辺連結度，点連結度を同時に最適増大させる問題

石井 利昌，永持 仁，茨木 俊秀

京都大学 工学研究科 数理工学教室
606-01 京都市左京区吉田本町
e-mail: ishii,naga,ibaraki@kuamp.kyoto-u.ac.jp

あらまし 辺連結度，点連結度を同時に最適増大させる問題とは，入力として，無向多重グラフ $G=(V,E)$ と，要求関数 $\{r_\lambda(x,y)\in Z^+|x,y\in V\}$, $\{r_\kappa(x,y)\in Z^+|x,y\in V\}$ (Z^+ は非負整数集合を表す) が与えられたとき，最小本数の辺を G に加えることで，全ての $x,y\in V$ 間の辺連結度および点連結度をそれぞれ $r_\lambda(x,y)$ 以上，かつ $r_\kappa(x,y)$ 以上にする問題である．本研究では，全ての $x,y\in V$ について $r_\kappa(x,y)=2$ である場合については，この問題が多項式時間で解けることを示す．

和文キーワード: 無向多重グラフ，連結度増加問題，辺連結度，点連結度，辺分離．

Augmenting Edge-Connectivity and Vertex-Connectivity Simultaneously in Undirected Graphs

Toshimasa ISHII, Hiroshi NAGAMOCHI and Toshihide IBARAKI

Department of Applied Mathematics and Physics,
Graduate School of Engineering,
Kyoto University
Kyoto 606-01, Japan.

Abstract Given an undirected multigraph $G=(V,E)$ and requirement functions $\{r_\lambda(x,y)\in Z^+|x,y\in V\}$ and $\{r_\kappa(x,y)\in Z^+|x,y\in V\}$ (where Z^+ is the set of nonnegative integers), the edge and vertex-connectivities augmentation problem asks to augment G by adding the smallest number of new edges to G so that for every $x,y\in V$, the edge-connectivity and vertex-connectivity between x and y are at least $r_\lambda(x,y)$ and $r_\kappa(x,y)$, respectively in the resulting graph G'. In this paper, we show that if $r_\kappa(x,y)=2$ holds for every $x,y\in V$, then the problem can be solved in polynomial time.

英文 key words: undirected multigraph, connectivity augmentation problem, edge-connectivity, vertex-connectivity, edge-splitting.

1 Introduction

Let $G = (V, E)$ stand for an undirected multigraph with a set V of *vertices* and a set E of *edges*, where an edge with end vertices u and v is denoted by (u, v). A singleton set $\{x\}$ may be simply denoted by x. For two disjoint subsets $X,\ Y \subset V$, we denote by $E_G(X, Y)$ the set of edges, one of whose end vertices is in X and the other is in Y, and by $c_G(X, Y)$ the number of edges in $E_G(X, Y)$. In particular, $E_G(u, v)$ implies the set of edges with end vertices u and v, and $c_G(u, v) = |E_G(u, v)|$. We denote $n = |V|$, $e = |E|$. For a subset $V' \subseteq V$ in G, $G - V'$ denotes the subgraph induced by $V - V'$. A *cut* is defined as a subset X of V with $\emptyset \neq X \neq V$, and the *size* of cut X is denoted by $c_G(X, V - X)$, which may also be written as $c_G(X)$. A cut with the minimum number is called a *(global) minimum cut*, and its size, denoted by $\lambda(G)$, is called the *edge-connectivity* of G. The local edge-connectivity $\lambda_G(x, y)$ for two vertices $x, y \in V$ is defined to be the minimum size of a cut in G that separates x and y, or equivalently the maximum number of edge-disjoint path between x and y [4]. For a subset X of V, $\{v \in V - X \mid (u, v) \in E$ for some $u \in X\}$ is called a *neighbor set* of X, denoted by $\Gamma_G(X)$. Let $p(G)$ denote the number of components in G. A *separator* is defined as a cut S of V such that $p(G - S) > p(G)$ holds and no $S' \subset S$ has this property (not necessarily $p(G) = 1$). If $G \neq K_n$, then a separator with the minimum size is called a *(global) minimum separator*, and and its size, denoted by $\kappa(G)$, is called the *vertex-connectivity* of G. If $G = K_n$, define $\kappa(G) = n - 1$. The local vertex-connectivity $\kappa_G(x, y)$ for two vertices $x, y \in V$ is defined to be the number of vertex-disjoint paths between x and y in G. For a separator S, there is a component X of G such that $S \subseteq X$, and we say that the components in $X - S$ are *the S-components.* Let

$$\beta(G) = \max\{p(G - S) | S \text{ is a minimum separator}\}.$$

A cut $T \subset V$ is called *tight* if $G[T]$ induces a connected graph and $\Gamma_G(T)$ is a minimum separator in G and no $T' \subset T$ has this property.

In this paper, for a given function $a : \binom{V}{2} \rightarrow R^+$ (resp., $b : \binom{V}{2} \rightarrow R^+$), we call G *a-edge-connected* (resp., *b-vertex-connected*) if $\lambda_G(x, y) \geq a(x, y)$ (resp., $\kappa_G(x, y) \geq b(x, y)$) holds for every $x, y \in V$, if there is no confusion.

Given a multigraph $G = (V, E)$ and a requirement function $r_\lambda : \binom{V}{2} \rightarrow Z^+$ (Z^+: the set of nonnegative integers), (resp., a requirement function $r_\kappa(x, y) : \binom{V}{2} \rightarrow Z^+$), the *edge-connectivity augmentation problem,* (resp., the *vertex-connectivity augmentation problem*) asks to augment G by adding the smallest number of new edges so that the resulting graph G' becomes r_λ-edge-connected (resp., r_κ-vertex-connected). When the requirement function r_λ (resp., r_κ) satisfies $r_\lambda(x, y) = k \in Z^+$ for each $x, y \in V$ (resp., $r_\kappa(x, y) = l \in Z^+$ for each $x, y \in V$), this problem is called the global edge-connectivity problem (resp., the global vertex-connectivity problem).

