組合せ最適化ゲーム

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摘要　あるクラスの組合せ最適化問題ゲームを記述する 01-整数計画問題を用いた協力ゲームの新しい定式化を導入する。多くの興味深いグラフ問題のゲームがこの定式化により記述できる。新しい定式化による重要な一般的結果として、このクラスに属するゲームがコア（協力ゲーム理論の解の概念の１つ）を持つかどうかが、対応する 01-整数計画問題が整数制約なしで整数最適解を持つことと同値であることを示す。この数学的条件がグラフ問題において成立するための性質を調べ、それらのコアにを求める算法、計算量に関する次のような問題について考察する：コアが存在するかどうかの決定、コアが存在する場合に１つのコアを求める、与えられた配分がコアかどうかの判定など。例えば、新しい定式化に基づく結果を用いれば、単位ネットワーク上のフローゲームのコアを凸分解する多項式時間の算法を構築することができる。

Combinatorial Optimization Games

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Abstract We introduce a general integer programming formulation for a class of combinatorial optimization games, which include many interesting problems on graphs. The formulation immediately allows us to improve the algorithmic result for finding imputations in the core (an important solution concept in cooperative game theory) of the network flow game on unit networks. An important result is a general theorem that the core for this class of games is nonempty if and only if a related linear program has an integer optimal solution. We study the properties for this mathematical condition to hold for several problems on graphs, and apply them to resolve algorithmic and complexity issues for their cores: decide whether the core is empty; if the core is empty, find an imputation in the core; given an imputation x, test whether x is in the core.

1 Introduction

In the theory of cooperative games, a cooperative game is given by a pair (N, v) of a set N of players and a characteristic function $v : 2^N \rightarrow R_+$ (R_+ is the set of nonnegative reals). The value $v(S)$ for a subset (i.e., a coalition) $S \subseteq N$ means the profit which can be obtained by cooperation of players in S. It is a natural question to ask how to distribute the entire profit $v(N)$ to each player in some fair way, taking into account the other game values $v(S)$, $S \subset N$. A vector $z : N \rightarrow R_+$ with $z(V) = v(V)$ is called an *imputation*, where $z(S)$ for $S \subseteq V$ represents $\sum_{u \in S} z(u)$. Among many methods proposed so far for finding a rational imputation is *core* [6]. The core is defined to be the set of imputations z such that $z(S) \geq v(S)$ holds for all $S \subseteq E$. The concept of core provides us a fundamental principle for such imputation to be rational, as it says that any subgroup of players would acquire at least as much payment as they can collectively obtain as a subgroup.

Recently, several authors have taken algorithmic and complexity issues as the main focus in solutions for game theory problems [1, 2, 5]. An interesting problem discussed by Kalai and Zemel [3, 4] is a network flow game. Consider a digraph $D = (V, E)$ with a source vertex s and a sink vertex t. They consider a cooperative game associated with the maximum flow from s to t by identifying each arc as a player, and defining the value $v(S)$ of a subset $S \subseteq E$ to be the value of a maximum flow in the subgraph $D[S] = (V, S)$. For a special case of the maximum flow game on unit networks, i.e., those with capacity ones, Kalai and Zemel [4] showed a very nice characterization theorem of the core, which is based on a linear programming: An imputation is in the core if and only if it is a convex combination of the characteristic vectors of minimum s-t cuts.

It is not accidental that the above special case of the maximum flow game always has a nonempty core and allows polynomial time solution algorithms. In Section 3, we introduce a general integer programming formulation of combinatorial optimization games, and show that the game has a nonempty core if and only if the corresponding linear programming relaxation has an integer optimal solution. In Section 3, with this general formulation, we introduce several interesting examples. We apply the theorem to discuss algorithmic and computational issues from graph theory related to the core of these games.

In Section 4, we discuss the relationship between two games which are dual in the sense of the corresponding linear programming relaxations, and relate this duality with algorithmic design issues. In Section 5, we consider the edge-connectivity game on undirected graphs, which does not have the standard integer programming formulation of Section 2. In Section 6, we conclude our paper with a summary table of the results in this paper.

