Corollary 4.2 *$K(2,D)$ can be embedded in 3 pages where the pagewidth and cumulative pagewidth are 2^D and $5\cdot 2^{D-1}$, respectively.*

4.2 Shuffle-exchange graph

The shuffle-exchange graph $S(D)$ is defined as follows.

$$\begin{cases} V(S(D)) = \{(v_0, v_1, \ldots, v_{D-1}) \mid v_i \in Z_2,\ 0 \le i < D\}, \\ E(S(D)) = \left\{\{(u_0, u_1, \ldots, u_{D-1}), (v_0, v_1, \ldots, v_{D-1})\} \;\middle|\; \begin{array}{l} u_{i+1 (\text{mod } D)} = v_{i (\text{mod } D)} \text{ for } 0 \le i < D \text{ or} \\ u_j = v_j \text{ for } 0 \le j < D-1 \text{ and } u_{D-1} \ne v_{D-1}. \end{array}\right\} \end{cases}$$

Edges defined by the first condition are called shuffle-edges. Also edges defined by the second condition are called exchange-edges. The set of shuffle-edges and exchange-edges of $S(D)$ are denoted by $E_s(S(D))$ and $E_e(S(D))$, respectively.

Theorem 4.3 *$S(D)$ can be embedded in 3 pages where the pagewidth and cumulative pagewidth are 2^{D-2} and $5\cdot 2^{D-3}$, respectively.*

Proof. We employ $\varphi_{2,D}$ as the vertex ordering of $S(D)$. If $\{u,v\} \in E_e(S(D))$, then $\rho(u) = \rho(v)$ so $|\varphi_{2,D}(u) - \varphi_{2,D}(v)| = 1$. Let H be a graph obtained from $S(D)$ by identifying vertices u, v for all $\{u,v\} \in E_e(S(D))$. Then H is isomorphic to the underlying multi-graph of $B(2, D-1)$. Thus we can assign the shuffle-edges of $S(D)$ to 3 pages without crossing according to the 3-page bookembedding of

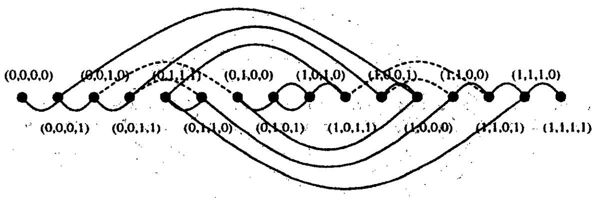

Figure 6: The 3-page bookembedding of $S(4)$.

$B(2, D-1)$. In this 3-page bookembedding of $\langle E_s(S(D))\rangle$, the pagewidth and cumulative pagewidth are similar to those of the 3-page bookembedding of $B(2, D-1)$.

For $i = 0, 1$, let P_i be the page to which the shuffle edges corresponding to elements of $A_D(T_i(2, D-1))$ are assigned. Also let $E_{e,i}(S(D)) = \{\{u, v\} \in E_e(S(D)) \mid \rho^{D-1}(u) = i\}$, for $i = 0, 1$. Now we assign elements of $E_{e,i}(S(D))$ to P_i for $i = 0, 1$. Clearly, crossings of edges do not occur. Also the pagewidth and cumulative pagewidth are changeless. Therefore the proposition holds. □

5 Conclusion

We have shown that d-ary de Bruijn and Kautz digraphs can be embedded in $(d+1)$ pages and shuffle-exchange graph can be embedded in 3 pages. For binary de Bruijn digraph, binary Kautz digraph and shuffle-exchange graph, these bookembeddings are optimal with respect to the number of pages, which follows from the nonplanarity of these networks. That is, the pagenumbers of these networks have been determined to be 3. For $d > 2$, it remains unknown whether $(d+1)$ pages are necessary for bookembeddings of d-ary de Bruijn and Kautz digraphs.

In this paper, we treat de Bruijn and Kautz networks as digraphs. The de Bruijn and Kautz graphs are the underlying graphs of these digraphs, respectively. Thus it is clear that d-ary de Bruijn and Kautz graphs can be embedded in $(d+1)$ pages. In these undirected versions, we can let the pagewidth and cumulative pagewidth be slightly smaller than the directed versions because de Bruijn and Kautz digraphs contain double arcs. The cumulative pagewidths of our bookembeddings of binary de Bruijn digraph (graph) and shuffle-exchange graph are smaller than those of the bookembeddings obtained in [7], respectively.

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