Watanabe and Nakamura [17] first proved that the global edge-connectivity augmentation problem can be solved in polynomial time for any given integer k. Their algorithm increases edge-connectivity one by one, each time augmenting edges on the basis of structural information of the current G. Currently, $O(e + k^2 n \log n)$ time algorithm due to Gabow [6] and $\tilde{O}(n^3)$ time randomized algorithm due to Benczúr [1], whose deterministic running time is $O(n^4)$, are the fastest among the existing algorithms. Different from the approach by Watanabe and Nakamura, Cai and Sun [2] first pointed out that the augmentation problem for a given k can be directly solved by applying the Lovász edge-splitting theorem. Based on this, Frank [5] gave an $O(n^5)$ time augmentation algorithm. Afterwards, Gabow [7] and Nagamochi and Ibaraki [15] improved it to $O(mn^2 \log(n^2/m))$ and $O(n^2(m + n \log n))$, respectively. Recently, Nagamochi and Ibaraki [16] gave an $O(n(m + n \log n) \log n)$ time algorithm. For a general r_λ, Frank [5] showed that the edge-connectivity augmentation problem can be solved in polynomial time by using Mader's edge-splitting theorem, and the time complexity is improved by Gabow [7] to $O(n^3 m \log(n^2/m))$.

As to the vertex-connectivity augmentation problem, the problem of adding the minimum number of new edges to make a k-vertex-connected graph $(k + 1)$-vertex-connected has been studied by several researchers. It is easy to see that $M(G) = \max\{\beta(G) - 1, \lceil t(G)/2 \rceil\}$ plays an lower bound on the optimal value to this problem, where $t(G)$ denotes the maximum number of pairwise disjoint tight sets in G. Eswaran and Tarjan [3] proved that the vertex-connectivity augmentation problem can be solved by adding $M(G)$ edges to G for $k = 1$, Watanabe and Nakamura stated the same result for $k = 2$ [18]. However, $M(G)$ may be smaller than the optimal value for $k \geq 3$. Recently Jordán presented an $O(n^5)$-time approximation algorithm for this problem [12, 13]. The difference between the number of new edges added by his algorithm and the optimal value is at most $(k - 2)/2$.

It is known that if requirement function r_κ satisfies $r_\kappa(x, y) = k$, $x, y \in V$ for some $k \in \{2,\ 3,\ 4\}$, then the global vertex-connectivity augmentation problem can be solved in polynomial time due to [3, 10], [18, 8], [11], where an input graph G may not be k-vertex-connected. However, whether there is an polynomial time algorithm for the global vertex-connectivity augmentation problem for an arbitrary k is an open question.

In this paper, we consider the problem of augmenting the edge-connectivity and the vertex-connectivity of a given graph G simultaneously by adding the smallest number of new edges. For a given function $a : \binom{V}{2} \to R^+$ (resp., $b : \binom{V}{2} \to R^+$), we say that G is *(a, b)-connected* if G is a-edge-connected and b-vertex-connected.

Given a multigraph $G = (V, E)$, a requirement function r_λ: $\binom{V}{2} \to Z^+$, a requirement function r_κ: $\binom{V}{2} \to Z^+$, the *edge and vertex-connectivities augmentation problem*, denoted by EVAP(r_λ, r_κ), asks to augment G by adding the smallest number of new edges to G so that the resulting graph G' becomes (r_λ, r_κ)-connected. Without loss of generality, $r_\lambda(x, y) \geq r_\kappa(x, y)$ is assumed for every $x, y \in V$. Clearly, EVAP(r_λ, r_κ) contains the edge-connectivity augmentation problem and the vertex-connectivity augmentation problem as its special cases.

When the requirement function r_κ satisfies $r_\kappa(x, y) = \ell \in Z^+$ for each $x, y \in V$, this problem is denoted by EVAP(r_λ, ℓ), if no confusion arises. In this paper, we show that the problem EVAP$(r_\lambda, 2)$ can be solved in polynomial time for any requirement function r_λ.

In Section 2, we introduce preliminaries and a lower bound on the number of edges that are necessary to make a given graph G (r_λ, r_κ)-connected. In Section 3, we describe an outline of an algorithm for making a given graph G $(r_\lambda, 2)$-connected by adding a new edge set whose size is equal to the lower bound. In Section $4-7$, we prove the correctness of each step in our algorithm.

2 Preliminaries

2.1 Definitions

For a multigraph $G = (V, E)$, its vertex set V and edge set E may be denoted by $V[G]$ and $E[G]$, respectively. For a subset $V' \subseteq V$ (resp., $E' \subseteq E$) in G, $G[V']$ (resp., $G[E']$) denotes the subgraph induced by V' (resp., E'). For $V' \subset V$ (resp., $E' \subseteq E$) in G, we denote $G[V - V']$ (resp., $G[E - E']$) simply by $G - V'$ (resp., $G - E'$). For an edge set F with $F \cap E = \emptyset$, we denote $G = (V, E \cup F)$ by $G + F$. A *partition* $X_1, \cdots, X_t$ of vertex set V means a family of nonempty disjoint subsets of V whose union is V, and a ***subpartition*** of V means a partition of a subset of V.