2 Packing and Covering Games

2.1 Definitions

Let A be an $m \times n$ $\{0,1\}$-matrix. Let 1_k and 0_k denote the column vectors with all ones and all zeros, respectively, of dimension k. We may denote these vectors by 1 and 0 for simplicity. Let $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ be the corresponding index sets, and let t denote the transposition. Consider the following linear program,

$$\begin{array}{lll} LP(c, A, \max): & \max y^t c & \\ s.t. & y^t A \leq 1_n^t, & y \geq 0_m, \end{array}$$

and its dual,

$$\begin{array}{lll} DLP(c, A, \max): & \min 1_n^t x & \\ s.t. & Ax \geq c, & x \geq 0_n, \end{array}$$

where c is an m-dimensional column vector $\in R^m$, y is an m-dimensional column vector of variables and x is an n-dimensional column vector of variables.

We denote the corresponding integer programming version of $LP(c, A, \max)$ by $ILP(c, A, \max)$. Since A is a $\{0,1\}$-matrix, the integrality constraints are equivalent to require y to have $\{0,1\}$ values. We define the *packing game* $Game(c, A, \max)$ as follows, where $\overline{S} = N - S$:

1. The player set is N.
2. For each subset $S \subseteq N$, $v(S)$ is defined as the value of the following integer program: $ILP(c, A_{M,S}, \max)$:

$$\begin{array}{ll} & \max y^t c \\ s.t. & y^t A_{M,S} \leq 1_{|S|}^t, \quad y^t A_{M,\overline{S}} \leq 0_{n-|S|}^t, \\ & y \in \{0,1\}^m, \end{array}$$

where $A_{T,S}$ is the submatrix of A with row set T and column set S, and $v(\emptyset)$ is defined to be 0.

We then introduce a *covering game* $Game(c, A, \min)$ for the minimization problem in the similar manner:

1. The player set is M.

2. For each subset $T \subseteq M$, $v(T)$ is defined as the value of the following integer program: $ILP(d, A_{T,N}, \min)$:

$$\begin{array}{ll} & \min d^t x \\ s.t. & A_{T,N}x \geq 1_{|T|}, \quad x \in \{0,1\}^n, \end{array}$$

where $v(\emptyset)$ is defined to be 0.

Since the value of the game is defined by a solution to the minimization problem, this is in fact a problem of sharing the cost of the game. Thus, we would revise the definition of core. An imputation $w : M \rightarrow R_+$ is in the *core* if $w(T) \leq v(T)$ holds for all $T \subseteq M$.

From definition, both packing game and covering game are monotone, i.e., $v(S') \leq v(S)$ holds for any subsets $S' \subseteq S$ of N (or M).

In this paper, we shall introduce a number of optimization games on graphs, which are formulated as the above maximization and/or minimization games, and study the following properties and questions concerning their cores.

1. *Nonemptyness*: Is the core of the game always nonempty?
2. *Convex characterization*: Can any imputation in the core be represented as a convex combination of some well-defined dual objects (such as minimum s-t cuts)?
3. *Testing nonemptiness*: Can it be tested in polynomial time whether a given instance of the game has nonempty core?
4. *Checking membership*: Can it be checked in polynomial time whether a given imputation belongs to the core?
5. *Finding a core member*: Is it possible to find an imputation in the core in polynomial time?

As our discussion will focus on games on graphs, the polynomiality is determined in terms of the input size of the graph (even though the sizes of the constraint matrices A in the above formulations are sometimes exponential in $|V|$ and $|E|$).

2.2 Main theorems

Before presenting our main theorem regarding when the core is nonempty, we simplify the conditions for being in the core.

