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A Linear-Time Algorithm for Four-Partitioning Four-Connected Planar Graphs

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Given a graph $G = (V, E)$, four distinct vertices $u_1,u_2,u_3,u_4 \in V$ and four natural numbers n_1, n_2, n_3, n_4 such that $\sum_{i=1}^{4} n_i = |V|$, we wish to find a partition V_1, V_2, V_3, V_4 of the vertex set V such that $u_i \in V_i$, $|V_i| = n_i$ and V_i induces a connected subgraph of G for each i, $1 \leq i \leq 4$. In this paper we give a simple linear-time algorithm to find such a 4-partition of G if G is a 4-connected planar graph and u_1, u_2, u_3, u_4 are located on the same face of G. Our algorithm is based on a "4-canonical decomposition" of G, which is a generalization of an st-numbering and a "canonical 4-ordering" known in the area of graph drawings.

4連結平面グラフを4分割する線型時間アルゴリズム

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グラフ $G = (V, E)$, 4点 $u_1,u_2,u_3,u_4 \in V$ および $\sum_{i=1}^{4} n_i = |V|$ なる4つの自然数 n_1, n_2, n_3, n_4 が与えられたとき、各i, $1 \leq i \leq 4$ について $u_i \in V_i$, $|V_i| = n_i$ かつ V_i による G の誘導部分グラフが連結であるような V の4分割 V_1, V_2, V_3, V_4 を求めたい。本論文では、G が4連結平面グラフ、かつ u_1, u_2, u_3, u_4 が G のひとつの面上にあるとき、上記の4分割を求める線型時間アルゴリズムを与える。我々のアルゴリズムは G の4正規分割を用いている。4正規分割はグラフ描画の分野で知られている st 順序付けや正規4順序付けの一般化である。

1 Introduction

Given a graph $G = (V, E)$, k distinct vertices $u_1, u_2, \cdots, u_k \in V$ and k natural numbers $n_1, n_2, \cdots, n_k$ such that $\sum_{i=1}^{k} n_i = |V|$, we wish to find a partition $V_1, V_2, \cdots, V_k$ of the vertex set V such that $u_i \in V_i$, $|V_i| = n_i$, and V_i induces a connected subgraph of G for each i, $1 \leq i \leq k$. Such a partition is called a *k-partition* of G. The problem of finding a k-partition of a given graph often appears in fault-tolerant routings [WK94, WTK95]. The problem is NP-hard in general [DF85], and hence it is very unlikely that there is a polynomial-time algorithm to solve the problem. Although not every graph has a k-partition, Györi and Lovász independently proved that every k-connected graph has a k-partition for any $u_1, u_2, \cdots, u_k$ and $n_1, n_2, \cdots, n_k$ [G78, L77]. However, their proofs do not yield any polynomial-time algorithm for actually finding a k-partition of a k-connected graph. For the case $k = 2$ and 3, the following algorithms have been known:

(i) a linear-time algorithm to find a bipartition of a biconnected graph [STN90, STNMU90];

(ii) an algorithm to find a tripartition of a triconnected graph in $O(n^2)$ time, where n is the number of vertices of a graph [STNMU90]; and

(iii) a linear-time algorithm to find a tripartition of a triconnected planar graph [JSN94].

On the other hand, polynomial-time algorithms have not been known for the case $k \geq 4$. †

In this paper we give a linear-time algorithm to find a 4-partition of a 4-connected planar graph G in case u_1, u_2, u_3, u_4 are located on the same face of G. Our algorithm first bipartitions the 4-connected graph G into two biconnected graphs having about $n_1 + n_2$ and $n_3 + n_4$ vertices respectively, then bipartitions each of them to two connected graphs, and, by adjusting the numbers of vertices in the resulting four graphs, we finally obtain a required 4-partition of G. To bipartition G into two biconnected graphs, we will newly define and use a "4-canonical decomposition" of G, which is a generalization of an st-numbering and a "canonical 4-ordering" known in the area of graph drawings [E79, K94, KH94].

The rest of the paper is organized as follows. In Section 2 we introduce our notations and give a linear-time algorithm to find a 4-canonical decomposition of a 4-connected planar graph. In Section 3 we present a linear-time algorithm to find a 4-partition of a 4-connected planar graph. Finally we put our discussions in Section 4.

2 4-Canonical Decomposition

In this section we introduce some definitions and prove that every 4-connected plane graph has a 4-canonical decomposition and it can be found in linear time.