We say that a cut X *separates* two disjoint subsets Y and Y' of V if $Y \subseteq X$ and $Y' \subseteq V - X$ (or $Y \subseteq V - X$ and $Y' \subseteq X$) hold. In particular, a cut X separates x and y if $x \in X$ and $y \in V - X$ (or $x \in V - X$ and $y \in X$) hold. A cut X *crosses* another cut Y if none of subsets $X \cap Y$, $X - Y$, $Y - X$ and $V - (X \cup Y)$ is empty. We say that a separator $S \subset V$ separates two disjoint subsets Y and Y' of $V - S$ if no two vertices $x \in Y$ and $y \in Y'$ are connected in $G - S$. In particular, a separator S separates vertices x and y in $V - S$ if x and y are contained in different components of $G - S$.

2.2 Edge-Splitting

In this section, we introduce an operation of transforming a graph, called *edge-splitting*, which is helpful to solve the edge-connectivity augmentation problem.

Given a multigraph $G = (V, E)$, a designated vertex $s \in V$, vertices $u, v \in \Gamma_G(s)$ and a nonnegative integer $\delta \leq \min\{c_G(s, u), c_G(s, v)\}$, we construct graph $G' = (V, E')$ from G by deleting δ edges from $E_G(s, u)$ and $E_G(s, v)$, respectively, and adding new δ edges to $E_G(u, v)$:

$c_{G'}(s, u) := c_G(s, u) - \delta$,
$c_{G'}(s, v) := c_G(s, v) - \delta$,
$c_{G'}(u, v) := c_G(u, v) + \delta$,
$c_{G'}(x, y) := c_G(x, y)$ for all other pairs $x, y \in V$.

We say that G' is obtained from G by *splitting* δ pair of edges (s, u) and (s, v) (or by splitting (s, u) and (s, v) by size δ), and denote the resulting graph G' by $G/(u, v; \delta)$. Clearly, for any cut X, if cut X separates s and $\{u, v\}$, then $c_{G/(u,v;\delta)}(X) = c_G(X) - 2\delta$ holds, and otherwise then $c_{G/(u,v;\delta)}(X) = c_G(X)$. A sequence of splittings is *complete* if the resulting graph G' does not have any neighbor of s.

The following theorem holds is proven by Mader [14].

Theorem 2.1 [14] *Let $G = (V, E)$ be a multigraph with a designated vertex $s \in V$ with $c_G(s) \neq 1, 3$ and $\lambda_G(x, y) \geq 2$ for each pair $x, y \in V$. Then for each edge $(s, u) \in E$ there is an edge $(s, v) \in E$ such that $\lambda_{G/(u,v;1)}(x, y) = \lambda_G(x, y)$ holds for every pair $x, y \in V - s$.* □

This says that if $c_G(s)$ is even, there always exists a complete splitting at s such that the resulting graph G' satisfies $\lambda_{G'-s}(x, y) = \lambda_G(x, y)$ for each pair $x, y \in V - s$.

2.3 Lower Bound

In this section, we consider the EVAP(r_λ, r_κ), and give a lower bound on the number of edges that is necessary to make a graph G (r_λ, r_κ)-connected. For a vertex set $X \subset V$, define

$$r_\lambda(X) \equiv \max\{r_\lambda(u, v) \mid u \in X, v \in V - X\},$$
$$r_\kappa(X) \equiv \max\{r_\kappa(u, v) \mid u \in X, v \in V - X - \Gamma_G(X), V - X - \Gamma_G(X) \neq \emptyset\}.$$

To make a graph G r_λ-edge-connected, it is necessary to add

(1) at least $r_\lambda(X) - c_G(X)$ edges between X and $V - X$ for each cut X.

Also, to make a graph G r_κ-vertex-connected, it is necessary to add

(2) at least $r_\kappa(X) - |\Gamma_G(X)|$ edges between X and $V - X - \Gamma_G(X)$ for each tight set X, or

(3) at least $p(G-S)-1$ edges to connect components of $G-S$ for a separator S.

(See Section 1 for definitions of $\Gamma_G(X)$ and $p(G-S)$.) Based on the observations (1) and (2), we need to add $\lceil\alpha(G)/2\rceil$ new edges to make G (r_λ, r_κ)-edge-connected, where

$$\alpha(G) = \max\left\{\sum_{i=1}^{p}(r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^{q}(r_\kappa(X_i) - |\Gamma_G(X_i)|)\right\}$$

among all subpartitions $\{X_1, \cdots, X_q\}$ of V with $V - X_i - \Gamma_G(X_i) \neq \emptyset$, $i = p+1, \cdots, q$. From (3), to make G r_κ-vertex-connected, at least $\beta(G)-1$ new edges are necessarily added to G. Then we easily have the next lemma.

Lemma 2.1 (The Lower Bound) *To make a given graph G (r_λ, r_κ)-connected, at least*

$$\gamma(G) \equiv \max\{\lceil\alpha(G)/2\rceil, \beta(G)-1\}$$

new edges must be added. □

3 The EVAP(r_λ,2)

In this paper, we show that the EVAP(r_λ,2) can be solved in polynomial time.

In what follows, we assume $r_\lambda(x,y) \geq r_\kappa(x,y) = 2$ for each $x, y \in V$. Now $\alpha(G)$ in Section 2.3 is rewritten by

$$\left\{\max\{\sum_{i=1}^{p}(r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^{q}(2 - |\Gamma_G(X_i)|)\right\}$$

where the maximization is taken over all subpartitions $\{X_1, \cdots, X_q\}$ of V such that $V - X_i - \Gamma_G(X_i) \neq \emptyset$ for $i = p+1, \cdots, q$.

In this paper, we show the following main theorem.

Theorem 3.1 *Given an undirected multigraph $G = (V, E)$ and a requirement function $\{r_\lambda(x,y) \in Z^+ | x, y \in V\}$, G can be made $(r_\lambda, 2)$-connected by adding $\gamma(G)$ new edges.* □

We will prove this theorem by presenting a polynomial time algorithm for making G $(r_\lambda, 2)$-connected by adding $\gamma(G)$ new edges.

A vertex v is called a *cut vertex* in G if $\{v\}$ is a minimum separator in G. An edge $e = (u, u')$ is called *admissible* (with respect to v) if there is a cut vertex v such that $v \neq u, u'$ and $p(G-v) = p((G-e)-v)$. For a subset F of edges in a graph G, we say that two edge e_1 and e_2 are *switched* in F if we delete $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ from F, and add edges (u_1, u_2) and (w_1, w_2) to F. Our algorithm for solving the EVAP(r_λ, 2) consists of the following four major steps.

I) **Vertex-augmenting:** Augment $G = (V, E)$ by adding a new vertex s and new edges between s and V so that the resulting graph $G_1 = (V \cup \{s\}, E \cup F_1)$ satisfies the requirements for the r_λ-edge-connectivity and the 2-vertex-connectivity (more precisely, $c_{G_1}(X) \geq r_\lambda(X)$ holds for each $\emptyset \neq X \subset V$, and $|\Gamma_{G_1}(X \cup s)| \geq 2$ holds for each $\emptyset \neq X \subset V$ with $V - X - \Gamma_{G_1}(X) \neq \emptyset$) and $F_1 = E_{G_1}(s)$ is minimal subject to these conditions.

Lemma 3.1 *$|F_1| = \alpha(G)$ holds.* □

II) **Edge-splitting:** Find a complete edge-splitting at s in G_1 which preserves the r_λ-edge-connectivity (after adding one edge (s, v) for an arbitrarily chosen non cut vertex v of G if $c_{G_1}(s)$ is odd). Let $G_2 = (V, E \cup F_2)$ denote the graph obtained by such a complete edge-splitting, ignoring the isolated vertex s. Mader's theorem guarantees the next.

Lemma 3.2 *G_2 is r_λ-edge-connected.* □

If G_2 is 2-vertex-connected, then we are done (since $|F_2| = \lceil\alpha(G)/2\rceil$ implies that G_2 is optimally augmented). Otherwise, go to III.

III) **Edge-switching:** Now G_2 has a cut vertex.

Lemma 3.3 *If G_2 has an admissible edge $e_1 \in F_2$, then there is another edge $e_2 \in F_2$ such that switching e_1 and e_2 decreases the number of tight sets by at least one while preserving the r_λ-edge-connectivity and the current local 2-vertex-connectivity.* □

By Lemma 3.3, we can switch some edges in F_2 so that the resulting graph $G_3 = (V, E \cup F_3)$ has no admissible edge in F_3 (hence G_3 has at most one cut vertex, as shown later).

If G_3 has no cut vertex, then we are done (since $|F_3| = \lceil\alpha(G)/2\rceil$ implies that G_3 is optimally augmented). Otherwise, go to IV.

IV) **Edge-augmenting:** Now G_3 has one cut vertex v.

Lemma 3.4 *$p(G_3 - v) = p(G-v) - \lceil\alpha(G)/2\rceil$.* □

Note that $\beta(G) \geq p(G-v)$. Then add another $\beta(G) - 1 - \lceil\alpha(G)/2\rceil$ new edges to G_3 so that the resulting graph $G_4 = (V, E \cup F_3 \cup F_4)$ becomes 2-vertex-connected. Finally, we are done (since $|F_3| + |F_4| = \beta(G) - 1$ implies that G_2 is optimally augmented).

In the following four sections, we prove that the correctness for each major step in this algorithm.

4 Correctness of Step I

In this section, we show the correctness of Step I. Step I can be carried out as follows:

I) **Vertex-augmenting:**

1. Add a sufficiently large number of edges between a new vertex s and V to G so that the resulting graph $G' = (V \cup \{s\}, E \cup F')$ satisfies

$$c_{G'}(X) \geq r_\lambda(X) \text{ for each } \emptyset \neq X \subset V, \quad (4.1)$$

$$|\Gamma_{G'}(X \cup s)| \geq 2 \text{ for each } \emptyset \neq X \subset V \text{ with } V - X - \Gamma_{G'}(X) \neq \emptyset. \quad (4.2)$$

(This can be done by adding $\max\{r_\lambda(x,y) \mid x, y \in V\}$ edges between s and each vertex $v \in V$.)

2. Discard new edges, one by one, as long as (4.1) and (4.2) remain valid. Denote the resulting graph by $G_1 = (V \cup \{s\}, E \cup F_1)$ (i.e., $F_1 = E_{G_1}(s, V)$). Note that if G is not connected, then $\kappa_{G_1}(x,y) \geq 2$ may not hold for some $x, y \in V$, since a subset $X \subset V$ which induces a component $G[X]$ of G satisfies $\Gamma_{G_1}(X) = \emptyset$ or $\{s\}$ (and hence $\kappa_{G_1}(x,y) \leq 1$ for $x \in X$ and $y \in V - X$). Clearly, the above 1. and 2. can be performed in polynomial time. We claim the next.

Lemma 3.1 $|F_1| = \alpha(G)$ holds.

The Proof of Lemma 3.1: Now $\lambda_{G_1}(x,y) \geq 2$ holds for every $x, y \in V$ from the assumption $r_\lambda(x,y) \geq r_\kappa(x,y)$.

First, we show $|F_1| \geq \alpha(G)$. Let $\mathcal{F}^* = \{X_1^*, \cdots, X_p^*, X_{p+1}^*, \cdots, X_q^*\}$ be a subpartition of V with $V - X_i^* - \Gamma_{G_1}(X_i^*) \neq \emptyset$ for $i = p+1, \cdots, q$ that attains $\alpha(G) = \sum_{i=1}^{p}(r_\lambda(X_i^*) - c_G(X_i^*)) + \sum_{i=p+1}^{q}(2 - |\Gamma_G(X_i^*)|)$. If $|F_1| < \alpha(G)$ holds, then there must be at least one cut $X_i^* \in \mathcal{F}^*$ that violates (4.1) or (4.2), contradicting construction of G_1.

Now we prove the converse, $|F_1| \leq \alpha(G)$ by showing several claims.

A cut $X \subset V$ is called *critical* in G_1 if $s \in \Gamma_{G_1}(X)$ holds and the removal of any edge $e \in E_{G_1}(s, X)$ violates (4.1) or (4.2). Clearly, a subset $X \subset V$ with $s \in \Gamma_{G_1}(X)$ is critical if and only if X satisfies at least one of the following conditions:

(1) $c_{G_1}(X) = r_\lambda(X)$.
(2) $c_{G_1}(s, X) = 1, |\Gamma_{G_1}(X) - s| = 1$, and $V - X - \Gamma_{G_1}(X) \neq \emptyset$.
(3) $\Gamma_{G_1}(X) = \{s\}$, $|\Gamma_{G_1}(s) \cap X| = 2$, and there is a vertex $v \in \Gamma_{G_1}(s) \cap X$ with $c_{G_1}(s, v) = 1$.

We will prove that G_1 has a set of critical cuts $X_1, \cdots, X_q$ such that

$$X_i \cap X_j = \emptyset, \text{ for } 1 \leq i < j \leq q, \quad \Gamma_{G_1}(s) \subseteq X_1 \cup \cdots \cup X_q, \quad (4.3)$$

which proves $|F_1| \leq \alpha(G)$. We call a critical cut X *v-minimal* if $v \in \Gamma_{G_1}(s) \cap X$ and there is no critical cut X' with $\{v\} \subseteq X' \subset X$. A subset X is called *critical of type* (1) (resp., (2), (3)) if it satisfies (1) (resp., (2), (3)).

First, we introduce some properties of critical cuts.

Claim 4.1 *Any critical cut X of type (3) is also critical of type (1).* □

From this claim, we can regard critical cuts of type (3) as those of type (1). The next property is known in [5].

Claim 4.2 *Let X and Y be critical cuts of type (1) in G_1. Then at least one of the following statements holds.*
(i) Both $X \cap Y$ and $X \cup Y$ are critical.
(ii) Both $X - Y$ and $Y - X$ are critical, and $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$. □

An analogous property holds for critical cuts of type (2).

Claim 4.3 *Let X and Y be critical cuts of type (2). If Y is v-minimal for some $v \in V - X$, then they do not cross each other.* □

Claim 4.4 *Let X be a critical cut of type (1), and Y be a critical cut of type (2) such that $\Gamma_{G_1}(s) \cap (Y - X) \neq \emptyset$. If X and Y cross each other then $c_{G_1}(X \cap Y, s) = 0$ holds and cut $Y - X$ is critical of type (1).* □

Now we are ready to prove that G_1 has a set of critical cuts $X_1, \cdots, X_q$ that satisfy (4.3). Let $N_1 \subseteq \Gamma_{G_1}(s)$ be the set of neighbors u of s such that there is a critical cut X of type (1) with $u \in X$. Let us choose a critical cut X_u of type (1) with $u \in X_u$ for each $u \in N_1$ so that $\sum_{X \in \{X_u | u \in N_1\}} |X|$ is minimized. Denote such a set $\{X_u \mid u \in N_1\}$ by $\mathcal{F}_1$. For $N_2 = \Gamma_{G_1}(s) - N_1$, we choose a u-minimal critical cut X_u for each $u \in N_2$, and let $\mathcal{F}_2 = \{X_u \mid u \in N_2\}$. Then we claim the next.

Claim 4.5 *$\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ consists of disjoint critical cuts whose union contains $\Gamma_{G_1}(s)$.*

Proof. Let $\mathcal{F}_1 = \{X_1, \cdots, X_p\}$ and $\mathcal{F}_2 = \{X_{p+1}, \cdots, X_q\}$ with each $\emptyset \neq X_i \subset V$. Clearly, $\Gamma_{G_1}(s) \subseteq \bigcup_{X_i \in \mathcal{F}} X_i$ holds from construction of $\mathcal{F}$.

We show that X_i and X_j are pairwise disjoint for each $X_i, X_j \in \mathcal{F}_1$. Assume that $\mathcal{F}_1$ contains X_i and X_j which are not pairwise disjoint. Note that $X_i \subset X_j$ does not hold from construction of $\mathcal{F}_1$. If X_i and

X_j cross each other, then Claim 4.2 implies that at least one of the following statements holds:
(i) Both $X_i \cap X_j$ and $X_i \cup X_j$ are critical.
(ii) Both $X_i - X_j$ and $X_j - X_i$ are critical, and $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$.
If the statement (i) holds, then $\mathcal{F}_1' = (\mathcal{F}_1 - X_i - X_j) \cup \{X_i \cup X_j\}$ would satisfy $N_1 \subseteq \mathcal{F}_1'$ and $\sum_{X \in \mathcal{F}_1'} |X| < \sum_{X \in \mathcal{F}_1} |X|$, contradicting the minimality of $\sum_{X \in \mathcal{F}_1} |X|$. If the statement (ii) holds, then $\mathcal{F}_1' = (\mathcal{F}_1 - X_i - X_j) \cup \{X_i - X_j, X_j - X_i\}$ satisfies $\sum_{X \in \mathcal{F}_1'} |X| < \sum_{X \in \mathcal{F}_1} |X|$ and $N_1 \subseteq \mathcal{F}_1'$ (by $c_{G_1}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$). This again contradicts the minimality of $\sum_{X \in \mathcal{F}_1} |X|$. Therefore X_i and X_j are pairwise disjoint for each $X_i, X_j \in \mathcal{F}_1$.

Claim 4.3 implies that X_i and X_j are pairwise disjoint for each $X_i, X_j \in \mathcal{F}_2$.

Finally, we show that X_i and X_j are pairwise disjoint for each $X_i \in \mathcal{F}_1$ and $X_j \in \mathcal{F}_2$. Note that $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$ holds from definition of N_1. Then $X_j \subset X_i$ does not hold. Also note that $X_i \subset X_j$ does not hold, otherwise $\Gamma_{G_1}(s) \cap X_i \neq \emptyset$ and $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$ imply $c_{G_1}(X_j, s) \geq c_{G_1}(X_i, s) + 1 \geq 2$, contradicting that X_j is of type (2). Assume that X_i and X_j cross each other. Now $\Gamma_{G_1}(s) \cap (X_j - X_i) \neq \emptyset$ holds. Therefore Claim 4.4 implies that $c_{G_1}(s, X_i \cap X_j) = 0$ holds and $X_j - X_i$ is a critical cut of type (1). This implies that any vertex in X_j cannot belong to N_2, contradicting $X_j \in \mathcal{F}_2$. □

Clearly $\mathcal{F}$ is a subpartition of V by Claim 4.5. Since $\Gamma_{G_1}(s) \subseteq X_1 \cup \cdots \cup X_q$ with $X_i \in \mathcal{F}$ holds, it holds

$$|F_1| = \sum_{i=1}^{p} (r_\lambda(X_i) - c_G(X_i)) + \sum_{i=p+1}^{q} (2 - |\Gamma_G(X_i)|),$$

for $\mathcal{F}_1 = \{X_1, \cdots, X_p\}$ and $\mathcal{F}_2 = \{X_{p+1}, \cdots, X_q\}$. From definition of $\alpha(G)$, we have $|F_1| \leq \alpha(G)$. □

5 Correctness of Step II

Let $G_1 = (V \cup \{s\}, E \cup F_1)$ be the graph obtained from a given graph G by Step I. In the Step II, a graph G_2 is constructed from G_1 as follows.

II) **Edge-splitting:** If $c_{G_1}(s)$ is odd, then we add one edge (s, v) for an arbitrarily chosen vertex $v \in V$ which is not a cut vertex in G. Find a complete edge-splitting at s in G_1 which preserves condition (4.1) (i.e., the r_λ-edge-connectivity). By Mader's theorem, there always exists such a complete edge-splitting at s, and it can be computed in polynomial time. Let $G_2 = (V, E \cup F_2)$ denote the graph obtained by such a complete edge-splitting, ignoring the isolated vertex s. Therefore, the next is immediate from Mader's theorem.

Lemma 3.2 G_2 satisfies (4.1) (i.e., G_2 is r_λ-edge-connected). □

However, at this point G_2 may have a cut vertex, even though G_1 satisfies (4.2). If G_2 is 2-vertex-connected, then we are done (since $|F_2| = \lceil \alpha(G)/2 \rceil$ implies that G_2 is optimally augmented and $\gamma(G) = |F_2|$). Otherwise, we go to Step III.

Theorem 2.1 implies that if $c_{G_1}(s)$ is even, then $G_1 = (V \cup \{s\}, E \cup F_1)$ has a complete splitting at s which preserves the r_λ-edge-connectivity, where the 2-vertex-connectivity may be violated.

In Step II, if $c_{G_1}(s)$ is odd, then we add one edge (s, v) to G_1 for an arbitrarily chosen vertex $v \in V$ which is not a cut vertex of G. Such choice of w will be useful for the correctness of Step IV in Section 7.

6 Correctness of Step III

Let $G_2 = (V, E \cup F_2)$ be the graph obtained in Step II. Now G_2 has a cut vertex and G_2 is 2-edge-connected. Moreover, since (4.2) holds in G_1, G_2 satisfies

$$G_2[X \cup \{v\}] \text{ contains at least one edge in } F_2 \text{ for any cut vertex } v \text{ in } G_2 \text{ and its } v\text{-component } X. \tag{6.1}$$

Before describing Step III, more precisely, we will give a proof for the next lemma. We restate Lemma 3.3 in a more precise form:

Lemma 3.3 Assume that G_2 has an admissible edge $e_1 \in F_2$ with respect to a cut vertex v of G_2. Let X be a v-component with $e_1 \notin E[G_2[X \cup \{v\}]]$, and e_2 be chosen arbitrarily from $F_2 \cap E[G_2[X \cup \{v\}]]$. Then switching e_1 and e_2 decreases the number of tight sets at least by one while preserving the r_λ-edge-connectivity. Moreover, the resulting graph G_2' by switching e_1 and e_2 still satisfies (6.1), and $\kappa_{G_2'}(x, y) \geq 2$ holds for any vertices x and y with $\kappa_{G_2}(x, y) \geq 2$. □

Based on this lemma, Step III repeats switching two edges in F_2 until the resulting graph has no admissible edge in F_2.

Let $G_3 = (V, E \cup F_3)$ be the resulting graph obtained by such a sequence of switching edges in F_2, where F_3 means the final F_2. By the following Claim 6.3, G_3 has at most one cut vertex.

If G_3 has no cut vertex, then we are done (since $|F_3| = \lceil \alpha(G)/2 \rceil$ implies that G_3 is optimally augmented). Otherwise, we go to Step IV.

Proof of Lemma 3.3: We prove Lemma 3.3 by showing some claims.

Claim 6.1 *Let $v \in V$ denote a cut vertex in G_2. Assume that a v-component T contains an admissible edge $e = (u, u')$ with respect to v. Then $G_2[T] - e$ contains a path P between u and u'.* □

Claim 6.2 *Any two cuts X and Y which are both tight in G_2 are pairwise disjoint.* □

Claim 6.3 *If G_2 has two cut vertices v_1 and v_2, then there are a v_1-component X_1 and a v_2-component X_2 such that $X_1 \cap X_2 = \emptyset$. Let edge e_1 be arbitrarily chosen from $F_2 \cap E[G_2[X_1 \cup \{v\}]]$. Then e_1 is an admissible with respect to v_2.* □

Claim 6.4 *Let $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ be the edges in the statement of Lemma 3.3. Then the graph $G'_2 = (V, E \cup F'_2)$ obtained by switching e_1 and e_2, where $F'_2 = F_2 \cup \{(u_1, u_2), (w_1, w_2)\} - \{e_1, e_2\}$, satisfies followings:*

(i) $\lambda_{G'_2}(x, y) \geq r_\lambda(x, y)$ for every $x, y \in V$.

(ii) $p(G'_2 - v) < p(G_2 - v)$.

(iii) $\kappa_{G'_2}(x, y) \geq 2$ for every $x, y \in V$ with $\kappa_{G_2}(x, y) \geq 2$.

(The statements (ii) and (iii) and Claim 6.2 imply that switching e_1 and e_2 decreases the number of tight sets in G_2 by at least one.)

Proof. (i) We assume that there is a cut X such that $c_{G'_2}(X) \leq r_\lambda(X) - 1$ holds. Note that $c_{G_2}(X) \leq c_{G'_2}(X)$ holds if cut X does not separate $\{u_1, u_2\}$ and $\{w_1, w_2\}$ in G'_2. Since $c_{G_2}(X) \geq r_\lambda(X)$ originally holds, cut X separates $\{u_1, u_2\}$ and $\{w_1, w_2\}$ and hence $c_{G'_2}(X) = c_{G_2}(X) - 2$ holds. Since the cut X crosses both v-components T_1 and T_2 in G_2, either $G_2[X]$ or $G_2[V - X]$ consists of at least two components. Without loss of generality, assume that $G_2[X]$ consists of at least two components. There are vertices $x^* \in X$ and $y^* \in V - X$ such that $r_\lambda(x^*, y^*) = r_\lambda(X) \geq c_{G'_2}(X) + 1$. Without loss of generality, assume that $x^* \in X \cap T_1$. Note that $c_{G_2}(X \cap T_2) \geq r_\lambda(X \cap T_2) \geq 2$ and $c_{G_2}(X \cap T_1) \geq r_\lambda(X \cap T_1) \geq r_\lambda(x^*, y^*) \geq c_{G'_2}(X) + 1$ hold. This implies $c_{G_2}(X) = c_{G_2}(X \cap T_1) + c_{G_2}(X \cap T_2) \geq (c_{G'_2}(X) + 1) + 2$, contradicting $c_{G'_2}(X) = c_{G_2}(X) - 2$.

(ii) It is sufficient to show that $G'_2[T_1 \cup T_2]$ is connected. Since the removal of the admissible edge e_1 does not increase the number of v-components, T_1 remains a v-component in $G_2 - e_1$. If T_2 remains a v-component in $G_2 - e_2$, then $G[T_1]$ and $G[T_2]$ are joined by the edges (u_1, u_2) and (w_1, w_2) obtained by switching e_1 and e_2 in G'_2. If T_2 consists of two components T_2^1 and T_2^2 in $G_2 - e_2$, then $u_2 \neq v \neq w_2$ holds and u_2 and w_2 are separated by T_2^1. Assume $u_2 \in T_2^1$ and $w_2 \in T_2^1$ without loss of generality. Now T_2^1 (resp., T_2^2) and T_1 are joined by the edges (u_1, u_2) (resp., (w_1, w_2)). This implies that $G'_2[T_1 \cup T_2]$ is a component since T_1 remains a v-component in $G_2 - e_1$. Therefore if v remains a cut vertex in G'_2, then $T_1 \cup T_2$ is a v-component (otherwise, clearly, $p(G_2 - v) = 1$).

(iii) Assume that there are vertices $x, y \in V$ such that $\kappa_{G_2}(x, y) = 2$ but $\kappa_{G'_2}(x, y) = 1$. Let $v' \in V$ denote a cut vertex in G'_2 that separates x and y. Clearly, $v' \neq v$ (because $v = v'$ would imply $\kappa_{G_2}(x, y) = 1$). Let $W_1, W_2, \cdots, W_q$ ($q \geq 2$) be the v'-components of G'_2, where $x \in W_1$ and $y \in W_2$. Since a cut vertex v' does not separate x and y in G_2, $e_1 \in E_{G_2}(W_1, W_2)$ or $e_2 \in E_{G_2}(W_1, W_2)$ holds. Also note that no edge other than e_1 and e_2 cannot belong to $E_{G_2}(W_1, W_2)$. We can easily see that $G_2[W_1 \cup W_2 \cup \{v'\}]$ contains u_1, w_1, u_2, and w_2. Then note that $u_i, w_i \in W_j$ cannot hold for any i, j with $1 \leq i \leq j \leq 2$. Otherwise (assume $u_1, w_1 \in W_1$ without loss of generality) then $e_2 \in E_{G_2}(W_1, W_2)$ holds (assume $u_2 \in W_1$ and $w_2 \in W_2$ without loss of generality). Now $(w_1, w_2) \in E_{G'_2}(W_1, W_2)$ holds and $G'_2[W_1]$ and $G'_2[W_2]$ are both connected from definition of W_1 and W_2, contradicting that cut vertex v' separates x and y in G'_2. Therefore, for each $i = 1, 2$, we have now $e_i = (u_i, w_i) \in E_{G_2}(W_1, W_2)$ or $u_i = v'$ or $w_i = v'$.

We first consider the case of $e_1 \in E_{G_2}(W_1, W_2)$. Then $v' \in T_1$ holds since $G_2[T_1] - e_1$ is connected by Claim 6.1. Hence $e_2 \in E_{G_2}(W_1, W_2)$ holds since $v' \in T_1$ implies $u_2 \neq v' \neq w_2$. Let $v \notin W_2$ and $u_1, u_2 \in W_1$ without loss of generality. Now $\Gamma_{G'_2}(T_2 \cap W_2) \cap (T_2 - W_2) = \emptyset$ holds since v' is a cut vertex of G'_2 and $v' \notin T_2$ hold. Note that $E_{G'_2}(T_2 \cap W_2, V - (T_2 \cap W_2)) = \{(w_1, w_2)\}$ since T_2 is a v-component of G_2 and $u_2 \in W_1$ holds. This implies $\Gamma_{G_2}(T_2 \cap W_2) = \{u_2\}$ holds and hence e_2 is a bridge of G_2 from $E_{G_2}(W_1, W_2) = \{e_1, e_2\}$, which contradicts $\lambda(G_2) \geq 2$.

We then consider the case of $e_1 \notin E_{G_2}(W_1, W_2)$ holds, i.e., $v' = u_1 \in T_1$ or $v' = w_1 \in T_1$ holds. This implies that $e_2 \in E_{G_2}(W_1, W_2)$ holds and $v' \notin T_2$. Therefore, this clearly leads to a contradiction, in a similar way to above case of $e_1 \in E_{G_2}(W_1, W_2)$. □

From above claim, Lemma 3.3 is proved. □

7 Correctness of Step IV

1. Let G_3 be obtained from G_2 by Step III. Now G_3 has one cut vertex v

Lemma 3.4 $p(G_3 - v) = p(G - v) - \lceil \alpha(G)/2 \rceil$. □

Now let $T_1, \cdots, T_q$ be v-components. We can make G_3 2-vertex-connected by add one edge between T_i and T_{i+1} for each $i = 1, \cdots, q - 1$. That is, Adding $p(G_3 - v) - 1$ edges to G_3 makes G_3 2-vertex-connected. Note that $p(G_3 - v) = p(G - v) - \lceil \alpha(G)/2 \rceil$ holds from Lemma 3.4 and

$\beta(G) \geq p(G-v)$ clearly holds. Therefore we can add another $\beta(G) - 1 - \lceil\alpha(G)/2\rceil$ new edges to G_3 so that the resulting graph $G_4 = (V, E \cup F_3 \cup F_4)$ becomes 2-vertex-connected. Finally, we are done (since $|F_3|+|F_4| = \beta(G)-1$ implies that G_4 is optimally augmented).

Before proving Lemma 3.4, we first introduce properties of G_3 in the following two claims.

Claim 7.1 *Now G_3 has no edge $e = (v, v') \in F_3$ incident to the cut vertex v.* □

Claim 7.2 $p(G-v) = p(G_3 - v) + |F_3|$ *holds. That is, deleting any edge $e \in F_3$ increases the number of v-components in G_3.*

Proof. If $p(G - v) < p(G_3 - v) + |F_3|$ holds, then there is at least one edge $e \in F_3$ with $p((G_3 - e) - v) = p(G_3 - v)$. Then e is admissible with respect to v since any edge in F_3 is not incident to v from Claim 7.1, contradicting construction of G_3. □

This claim implies that since G_3 has no edge in F_3 incident to the cut vertex v, a graph $H = (W, F_3)$ is a forest, where a vertex set W of H is obtained by removing the cut vertex v and contracting each component of $G - v$ to one vertex.

Now Claim 7.2 implies Lemma 3.4 since $|F_3| = \lceil\alpha(G)/2\rceil$ holds from construction, proving correctness of Step IV. □

Theorem 7.1 *The EVAP$(r_\lambda, 2)$ can be solved in polynomial time.* □

Very recently, we proved that the EVAP$(4, 3)$ is polynomially solvable. Unfortunately, the above lower bound $\gamma(G)$ does not always attain the optimal value to this problem. The result will be reported somewhere else.

References

[1] A. A. Benczúr, *Augmenting undirected connectivity in $\tilde{O}(n^3)$ time*, Proceedings 26th ACM Symposium on Theory of Computing, 1994, pp. 658-667.

[2] G.-R. Cai and Y.-G. Sun, *The minimum augmentation of any graph to k-edge-connected graph*, Networks, Vol.19, 1989, pp. 151-172.

[3] K. P. Eswaran and R. E. Tarjan, *Augmentation problems*, SIAM J. Computing, Vol.5, 1976, pp. 653–665.

[4] L. R. Ford and D. R. Fulkerson, Flows in Networks, Princeton University Press, Princeton, N. J., 1962.

[5] A. Frank, *Augmenting graphs to meet edge-connectivity requirements*, SIAM J. Discrete Mathematics, Vol.5, 1992, pp. 25-53.

[6] H.N. Gabow, *Applications of a poset representation to edge connectivity and graph rigidity*, Proc. 32nd IEEE Symp. Found. