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Edge Guards in Straight Walkable Polygons

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Abstract

We study a variation of the art gallery problem with the galleries restricted to straight walkable polygons and with the guards allowed to patrol individual edges of a polygon. Given a simple polygon P with vertices s and g, polygon P is said *straight walkable* if we can move two points monotonically on two polygonal chains of P from s to g, one clockwise and the other counterclockwise, such that two points are always mutually visible. For instance, monotone polygons and spiral polygons are straight walkable. We show that $\lfloor (n+2)/5 \rfloor$ edge guards are occasionally necessary and always sufficient to watch an n-vertex gallery of this type. Furthermore, we also show that if the given polygon is straight walkable and rectilinear, then $\lfloor (n+3)/6 \rfloor$ edge guards are necessary and sufficient.

ストレートウォーカーブル多角形におけるエッジガードについて

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本研究ではストレートウォーカーブル多角形におけるエッジガード問題を調べる。この問題は美術館問題の一つの変種である。エッジガードは多角形の辺をパトロールできるガードである。単純多角形Ｐに頂点ｓとｇが与えられるときに、ｓとｇはＰの境界線を二つの輪に分ける。二人のガードが二つの輪に沿ってｓからｇまで互い見えるようにパトロールすることを考える。パトロールの途中後戻りしないでｓからｇに到達することができれば、Ｐをストレートウォーカーブル多角形と言う。例えば、単調多角形と螺旋多角形はストレートウォーカーブル多角形である。本研究ではｎ頂点のストレートウォーカーブル多角形に対し$\lfloor (n+2)/5 \rfloor$エッジガードが必要十分であるを証明する。さらに、直交ストレートウォーカーブル多角形に対し$\lfloor (n+3)/6 \rfloor$エッジガードが必要十分であることも示す。

1 Introduction

The **Art Gallery problem** posed by V. Klee asks for a minimum number of guards such that each point of an n-wall art gallery can be seen by at least one guard. The room is a polygon and each guard is a stationary point who can see any point connected to it by a line segment that does not go outside the polygon. The problem is shown to be NP-hard by Lee and Lin [9]. However, Chavatal showed that $\lfloor n/3 \rfloor$ guards are occasionally necessary and always sufficient for a simple polygon. For a survey on art gallery problems, see [10, 11].

Many variants of the art gallery problem have also been studied. Of particular interest to us among these problems is the problem of placing *edge guards* in a polygon P of n vertices [10]. Edge guards are the guards who are allowed to patrol individual edges of P. A region is said to be covered by a guard if the guard can see that region. In this paper, we consider the edge guard problem for straight walkable polygons, a new class of polygons defined and studied by Icking and Klein [8]. Given a simple polygon P with vertices s and g, polygon P is said *straight walkable* if we can move two points monotonically on two polygonal chains of P from s to g, one clockwise and the other counterclockwise, such that two points are always mutually visible. For instance, monotone polygons and spiral polygons are straight walkable. We prove that for a straight walkable polygon with n vertices, $\lfloor (n+2)/5 \rfloor$ edge guards are occasionally necessary and always sufficient. Furthermore, we also show that if the given polygon is straight walkable and rectilinear, then $\lfloor (n+3)/6 \rfloor$ edge guards are necessary and sufficient. Besides, both of our algorithms require linear time to place edge guards.

Previous research has been focussed on placing edge guards in monotone polygons. In Aggarwal's thesis, a 15-page proof was given to show the tight bound of $\lfloor (n+2)/5 \rfloor$ line guards for monotone polygons. *Line guards* are the guards who are allowed to patrol any line segment wholly contained in the polygon. Later, Bjorling-Sachs and Souvaine showed the same result using edge guards [3]. However, their proof even contains 53 pages! For rectilinear monotone polygons, a tight bound of $\lfloor (n+3)/6 \rfloor$ edge guards was also established by Bjorling-Sachs [4]. Again, the proof is very long and contains 37 pages.