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. Throughout the paper we denote by n the number of vertices in G, that is, $n = |V|$. An edge joining vertices u and v is denoted by (u, v). The *degree* of a vertex v is the number of neighbors of v in G. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or single-vertex graph K_1. G is called a *k-connected* graph if $\kappa(G) \geq k$. We call a vertex of G a *cut vertex* if its removal results in a disconnected or single-vertex graph. For $W \subseteq V$, we denote by $G - W$ the graph obtained from G by deleting all vertices in W and all edges incident to them.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane* graph is a planar graph with a fixed embedding. The *contour* $C(G)$ of a biconnected plane graph G is the clockwise (simple) cycle on the boundary of the external face. We write $C(G) = w_1, w_2, \cdots, w_h, w_1$ if the vertices $w_1, w_2, \cdots, w_h$ on $C(G)$ appear in this order. A *chord* in a biconnected plane graph G is a path which connects two inconsecutive vertices w_p and w_q, $p < q$, on $C(G)$ without passing through

†A polynomial-time algorithm for any k is claimed in [MM94], but is not correct [G96].

any other vertices on $C(G)$ and lies on an internal face. Thus the definition of a chord depends on which vertex is considered as the starting vertex w_1 of $C(G)$. The vertices w_p and w_q are called the *ends* of the chord. The chord is said to be *minimal* if none of $w_{p+1}, w_{p+2}, \cdots, w_{q-1}$ is an end of a chord. Let $\{v_1, v_2, \cdots, v_{p-1}, v_p\}$ be a set of three or more consecutive vertices on $C(G)$ such that the degrees of the first and the last vertices are at least three and the degrees of all intermediate vertices $v_2, v_3, \cdots, v_{p-1}$ are two. Then we call the set $\{v_2, v_3, \cdots, v_{p-1}\}$ a *handle* of G. For a cycle C in a plane graph G, we denote by $I(C,G)$ the subgraph of G inside C, that is, the plane subgraph of G induced by the set of vertices inside (or on) the cycle C. Clearly $I(C,G)$ is biconnected if G is biconnected. We have the following lemma.

Lemma 2.1 *Assume that G is a 4-connected plane graph and that $C = w_1, w_2, \cdots, w_h, w_1$ is a cycle in G such that $I(C,G)$ is not a cycle. If $I(C,G)$ has a chord, then let w_p and w_q be the two ends of any minimal chord, otherwise let $w_p = w_1$ and $w_q = w_h$. Then the following* (a) *and* (b) *hold.*

(a) *If $W = \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ is a handle of $I(C,G)$, then $I(C,G) - W$ is a biconnected graph.*

(b) *Otherwise, there is a set $W = \{w_{p'}, w_{p'+1}, \cdots, w_{q'}\}$ of one or more consecutive vertices on C such that*

  (i) $p < p' \leq q' < q$, *and*

  (ii) *none of the vertices in W except the first vertex $w_{p'}$ and the last one $w_{q'}$ has a neighbor in the proper outside of C.*

Moreover $I(C,G) - W$ is a biconnected graph for any such set W.

Proof. Since (a) is obvious, we give only a proof of (b). Clearly a singleton set $W = \{w_{p'}\}$ for any p', $p < p' < q$, satisfies (i) and (ii). We now prove that $I(C,G) - W$ is a biconnected graph for any set W satisfying (i) and (ii).

Suppose for a contradiction that $G' = I(C,G) - W$ is not a biconnected graph and hence G' has a cut vertex v. If v is in the proper inside of C, then G is not 4-connected, a contradiction. If v is on C, then either a vertex in W is an end of a chord in $I(C,G)$, or W is included in a handle, or G is not 4-connected, a contradiction. $\mathcal{Q.E.D.}$

Let $G = (V, E)$ be a connected graph, and let $(s,t) \in E$. We say that an ordering $\pi = v_1, v_2, \cdots, v_n$ of the vertices of G is an *st-numbering* of G if the following conditions are satisfied:

(st1) $v_1 = s$ and $v_n = t$; and

(st2) each $v_i \in V - \{v_1, v_n\}$ has two neighbors v_p and v_q such that $p < i < q$.

Not every connected graph has an st-numbering, but the following lemma holds.