Lemma 1 *For Game$(c, A, \max)$, define $S_i = \{j \in N \mid A_{ij} = 1\}$ for $i \in M$. Then $v(S_i) \geq c_i$ holds for all $i \in M$.*

Proof: By definition, $v(S_i)$ is the optimal value to the following integer program: $ILP(c, A_{M,S_i}, \max)$:

$$\begin{array}{ll} & \max y^t c \\ s.t. & y^t A_{M,S_i} \leq 1^t_{|S_i|}, \quad y^t A_{M,\overline{S_i}} \leq 0^t_{n-|S_i|}, \\ & y \in \{0,1\}^m. \end{array}$$

Set $y_i := 1$, and $y_k := 0$ for $k \in \{1, 2, \ldots, m\} - \{i\}$. This vector y is an integer feasible solution with objective value c_i. □

Lemma 2 *A vector $z : N \rightarrow R_+$ is in the core of Game$(c, A, \max)$ if and only if*

1. *$z(N) = v(N)$ (i.e., z is an imputation),*
2. *$z(S_i) \geq c_i$ for all $i \in M$, where $S_i = \{j \in N \mid A_{ij} = 1\}$ (i.e., z is feasible to $DLP(c, A, \max)$ of $LP(c, A, \max)$.*

Proof: The necessity follows from Lemma 1. To prove its sufficiency, we show that $z(S) \geq v(S)$ holds for all $S \subseteq N$. Consider an optimal solution y^* to the integer program: $ILP(c, A_{M,S}, \max)$:

$$\begin{array}{ll} & \max y^t c \\ s.t. & y^t A_{M,S} \leq 1^t_{|S|}, \quad y^t A_{M,\overline{S}} \leq 0^t_{n-|S|}, \\ & y \in \{0,1\}^m, \end{array}$$

which yields $v(S)$ Let $I = \{i \in M \mid y^*_i = 1\}$; i.e., $v(S) = \sum_{i \in I} c_i$. From $(y^*)^t A_{M,\overline{S}} \leq 0^t_{n-|S|}$, it holds $S_i \subseteq S$ for all $i \in I$. Furthermore $(y^*)^t A_{M,S} = \sum_{i \in I} A_{i,S} \leq 1^t_{|S|}$ implies that all S_i, $i \in I$, are disjoint. Therefore, $z(S) \geq \sum_{i \in I} z(S_i)$. On the other hand, $v(S) = \sum_{i \in I} c_i \leq \sum_{i \in I} z(S_i)$ by the assumption 2 on z. Hence $v(S) \leq z(S)$. □

Lemma 2 leads to the following theorem:

Theorem 1 *The core for Game$(c, A, \max)$ is nonempty if and only if $LP(c, A, \max)$ has an integer optimal solution. In such case, a vector $z : N \rightarrow R_+$ is in the core if and only if it is an optimal solution to $DLP(c, A, \max)$.*

Proof: Let $z : N \rightarrow R_+$ be a vector in the core. In Lemma 2, the first condition states that $z(N)$ is equal to the optimal value of $ILP(c, A, \max)$. The second condition of Lemma 2 states that z is a feasible solution to $DLP(c, A, \max)$, the dual of $LP(c, A, \max)$. Now if the two conditions of Lemma 2 holds, then $z(N) = v(N) \leq opt(ILP(c, A, \max)) \leq opt(LP(c, A, \max)) \leq opt(DLP(c, A, \max))$ (by the duality theory of linear programming) $\leq z(N)$, and we have

$opt(ILP(c, A, \max)) = opt(LP(c, A, \max))$, where $opt(P)$ denotes the optimum value of problem P.

On the other hand if $opt(ILP(c, A, \max)) = opt(LP(c, A, \max))$, then let $z : N \to R_+$ be an optimal solution of $DLP(c, A, \max)$. Then $z(N) = opt(DLP(c, A, \max)) = opt(LP(c, A, \max)) = opt(ILP(c, A, \max)) = v(N)$ implies $z(N) = v(N)$ (i.e., condition 1 of Lemma 2). The condition 2 also holds since z is feasible to $DLP(c, A, \max)$. Then, z is in the core by Lemma 2.

The second statement follows from the above argument. □

Similarly, we have the following lemma and theorem for the minimization game.

Lemma 3 *A vector $w : M \to R_+$ is in the core of $Game(d, A, \min)$ if and only if*

1. *$w(M) = v(M)$ (i.e., w is an imputation),*
2. *$w(T_j) \leq d_j$ for all $j \in N$, where $T_j = \{i \in M \mid A_{ij} = 1\}$ (i.e., z is feasible to the dual $DLP(d, A, \min)$ of $LP(d, A, \min)$).* □

Theorem 2 *The core for $Game(d, A, \min)$ is nonempty if and only if $LP(d, A, \min)$ has an integer optimal solution. In such case, a vector $w : M \to R_+$ is in the core if and only if it is an optimal solution to $DLP(d, A, \min)$.* □

3 A Selection of Examples

There are many interesting optimization games on graphs, which can be formulated as packing and covering games in Section 2. Kalai surveys many games of this kind [2], including the maximum flow game. We will focus on the following games.