Bjorling-Sachs and Souvaine's proof for monotone polygons depends on the triangulation algorithm of Garey, Johnson, Preparata and Tarjan [6]. They converted the triangulating algorithm into an algorithm for placing edge guards, which results in a long proof. In the next section, we first give a method to partition a straight walkable polygon P into $n-2$ triangular regions. The partition is generally not a triangulation of polygon P. But, an important property of this partition is that its dual is a path. In contrast, the dual of a triangulation of a monotone polygon is not necessarily a path. Then we cover five triangular regions by one guard, which yields a total coverage by $\lfloor (n+2)/5 \rfloor$ guards.

2 Straight Walkable Polygons

We define notation for the rest of the paper; much of our notation is borrowed from [8]. A *simple polygon* is the polygon without selfintersections or holes. When two vertices s and g of polygon P are given, the boundary of P consists of two polygonal chains, L and R, with common endpoints s and g. Both chains L and R are oriented from s to g. Points on L (R) are denoted by p, p', p_1, etc. (q, q', q_1, etc.). For a vertex v of a polygonal chain, $Succ(v)$ denotes the vertex of the chain immediately succeeding v, and $Pred(v)$ the vertex immediately preceding v. For convenience, we assume that polygon P is in a general position in the plane. That is, no three vertices of P are collinear, and no three edge extensions have a point in common.

A vertex of P is *reflex* if its interior angle is greater than 180^0; otherwise, it is *convex*. An important definition for reflex vertices is that of *ray shots*: the backward ray shot (or hit point) from a reflex vertex v of chain L or R, denoted by *Backw(v)*, is the first point of P hit by a "bullet" shot at v in the direction from $Succ(v)$ to v, and the forward ray shot *Forw(v)* is the first point hit by the bullet shot at v in the direction from v to $Pred(v)$.

A *walk instruction* is one of the following elementary motions. (i) Both guards move forward along segments of single edges. (ii) One guard moves forward but the other moves backward along segments of single edges. As a special case of (i) or (ii), one guard may stand still while the other moves.

A *general walk* on P is a pair of functions:

$$l : [0,1] \to L,\ r : [0,1] \leftarrow R,$$

where $l(0) = r(0) = s$, $l(1) = r(1) = g$, and $l(t)$ and $r(t)$ are mutually visible for all t. Any line segment $\overline{l(t)r(t)}$ is called a *walk line segment* of the walk. The point $r(t)$ is the *walk partner* of $l(t)$, and vice versa. A walk is *straight* if l and r are monotonic functions. A straight *counter-walk* is a pair of monotonic, continuous functions with $l(0) = r(1) = s$ and $l(1) = r(0) = g$. Polygon P is said *walkable* if it admits a walk.

For polygon P with two marked vertices s and g, a straight walk can be found in the following way [8]. First, we check if L and R are mutually weakly visible. If not, P is not straight walkable. Otherwise, we proceed to compute for each vertex v, a closed interval $[lo(v), hi(v)]$ of the polygonal chain, L or R, opposite to v such that according to the shot restrictions of reflex vertices, any possible walk partner of v must be contained in this interval. The interval $[lo(v), hi(v)]$ is called the *walkable interval* of v. P is straight walkable if and only if none of the walk intervals is empty. Icking and Klein developed an $O(n \log n)$ time algorithm to determine and find a straight walk [8]. Later, the time complexity was improved to $O(n)$ by Herffernan [7].

The functions lo and hi are defined on the vertices of P. The following definition is given for L, and the case of vertices of R is symmetric. (The operation *min* and *max* are defined with respect to the ordering on the chain L.)

Definition 1 *[8] For a vertex $p \in L$, we define:*
$hiP(p) = min \{ q | q \text{ vertex of } R \text{ and } L \ni Backw(q) > p \}$
$hiS(p) = min \{ Forw(p') \in R | p' \text{ vertex of } L_{>p} \}$
$hi(p) = min \{ hiP(p), hiS(p), g \}$
$loP(p) = max \{ q | q \text{ vertex of } R \text{ and } L \ni Forw(q) > p \}$
$loS(p) = max \{ Backw(p') \in R | p' \text{ vertex of } L_{<p} \}$
$lo(p) = max \{ hiP(p), hiS(p), g \}$

Icking and Klein also gave a way to actually find a straight walk. For instance, we choose $lo(p)$ as a walk partner for each vertex $p \in L$, and $hi(q)$ for each vertex $q \in R$. Note that two chosen walk line segments may overlap each other but can not cross. (Overlapped walk line segments can be avoided if we carefully choose $hi(q)$ for some vertices q of R, see [8]). Since a walk line segment is chosen for each vertex, polygon P is partitioned into quadrilaterals and triangles, which gives us a sequence of walk instructions. See Fig. 1(a). We call the partition, which is produced by assigning $lo(p)$ as a walk partner for $p \in L$ and $hi(q)$ for $q \in R$, the *walk partition* of polygon P.

As stated in Section 1, our goal is to partition a straight walkable polygon into triangular regions. For this purpose, we first note that there are two quadrilaterals in the walk partition of Fig. 1(a) and both of them are concave. (Because of the choice of $lo(p)$ as a walk partner for each vertex $p \in L$ and $hi(q)$ for each vertex $q \in R$, it is impossible for a quadrilateral in the walk partition to be convex.) Within a guadrilateral, segment $\overline{vBackw(v)}$ or $\overline{vForw(v)}$ is a walk line segment, where v denotes the reflex vertex of the quadrilateral. Thus, any quadrilateral can further be divided into two triangles by introducing the walk segment $\overline{vBackw(v)}$ or $\overline{vForw(v)}$. We call the resulting partition, which has only triangular regions, the *modified walk partition* of polygon P. See Fig. 1(b) for an example. Note that the modified walk partition is generally not a triangulation of polygon P. Polygon vertices may lie on edges of triangular regions. On the other hand, there may exist at most two vertices in a triangle that are not polygon vertices. They are instead $Forw(v)$ or $Backw(v)$ of reflex vertices v.

Lemma 1 *The number of triangles in the modified walk partition of polygon P is $n-2$.*

Proof. Consider first the simplest case where neither $Forw(v)$ nor $Backw(v)$ appears in the modified walk partition. In this case, the modified walk partition is a triangulation of polygon P. Thus, the number of triangles is $n-2$. If there exist some $Forw(v)$s and $Backw(v)$s in the modified walk partition, then the walk segments assigned with them divide polygon P into disjoint pieces. If we consider each piece P_i as a polygon, then P_i is triangulated by the walk segments within it and the number of triangles in P_i is $size(P_i)-2$. By noticing that $Forw(v)$ and $Backw(v)$ are generated from a reflex vertex v, we have $\sum_i(size(P_i)-2)=n-2$. □

Since polygon P is straight walkable, we have the following immediate consequence.

Corollary 1 *The dual graph of the modified walk partition of a straight walkable polygon P, with a node for each triangle and an arc connecting two nodes whose triangles share a walk segment (or some part of a walk segment), is a path.*

Now we give the main result of this paper.

Theorem 1 $\lfloor(n+2)/5\rfloor$ *edge guards are occasional necessary and always sufficient to cover a straight walkable polygon of n vertices.*

Proof. Suppose that P is a straight walkable polygon of n vertices. The necessity of $\lfloor(n+2)/5\rfloor$ edge guards has been simply established by a spiral polygon [12] or a monotone polygon [3]. We now prove that $\lfloor(n+2)/5\rfloor$ edge guards are sufficient to cover polygon P.

From Lemma 1 and Corollary 1, we obtain a partition of polygon P, whose dual graph is a path. Our method for placing edge guards in the modified walk partition is to cover five consecutive triangles by one guard, which yields a total coverage by $\lceil(n-2)/5\rceil=\lfloor(n+2)/5\rfloor$ guards. Note that we should place the guards on the edges belonging to L or R, but not on the walk line segments. In the following, if a guard is said to place at a vertex of P, then it means that the guard is placed on either edge adjacent to the vertex. If a guard is said to place on some part of an edge of P, then it means that the guard is placed on the whole edge.

Let P_5 denote the union of five consecutive triangles. Most vertices of P_5 coincide with the vertices of polygon P, but there may exist at most two vertices in P_5, which are not the vertices of P. This is because only the first and last walk segments in the group of five triangles can introduce a new vertex, respectively. See Fig. 2. Since five consecutive triangles (of the modified walk partition) do not give a triangulation of polygon P_5, the number of vertices of P_5 is not always seven. In some special cases, P_5 may have eight vertices.

If no three vertices of P_5 are collinear, then P_5 has exactly seven vertices. Otherwise, a contradiction generates if P_5 has six vertices (then a new walk partition of P_5 with four triangles can be easily found), or if P_5 has eight vertices (then any walk partition of P_5 should have six triangles).

In P_5 there may exist three collinear vertices. This is because P_5 can contain two vertices not belonging to P. (Recall that no three vertices of polygon P are collinear.) These two vertices, denoted by v_f and of v_l (Fig. 2), are introduced by the first and last walk segments, respectively. First, It is possible for v_l to be collinear with a pair of other two vertices, see Fig. 2. However, the situation is different for v_l. It is imposible for v_f to be collinear with a pair of other two vertices of P_5. Suppose that $v_f = Forw(v)$ or $Backw(v)$ for some vertex v. Then either v lies on the first walk line segment of P_5, or v is not contained in P_5. If v_f is collinear with a pair of other two vertices of P_5, say v_1 and v_2, then v must not lie on that line (otherwise at least one of v_1 and v_2 can not be a vertex of P_5). Therefore, the three lines extending the edge containing v_f, the walk segment $\overline{v_f v}$ and the edge $\overline{v_1 v_2}$ have point v_f in common. It contradicts with the assumption that polygon P is in a general position in the plane. Note that the difference of v_f and v_l generates from our method used for determining the modified walk partition. Further, it is impossible for v_l to be collinear with two pairs of other vertices, since otherwise three edge extensions of P should have a point in common. In summary, P_5 has either seven or eight vertices.

According to the number of vertices of P_5, we consider two different cases for placing edge guards in P_5.

Case 1 P_5 has seven vertices. Let L_k denote the numbers of vertices of P_5 belonging to the left chain L. Clearly, $1 \leq L_k \leq 6$. By symmetry, we only consider the cases where $L_k = 1, 2$, and 3.

Case 1.1 $L_k = 1$ or 2. That is, the left chain consists of only one vertex or edge. Clearly, the guard placed on that vertex or edge covers P_5.

Case 1.2 $L_k = 3$. Let l_i $(i = 1, 2, 3)$ and r_j $(j = 1, 2, 3, 4)$ denote the vertices lying on L and R, respectively. In the following, we distinguish two different cases. Our first case handles all configurations where either r_1 or r_4 is visible to l_2 in P_5 (i.e., the line segment $\overline{l_2 r_1}$ or $\overline{l_2 r_4}$ is entirely contained in P_5.) If r_1 and r_4 are not visible to l_2, then vertices r_2 and r_3 must be reflex, and l_2 must lie in the intersection of three lines extending edges $\overline{r_1 r_2}$, $\overline{r_2 r_3}$ and $\overline{r_3 r_4}$. Thus, r_2 and r_3 are visible to l_2, to which our second case devotes.

Case 1.2.1 Either r_1 or r_4 is visible to l_2 in P_5. If r_1 is visible to l_2, then all vertices r_i are visible to some points of edge $\overline{l_2 l_3}$. Since all vertices of P_5 are visible to some points of $\overline{l_2 l_3}$, a guard on edge $\overline{l_2 l_3}$ covers P_5 [2]. Similarly, if r_4 is visible to l_2, then a guard on edge $\overline{l_1 l_2}$ suffices.

Case 1.2.2 Both r_2 and r_3 are visible to l_2 in P_5. In this case, l_1 is visible to r_2, and l_3 is visible to r_3. Thus, a guard on edge $\overline{r_2 r_3}$ covers polygon P_5.

Case 2 P_5 has eight vertices. In this case, there are three collinear vertices in P_5, and these vertices lie on an edge of the last triangle. Let $\triangle$ denote the last triangle, and let $P_4 = P_5 - \triangle$. Then, P_4 has six vertices. Again, let L_k denote the numbers of vertices of P_4 belonging to the left chain L. Since $1 \leq L_k \leq 6$, we only consider the cases where $L_k = 1, 2$ and 3.

Case 2.1 $L_k = 1$ or 2. That is, the left chain consists of only one vertex or edge. Clearly, the guard on that vertex or edge covers P_4 and $\triangle$.

Case 2.2 $L_k = 3$. Let l_i $(i = 1, 2, 3)$ and r_j $(j = 1, 2, 3)$ denote the vertices lying on L and R, respectively. In this case, either l_2 is visible to r_1, or r_2 is visible to l_1 in P_4. If l_2 is visible to r_1, then an guard on edge $\overline{l_2 l_3}$ covers polygon P_4 and triangle $\triangle$. Similarly, if r_2 is visible to l_1, then a guard on edge $\overline{r_2 r_3}$ suffices. □

Remark. From the above proof (Case 1), it is not difficult to find a triangulation for a straight walkable polygon with seven vertices, such that the triangulation dual is a path.

3 Rectilinear Walkable Polygons

We have shown in Section 2 that $\lfloor (n+2)/5 \rfloor$ edge guards are necessary and sufficient to cover a straight walkable polygon. What is then the bound if straight walkable polygons are also rectilinear? A polygon is *rectilinear* if its edges are all parallel to a pair of orthogonal coordinate axes. We will show that $\lfloor (n+3)/6 \rfloor$ edge guards are necessary and sufficient to cover a rectilinear walkable polygon P of n vertices. In [4], Bjorling-Sachs gave a 37-page proof for rectilinear monotone polygons. His proof depends on Sack's monotone quadrilateralization algorithm [10]. However, the quadrilaterals produced by Sack's algorithm are not rectilinear. It is the reason why the proof is so long. In the following, we first divide polygon P into disjoint rectilinear pieces, each with eight vertices, and then cover each piece by one guard.

Theorem 2 $\lfloor (n+3)/6 \rfloor$ *edge guards are occasional necessary and always sufficient to cover a rectilinear walkable polygon of n vertices.*

Proof. Suppose that P is a rectilinear walkable polygon of n vertices. Again, the necessity of $\lfloor (n+3)/6 \rfloor$ edge guards can be simply established by a rectilinear spiral (or a rectilinear monotone polygon [4]). Let r be the number of reflex vertices. We now prove that $\lceil (r+1)/3 \rceil$ edge guards are always sufficient to cover polygon P. Since $n = 2r + 4$ [10], $\lceil (r+1)/3 \rceil = \lfloor (n+3)/6 \rfloor$.

For polygon P, we can assume that both the first and last walk segments are vertical. We first index the reflex vertices of P in the walk ordering of two guards, and then partition P into pieces by drawing a vertical (walk) line segment at every third reflex vertex. Each of resulting pieces then has at most eight vertices (or at most two reflex vertices), which will be covered by an edge guard. Hence, we obtain the sufficiency of $\lceil (r+1)/3 \rceil$ edge guards for rectilinear walkable polygons.

We now show that each piece with eight vertices can be covered by one guard. There are only six types of rectilinear walkable pieces with eight vertices, as shown in Fig. 3. (Symmetric cases are omitted.) In Fig. 3a and Fig. 3b, two reflex vertices are consecutive. The guard placed on the edge (bold line) having two reflex vertices as its endpoints covers the whole piece. In Fig. 3c and Fig. 3d, there is a chain that consists of only one edge. The guard placed on that edge covers the piece. In Fig. 3e and Fig. 3f, two reflex vertices lies in different chains. The guard placed on a vertical edge in Fig. 3e or a horizontal edge in Fig. 3f that is adjacent to either reflex vertex covers the piece. Note that no guard is placed on two (dotted) walk line segments of the piece in Fig. 3. □

4 Concluding Remarks

In this paper, we have shown the tight bounds of $\lfloor (n+2)/5 \rfloor$ and $\lfloor (n+3)/6 \rfloor$ edge guards for straight walkable polygons and rectilinear walkable polygons, respectively. Our key idea is to divide a polygon into a partition, whose dual is a path. Finally, we pose an open problem on edge guards in simple polygons. It is conjectured that $\lfloor (n+1)/4 \rfloor$ is a tight bound for edge guards in simple polygons. Two types of polygons, discovered by Paige and Shermer [11], established the lower bound of $\lfloor (n+1)/4 \rfloor$ guards. One type of polygons has seven vertices and requires two edge guards while the other has eleven vertices and requires three edge guards. A possible way to show the upper bound is first to partition an n-vertex polygon into $n-2$ triangular regions and then cover every four triangles by one guard. The resulting bound would be $\lceil (n-2)/4 \rceil = \lfloor (n+1)/4 \rfloor$.

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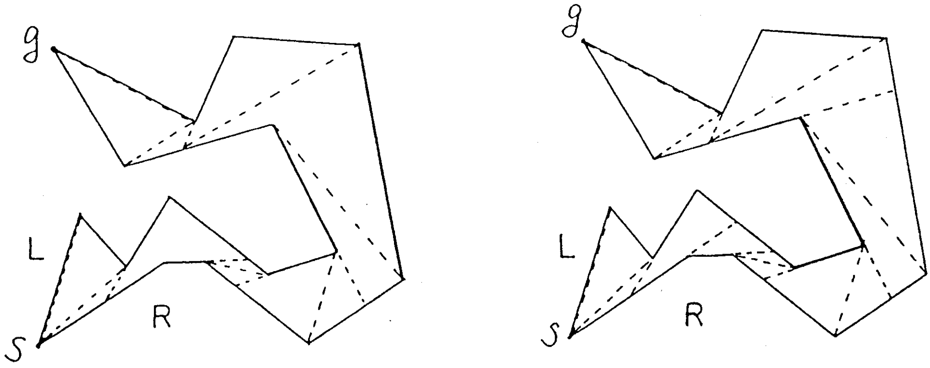

(a) The walk partition due to Icking and Klein. (b) The modified walk partition.

Fig. 1. Construction of a straight walk.

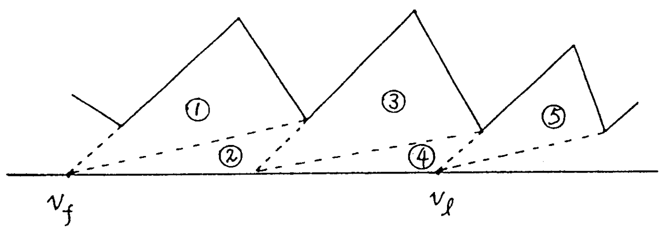

Fig. 2. P_5 may have eigth vertices.

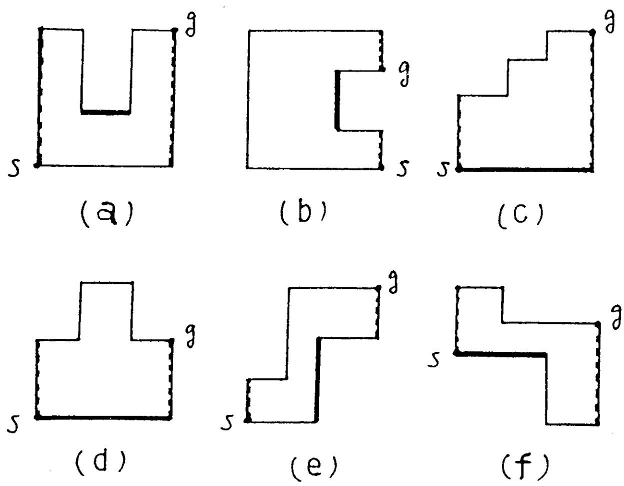

Fig. 3. Edge guards in rectilinear walkable polygons with eight vertices.