Lemma 2.2 [E79] *Let G be a biconnected graph, and let (s,t) be any edge of G. Then G has an st-numbering $\pi = v_1, v_2, \cdots, v_n$ such that $v_1 = s$ and $v_n = t$, and π can be found in linear time.*

A bipartition of a biconnected graph can be found by an st-numbering as follows [STNMU90, STN90]. Let $G = (V, E)$ be a biconnected graph, let $u_1, u_2 \in V$ be two designated distinct vertices and let n_1,n_2 be two natural numbers such that $n_1 + n_2 = n$. Add an edge (u_1, u_2) to G if $(u_1, u_2) \notin E$, and let G' be the resulting graph. Since G' is biconnected, by Lemma 2.2 G' has an st-numbering $v_1(= u_1), v_2, \cdots, v_n(= u_2)$. Clearly the following fact holds:

(st3) both $\{v_1, v_2, \cdots, v_i\}$ and $\{v_{i+1}, v_{i+2}, \cdots, v_n\}$ induce connected subgraphs of G for each i, $1 \leq i < n$.

Thus, choosing $i = n_1$, one can find a required bipartition of G in linear time.

Generalizing an st-numbering in a sense, we define a "4-canonical decomposition" of a 4-connected planar graph G and in the succeeding section we give an algorithm to find a 4-partition of G by using the "4-canonical decomposition." We now give the definition of a 4-canonical decomposition.

Assume that $G = (V, E)$ is a 4-connected planar graph with four designated distinct vertices u_1, u_2, u_3, u_4 on the same face of G. We may assume that u_1, u_2, u_3, u_4 lie on the contour $C(G)$ of G, since, for any face F of G, we can re-embed G so that F becomes the external face. We may furthermore assume that the four vertices u_1, u_2, u_3, u_4 appear on $C(G)$ of G in this order. Moreover we may assume that $(u_1, u_2), (u_3, u_4) \in E$; otherwise, consider as G the new graph obtained from G by adding edges (u_1, u_2) and (u_3, u_4). For a set $W_1, W_2, \cdots, W_i$ of pairwise disjoint subsets of V, we denote by G_i the subgraph of G induced by $W_1 \cup W_2 \cup \cdots \cup W_i$, and by $\overline{G_i}$ the subgraph of G induced by $V - W_1 \cup W_2 \cup \cdots \cup W_i$, that is, $\overline{G_i} = G - W_1 \cup W_2 \cup \cdots \cup W_i$. We say that a partition $\Pi = W_1, W_2, \cdots, W_l$ of V is a *4-canonical decomposition* of G if the following three conditions (co1)–(co3) are satisfied:

(co1) both G_i and $\overline{G_i}$ are biconnected for each i, $1 \leq i < l$;

(co2) W_1 is the set of vertices on the inner face containing edge (u_1, u_2), and W_l is the set of vertices on the inner face containing edge (u_3, u_4); and

(co3) one of the following three conditions holds for each i, $1 < i < l$:

- (a) W_i is a handle of $\overline{G_{i-1}}$;
- (b) W_i consists of exactly one vertex v on both $C(G_i)$ and $C(\overline{G_{i-1}})$;
- (c) W_i is a handle of G_i.

We have the following two lemmas.

Lemma 2.3 *Let $G = (V, E)$ be a 4-connected plane graph with four designated distinct vertices u_1, u_2, u_3, u_4 appearing on $C(G)$ in this order. Then G has a 4-canonical decomposition $\Pi = W_1, W_2, \cdots, W_l$. Furthermore Π can be computed in linear time.*

Proof. Omitted.

Lemma 2.4 *Let $W_1, W_2, \cdots, W_l$ be a 4-canonical decomposition of a 4-connected plane graph G. Then the following (a) and (b) hold for any i, $1 < i < l$:*

(a) *If W_i satisfies (a) of (co3), then, for any $W_i' \subseteq W_i$, $G_i - W_i'$ is biconnected.*

(b) *If W_i satisfies (c) of (co3), then, for any $W_i' \subseteq W_i$, $\overline{G_{i-1}} - W_i'$ is biconnected.*

Proof. We give only a proof for (a) since the proof for (b) is similar. Let W_i be a handle of $\overline{G_{i-1}}$. Then each vertex $w_j \in W_i$ has at least two neighbors in G_{i-1}. Let W_i' be any subset of W_i. Since the graph G_{i-1} is biconnected, the graph $G_i - W_i'$ induced by $W_1 \cup W_2 \cup \cdots \cup W_{i-1} \cup (W_i - W_i')$ is also biconnected. $\mathcal{Q.E.D.}$

3 4-Partition of 4-Connected Plane Graph

In this section we give our algorithm to find a 4-partition of a 4-connected plane graph G. Assume that the four designated distinct vertices u_1, u_2, u_3, u_4 appear on $C(G)$ in this order and n_1, n_2, n_3, n_4 are natural numbers such that $\sum_{i=1}^{4} n_i = n$.

Algorithm Four-Partition
Find a 4-canonical decomposition $\Pi = W_1, W_2, \cdots, W_l$ of G;
Let i be the minimum integer such that $\sum_{j=1}^{i} |W_j| \geq n_1 + n_2$;
Let $r = \sum_{j=1}^{i} |W_j| - (n_1 + n_2)$;
There are the following two cases (1) $r = 0$, and (2) $r \geq 1$;
Case 1: $r = 0$.
{In this case, G_i contains $n_1 + n_2$ vertices, and $\overline{G_i}$ contains $n_3 + n_4$ vertices.}
Find a bipartition V_1, V_2 of the biconnected graph G_i such that $u_1 \in V_1$, $u_2 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce connected subgraphs;
Find a bipartition V_3, V_4 of the biconnected graph $\overline{G_i}$ such that $u_3 \in V_3$, $u_4 \in V_4$, $|V_3| = n_3$, $|V_4| = n_4$, and both V_3 and V_4 induce connected subgraphs;
Return V_1, V_2, V_3, V_4 as a 4-partition of G.
Case 2: $r \geq 1$.
{ In this case, G_i contains n_1+n_2+r vertices, and $\overline{G_i}$ contains n_3+n_4-r vertices. $\overline{G_i} = \overline{G_{i-1}} - W_i$.

Since $r \geq 1$, $|W_i| \geq 2$ and hence W_i is a handle of either $\overline{G_{i-1}}$ or G_i.}

Assume that W_i is a handle of $\overline{G_{i-1}}$ as illustrated in Fig 2(a), otherwise, interchange the roles of u_1, u_2 and u_3, u_4;

Let $C(\overline{G_{i-1}}) = w_1, w_2, ..., w_h, w_1$ where $w_1 = u_4$;

Let $W_i = \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$;

Find an st-numbering $v_1, v_2, \cdots, v_{n_3+n_4-r}$ of $\overline{G_i}$ such that $s = u_4$ and $t = u_3$;

Let $w_p = v_{p'}$ and $w_q = v_{q'}$;

Assume that $p' < q'$, otherwise, interchange the roles of u_3 and u_4;

There are the following three subcases (a) $n_4 \leq p'$, (b) $p' + r \leq n_4$, and (c) $p' < n_4 < p' + r$;

Subcase 2(a): $n_4 \leq p'$.

{In this subcase, the last r vertices in the handle W_i are added to $\overline{G_i}$ as the deficient r vertices.}

Let $V_4 = \{v_1, v_2, \cdots, v_{n_4}\}$ be the first n_4 vertices in the st-numbering of $\overline{G_i}$;

Let $V_3' = \{v_{n_4+1}, v_{n_4+2}, \cdots, v_{n_4+n_3-r}\}$ be the remaining $n_3 - r$ vertices in $\overline{G_i}$;

{By the fact (st3) of an st-numbering both V_4 and V_3' induce connected graphs.}

Let $W_i' = \{w_{q-1}, w_{q-2}, \cdots, w_{q-r}\}$ be the set of the last r vertices in W_i;

Let $V_3 = V_3' \cup W_i'$;

{Since w_{q-1} is adjacent to w_q in V_3', V_3 induces a connected graph of n_3 vertices.}

Let $G_{12} = G_i - W_i'$;

{ G_{12} is biconnected by Lemma 2.4(a), and has $n_1 + n_2$ vertices.}

Find a bipartition V_1, V_2 of G_{12} such that $u_1 \in V_1$, $u_2 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce connected subgraphs;

Return V_1, V_2, V_3, V_4 as a 4-partition of G.

Subcase 2(b): $p' + r \leq n_4$.

{In this subcase, the first r vertices in W_i are added to $\overline{G_i}$ as the deficient r vertices.}

Let $V_4' = \{v_1, v_2, \cdots, v_{n_4-r}\}$ be the set of the first $n_4 - r$ vertices of $\overline{G_i}$, where $w_p = v_{p'} \in V_4'$;

Let $V_3 = \{v_{n_4-r+1}, v_{n_4-r+2}, \cdots, v_{n_4+n_3-r}\}$ be the remaining n_3 vertices of $\overline{G_i}$;

Let $W_i' = \{w_{p+1}, w_{p+2}, \cdots, w_{p+r}\}$;

Let $V_4 = V_4' \cup W_i'$;

{V_3 and V_4 induce connected graphs having n_3 and n_4 vertices, respectively.}

Let $G_{12} = G_i - W_i'$;

Find a bipartition V_1, V_2 of the biconnected graph G_{12} such that $u_1 \in V_1$, $u_2 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce connected subgraphs;

Return V_1, V_2, V_3, V_4 as a 4-partition of G.

Subcase 2(c): $p' < n_4 < p' + r$.

{In this subcase, the first $n_4 - p'$ and the last $p' + r - n_4$ vertices in W_i are added to $\overline{G_i}$ as the deficient r vertices.}

Let $W_{i4}' = \{w_{p+1}, w_{p+2}, \cdots, w_{p+n_4-p'}\}$ be the set of the first $n_4 - p'$ vertices in W_i;

Let $W'_{i3} = \{w_{q-1}, w_{q-2}, \cdots, w_{q-(p'+r-n_4)}\}$ be the set of the last $p' + r - n_4$ vertices in W_i;
$\{W'_{i4} \cap W'_{i3} = \phi$ since $|W'_{i4}| + |W'_{i3}| = r \leq |W_i|$. $|W'_{i4} \cup W'_{i3}| = r\}$;
Let $V_4 = \{v_1, v_2, \cdots, v_{p'}\} \cup W'_{i4}$;
Let $V_3 = \{v_{p'+1}, v_{p'+2}, \cdots, v_{n_4+n_3-r}\} \cup W'_{i3}$;
$\{ |V_4| = n_4, |V_3| = n_3, w_p = v_{p'} \in V_4, w_q \in V_3$, and hence both V_4 and V_3 induce connected graphs.$\}$
Let $G_{12} = G_i - W'_{i4} \cup W'_{i3}$;
Find a bipartition V_1, V_2 of the biconnected graph G_{12} such that $u_1 \in V_1$, $u_2 \in V_2$, $|V_1| = n_1$, $|V_2| = n_2$, and both V_1 and V_2 induce connected subgraphs;
Return V_1, V_2, V_3, V_4 as a 4-partition of G.

□

Clearly the running time of the above algorithm is $O(n)$. Thus we have the following theorem.

Theorem 3.1 *A 4-partition of any 4-connected planar graph G can be found in linear time if the four vertices u_1, u_2, u_3, u_4 are located on the same face of G.*

As a byproduct we have the following lemma.

Lemma 3.2 *For any given internally triangulated 4-connected plane graph $G = (V, E)$, two distinct edges (u_1, u_2) and (u_3, u_4) on $C(G)$, and two numbers n_1, n_2 such that $n_1 + n_2 = n$ and $n_1, n_2 \geq 3$, there exists a partition V_1, V_2 of V such that $u_1, u_2 \in V_1$, $u_3, u_4 \in V_2$, $|V_1| = n_1, |V_2| = n_2$, and both V_1 and V_2 induce biconnected subgraphs of G.*

Proof. By Lemma 2.3, G has a 4-canonical decomposition $\Pi = W_1, W_2, \cdots, W_l$. Since G is internally triangulated, neither (a) nor (c) of (co3) holds and hence (b) must hold for each $W_i, i = 2, 3, \cdots, l-1$. Thus all W_i's except W_1 and W_l are singleton sets, each of W_1 and W_l contains exactly three vertices, and hence $l = n - 4$. For j, $1 < j < l$, the vertex in W_j has four or more neighbors, two of which are in $W_1 \cup W_2 \cup \cdots \cup W_{j-1}$ and other two of which are in $W_{j+1} \cup W_{j+2} \cdots \cup W_l$. Thus, for j, $1 < j < l$, both $W_1 \cup W_2 \cup \cdots \cup W_j$ and $V - (W_1 \cup W_2 \cup \cdots \cup W_j)$ induce biconnected graphs. Hence, it suffices to choose $j = n_1 - 2$. $\mathcal{Q.E.D.}$

4 Conclusion

In this paper we give a linear-time algorithm to find a 4-partition of a 4-connected planar graph G in case four vertices u_1, u_2, u_3, u_4 are located on the same face of G. It is remained as future work to find efficient algorithms for finding a k-partition of a k-connected (not always planar) graph for $k \geq 4$.

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