1. Maximum flow game in unit networks, s-t edge connectivity game in undirected graphs, s-t vertex connectivity game in undirected graphs, and maximum matching game in bipartite graphs.
2. Maximum r-arborescence game.
3. Maximum matching game and minimum vertex cover game.
4. Maximum independent set game and minimum edge cover game.
5. Minimum coloring game.

3.1 The Network Flow Game and Its Variants

Let us consider the maximum flow game in a unit directed network $D = (V, E)$ with source $s \in V$ and sink $t \in V$, which is also denoted simply by $D = (V, E, s, t)$.

Lemma 4 *Given a digraph $D = (V, E, s, t)$, the s-t arc-connectivity of D (i.e., the value of a maximum flow) is equal to the size of a minimum s-t cut in D.* □

Now let us define the s-t arc-connectivity on a digraph $D = (V, E, s, t)$. In this game, each player controls an arc, and the value $v(S)$ of a subset $S \subseteq E$ is defined to be the maximum flow (=the size of the maximum number of arc-disjoint paths) from s to t on the subnetwork $D[S] = (V, S)$. This game (E, v) can be represented by a packing game $Game(c, A, \max)$ of Section 2. Let $\mathcal{P}$ be the set of paths from source s to sink t (called s-t paths, for short) in G, and $A = A_{\mathcal{P},E}$ be the path-arc incidence matrix, where $A_{ij} = 1$ if and only if the arc j is in the s-t path i. Then (E, v) is given by $Game(1_{|\mathcal{P}|}, A, \max)$.

In some cases, there may be *dummy players* in the game in the sense that those players j always get $z(j) = 0$ in an imputation $z : N \to R_+$. To make $Game(1_{|M|}, A, \max)$ more general for the purpose of utilizing it in discussing the s-t vertex-connectivity game and some other games later, we introduce a set $\hat{E} \subseteq E$ of dummy players. A set $\hat{E}$ of arcs (for dummy players) is called *valid* if $F = E - \hat{E}$ contains at least one minimum s-t cut $C \subseteq E$ of D.

Theorem 3 *For a digraph $D = (V, E, s, t)$ and a set $\hat{E}$ of dummy players, the nontrivial s-t arc connectivity game has nonempty core if and only if $\hat{E}$ is valid.* □

Theorem 4 *Let $z : E \to R_+$ be an imputation of the nontrivial s-t arc-connectivity game on a digraph $D = (V, E, s, t)$ with a valid set $\hat{E}$ of dummy players. Then z is in the core with respect to $\hat{E}$ (i.e., $z(e) = 0$, $e \in \hat{E}$) if and only if it is a convex combination of the characteristic vectors for minimum s-t cuts C contained in $F = E - \hat{E}$.* □

Corollary 1 *For a valid set $\hat{E}$ of dummy players, testing nonemptiness, checking membership and finding a core member of the s-t arc-connectivity game, can all be answered in polynomial time.* □

We emphasize at this point that the results in Theorems 3 and 4 can be extended to other optimization games on graphs, which can be reducible to the maximum flow game in a directed network. Those problems include:

P1 s-t edge-connectivity game in an undirected graph $G = (V, E, s, t)$, where players are on edges and $v(S)$, $S \subseteq E$ is defined to be the size of maximum flow from s to t in the induced network $G[S]$,

P2 s-t vertex-connectivity game in a digraph $D = (V, E, s, t)$ (resp. undirected graph $G = (V, E, s, t)$), where players are on vertices except s and t, and $v(S)$, $S \subseteq V - \{s, t\}$ is defined to be the maximum number of arc (resp., edge) disjoint paths from s to t in the induced digraph $D[S]$ (resp., graph $G[S]$),

P3 maximum matching game with edge players on a bipartite graph $G = (V_1, V_1, E)$, where $v(S)$, $S \subseteq E$ is defined to be the size of maximum matching in the induced graph $G[S]$.

Using standard reduction techniques for network flow problems and with aid of dummy players, we can show the followings.

Corollary 2 *For a game in the above P1 (resp., P2 and P3), the core is always nonempty, and if the game is not trivial, the core is a convex combination of a set of characteristic vectors of minimum s-t cuts (resp., minimum s-t vertex-cuts for P2 and minimum vertex-covers for P3). Furthermore, testing nonemptiness, checking membership and finding a core member of all these games, can be answered in polynomial time.* □

3.2 The Arborescence Game

The maximum r-arborescence game and minimum r-cut game is played on a digraph $D = (V, E)$ with a root $r \in V$. For each subset $S \subseteq E$ of arcs (i.e., players), the game value $v(S)$ is defined to be the size of the maximum number of d arc-disjoint r-arborescences on the subgraph $G[S] = (V, S)$. This game can be formulated as a packing game $Game(1_{|M|}, A, \max)$ by matrix A such that the rows correspond to all r-arborescences and the columns correspond to all arcs; $A_{ij} = 1$ if and only if arc j is in the i-th r-arborescence.

A set $\hat{E}$ of arcs (for dummy players) is called *valid* if $F = E - \hat{E}$ contains at least one minimum r-cut $C \subseteq E$ of D. Analogously with Theorems 3 and 4, we have the following results.

Theorem 5 *For a digraph $D = (V, E)$ with root $r \in V$ and a set $\hat{E}$ of dummy players, the maximum r-arborescence game with $v(E) > 0$ has nonempty core if and only if $\hat{E}$ is valid.* □

Theorem 6 *Let $z : E \to R_+$ be an imputation of the maximum r-arborescence game on a digraph $D = (V, E)$ with root $r \in V$ and a valid set $\hat{E}$ of dummy players and let $v(E) > 0$. Then z is in the core with respect to $\hat{E}$ (i.e., $z(e) = 0$, $e \in \hat{E}$) if and only if it is a convex combination of the characteristic vectors for minimum r-cuts C contained in $F = E - \hat{E}$.* □

Corollary 3 *For a set $\hat{E}$ of dummy players, testing nonemptiness, checking membership and finding a core member of the maximum r-arborescence game, can all be answered in polynomial time.* □

3.3 Matching and Vertex Cover

Given a graph $G = (V, E)$, we define the maximum matching game by a game such that the players are on vertices and the game value $v(S)$ for a subset $S \subseteq V$ is the maximum matching size in the subgraph $G[S]$ induced by S. Similarly, the minimum vertex cover game is defined by a game such that players are on edges and $v(S)$ for $S \subseteq E$ is the minimum vertex cover size in the subgraph $G[S] = (V, S)$. These games are formulated by packing game $Game(1_{|E|}, A, \max)$ and covering game $Game(1_{|V|}, A, \min)$, respectively, where the constraint matrix A is the edge-vertex incidence matrix of G in which $A_{ij} = 1$ if and only if edge i and vertex j are incident.

3.3.1 Matching

By Lemma 2, an imputation z is in the core of the matching game if and only if $z(u) + z(u') \geq 1$ holds for all edges $(u, u') \in E$. Based on this observation, we can easily find two classes of graphs for which the cores are always nonempty. The first class of graphs for which the size of a minimum vertex cover is the same as the size of a maximum matching, and the class of graphs with perfect matching. However, the next theorem says that these are essentially all graphs which have nonempty cores for the maximum matching game.

Theorem 7 *An undirected graph $G = (V, E)$ has a nonempty core for the maximum matching game if and only if there exists a subset $V_1 \subseteq V$ such that*

1. *the subgraph $G_1 = G[V_1]$ induced by V_1 has a minimum vertex cover W with the same size as its maximum matching,*

2. *the subgraph $G_2 = G[V - V_1]$ induced by $V - V_1$ has a perfect matching,*

3. *all the remaining edges* $(u,u') \in E$ *between* G_1 *and* G_2 *satisfy* $u \in W$ *for the vertex cover* W *in 1.* □

Corollary 4 *For the core of the maximum matching game, testing nonemptiness, checking membership and finding a core member, can be done in polynomial time.* □

3.3.2 Vertex Cover

We characterize the class of graphs that have a nonempty core of the minimum vertex cover game.

Theorem 8 *The core for the minimum vertex cover game on graph* $G = (V, E)$ *is nonempty if and only if the size of a maximum matching is equal to the size of a minimum vertex cover.* □

Based on this, we have the following results.

Theorem 9 *For the core of the minimum vertex cover game, testing nonemptiness, checking membership and finding a core member, can be done in polynomial time.* □

Theorem 10 *Assume that the minimum vertex cover game on an undirected graph* $G = (V, E)$ $(E \neq \emptyset)$ *has nonempty core. Then an imputation is in the core if and only if it is a convex combination of the characteristic vectors of maximum matchings in* G. □

3.4 Edge Cover and Independent Set

For an undirected graph $G = (V, E)$, we can define a mutually dual pair of the minimum edge cover game and the independent set game by $Game(1_{|E|}, A', \min)$ and $Game(1_{|V|}, A', \max)$, respectively, where the constraint matrix A' is the vertex-edge incidence matrix of G (i.e., the transposition of the matrix A used for the pair of the maximum matching game and the minimum vertex cover game). Thus, for the minimum edge cover game, the players are on vertices, and the game value $v(S)$ for $S \subseteq V$ is the minimum number of edges that cover all vertices in S, i.e.,

$$\min\{|F| \mid F \cap E(u) \neq \emptyset, \ \forall u \in S\},$$

where $E(u)$ denotes the set of edges in E which are incident to u. Note that $v(S)$ is not necessarily the size of a minimum edge cover in the subgraph $G[S]$ induced by vertex set S.

Similarly, the players for the maximum independent set game are on edges and the game value $v'(T)$ for $T \subseteq E$ is the size of a maximum independent set in the subgraph $G[V\langle T\rangle]$ induced by $V\langle T\rangle$, where $V\langle T\rangle$ is defined by

$$V\langle T\rangle = \{i \in V \mid i \text{ is adjacent only to edges in } T\}.$$

(Note that $v'(T)$ is not the size of a maximum independent set in the subgraph $G[T]$.)

3.4.1 Edge Cover

We first observe that the minimum edge cover game is equivalent to the maximum matching game in the following sense.

Lemma 5 *For an undirected graph* $G = (V, E)$, *which has no isolated vertex, let* v *and* $\overline{v}$ *be the game values of the minimum edge cover game and the maximum matching game on* G, *respectively. Then* $v(S) + \overline{v}(S) = |S|$ *holds for all* $S \subseteq V$. □

Theorem 11 *Let* $G = (V, E)$ *an undirected graph with no isolated vertex. Then* $w : V \to R_+$ *is in the core of the minimum edge cover game on* G *if and only if* $\overline{w} = 1_{|V|} - w$ *is in the core of the maximum matching game on* G. □

This theorem and Theorem 7 claim that the graphs with nonempty cores for the minimum edge cover game is exactly same as those for the maximum matching game.

3.4.2 Independent Set

We first prove that the counterpart of Theorem 8 is also true for the maximum independent set game.

Theorem 12 *Let* $G = (V, E)$ *be an undirected graph with no isolated vertex. Then the core for the maximum independent set game on graph* G *is nonempty if and only if the size of a maximum independent set is equal to the size of a minimum edge cover in* G. □

Theorem 13 *For the core of the maximum independent set game, testing nonemptiness, checking membership and finding a core member, can be done in polynomial time.* □

Theorem 14 *Given an undirected graph* $G = (V, E)$ *with* $V \neq \emptyset$ *but no isolated vertex, an imputation is in the core of the maximum independent set game if and only if it is a convex combination of the characteristic vectors of minimum edge covers.* □

One may define the maximum clique problem in an undirected graph $G = (V, E)$ as the maximum independent set problem on its complement graph $\overline{G} = (V, \overline{E})$. Obviously, such clique game is given by a packing game $Game(1_{|\overline{E}|}, A'', \max)$ which has players on the edges in $\overline{G}$, where A'' is the vertex-edge incidence matrix A'' of the complement graph $\overline{G}$. Therefore, all the results in this subsection can be generalized to the maximum clique problem.

3.5 Chromatic Number

Let $\chi(G')$ denote the chromatic number of an undirected graph G' (i.e., the minimum number of maximal independent set which together covers all vertices of G'). For the minimum coloring game on a graph $G = (V, E)$, we define the game value $v(S)$, $S \subseteq V$ as $\chi(G[S])$, i.e., the size of a minimum coloring of the subgraph $G[S]$ induced from G by S. This game can be represented by a covering game $Game(1_{|\mathcal{I}|}, A, \min)$, the rows of the matrix A correspond to the vertices in a graph G, and the columns correspond to maximal independent sets, where $\mathcal{I}$ denotes the set of all maximal independent sets in G.

By Lemma 3, a vector $w : V \to R_+$ is in the core of the minimum coloring game if and only if $w(V) = \chi(G)$ and $w(S) \leq 1$ for any independent set $S \subseteq V$. Let $\omega(G)$ denote the size of a maximum clique in G, which satisfies $\omega(G) \leq \chi(G)$, as widely known in the coloring problem. We can easily observe that the characteristic vector I_C of a maximum clique $C \subseteq V$ is a core of the coloring game if $\omega(G) = \chi(G)$ holds. Therefore the minimum coloring game on such a graph has nonempty core. However, the converse is not true. That is, there is a graph $G = (V, E)$ such that $\omega(G) < \chi(G)$ but the core of the coloring game is nonempty. For example, for a graph G with $\omega(G) < \chi(G)$ and $\alpha(G)\chi(G) = |V|$, the imputation z defined by $z(u) := \chi(G)/|V|$ for all $u \in V$ is in the core, where $\alpha(G)$ is the stable number of G. Therefore, in general, the coloring game has no convex characterization by a set of maximum cliques. Also, from such a graph G, construct the graph $G' = G + K_{\chi(G)}$ by adding complete graph $K_{\chi(G)}$ via a single common vertex. Then G' satisfies $\omega(G') = \chi(G')$, but has nonempty core. That is, in general, the coincidence of the optimum values of $ILP(1_m, A, \max)$ and $IPL(1_n, A, \min)$ does not imply the convex characterization of the core of a $Game(1_n, A, \min)$.

We define for each edge $e = (i, j) \in E$ its characteristic vector $I_e : V \to \{0, 1\}$ by $I_e(k) := 1$ if $k \in \{i, j\}$ and 0 otherwise.

Theorem 15 *If a graph $G = (V, E)$ is bipartite and $E \neq \emptyset$, an imputation $w : V \to R_+$ is in the core of the minimum coloring game if and only if it is a convex combination of the characteristic vectors of edges in E; i.e., $w = \sum_{e \in E} \lambda_e I_e$, with $\sum_{e \in E} \lambda_e = 1$ and $\lambda_e \geq 0$ for all $e \in E$. This can be tested in polynomial time.* □

For general graphs, expectably, the problem is NP-complete.

Theorem 16 *For the minimum coloring game, it is NP-complete to decide whether the core is empty or not. It is also NP-complete to decide whether a given imputation is in the core or not.* □

A graph G is called *perfect* if $\omega(G[S]) = \chi(G[S])$ for all $S \subseteq V$.

Theorem 17 *Let $G = (V, E)$ be a perfect graph. Then the core of the minimum coloring game is always nonempty. Furthermore it can be tested in polynomial time whether an imputation w is in the core or not.* □

Given a game (N, v) with $v : 2^N \to R_+$ and a subset S of players with $\emptyset \neq S \subseteq N$, the game (S, v_S) with $v_S(S') = v(S')$ for $S' \subseteq S$ is called the *subgame* of (N, v) induced by S. A game (N, v) is called *totally balanced* if any subgame (S, v_S), $S \subseteq N$ has nonempty cores [3].

Theorem 18 *A graph $G = (V, E)$ is perfect if and only if the minimum coloring game on G is totally balanced.* □

4 Totally Balanced Games

In this section, we discuss the relationship of the total balancedness between $Game(1_m, A, \max)$ and $Game(1_n, A, \min)$.

Theorem 19 *If $Game(1_m, A, \max)$ is totally balanced, then the core for $Game(1_n, A, \min)$ is nonempty.* □

A weaker condition such as a nonempty core only for $Game(1_m, A, \max)$ would not give the same results. In addition, a stronger result that $Game(1_n, A, \min)$ is totally balanced would not hold either. However, for the opposite direction, the result is stronger as will be stated in Theorem 20.

Lemma 6 *If $Game(1_n, A, \min)$ is totally balanced, then the core for $Game(1_m, A, \max)$ is non-empty.* □

Lemma 6 provides an alternative proof of Theorem 18.

The condition that $Game(1, A, \min)$ is totally balanced cannot be relaxed to the nonemptiness of the core of $Game(1, A, \min)$. However, we can make the conclusion stronger.

Theorem 20 *If $Game(1_n, A, \min)$ is totally balanced, then $Game(1_m, A, \max)$ is also totally balanced.* □

5 Edge-connectivity in Graphs

Here we consider the edge-connectivity game on an undirected graph $G = (V, E)$. In this game, players are on edges, and $v(S)$ for $S \subseteq E$ is defined by $\lambda(G[S])$, where notation $G[S] = (V, S)$ is used and $\lambda(H)$ denotes the edge-connectivity of a graph H. Unfortunately this game does not have the standard formulation of Section 2. However, it is possible to ask all questions discussed so far.

We first see that this game always has nonempty core because the characteristic vector I_C for a minimum cut $C \subseteq E$ in G is in the core; for each $S \subseteq E$, $z = I_C$ satisfies $z(S) = |S \cap C| \geq v(S)$ (since $S \cap C$ is a cut in $G[S]$).

Although the edge-connectivity game is very similar to the s-t edge connectivity game studied in Section 4.1, the convex characterization of the core does not hold in general.

Theorem 21 *Let k denote the edge-connectivity of an undirected graph $G = (V, E)$. To test whether an imputation z is in the core of the edge-connectivity game can be done in polynomial time for $k \leq 2$, but is co-NP-complete if $k = 3$.* □

6 Conclusion

The computational issues in game theory have received much attention recently, and have been a motivation of our investigation into the classes of optimization games on graphs. We conclude the paper by giving a table of the results considered for the five properties/questions raised in Section 2.

References

[1] X. Deng and C. Papadimitriou, "On the complexity of cooperative game solution concepts," *Math. Oper. Res.*, 19, 2, pp. 257–266, 1994.

[2] E. Kalai, "Games, Computers, and O.R.," *ACM/SIAM SODA*, pp. 468-473, 1995.

[3] E. Kalai, and E. Zemel, "Totally balanced games and games of flow," *Math. Oper. Res.*, 7, pp. 476-478, 1982.

[4] E. Kalai, and E. Zemel, "Generalized network problems yielding totally balanced games," *Oper. Res.*, 30, pp. 998-1008, 1982.

[5] H. Nagamochi, D. Zeng, N. Kabutoya, and T. Ibaraki, "Complexity of the minimum base games on matroids," to appear in *Math. Oper. Res.*.

[6] L. S. Shapley, "On balanced sets and cores" *Naval Research Logistics Quarterly*, 14, pp. 453-460, 1967.

Table 1. Summary of the results for optimization games on graphs.

Games	Core nonemptiness	Convex characterization of the core	Testing nonemptiness of the core	Checking if an imputation is in the core	Finding an imputation in the core
Max flow (G, D)	yes	yes	—	P	P
s-t connectivity (G, D)	yes	yes	—	P	P
r-arborescence (D)	yes	yes	—	P	P
Max matching (G)	no	no	P	P	P
Min vertex cover (G)	no	yes	P	P	P
Min edge cover (G)	no	no	P	P	P
Max indep. set (G)	no	yes	P	P	P
Max clique (G)	no	yes	P	P	P
Min coloring (G)	no	no	NPC	NPC	NPH
Edge-connectivity (G)	yes	no	—	NPC	P

D: digraphs, G: undirected graphs, P: polynomial time, NPC: NP-complete, NPH: NP-hard, —: trivial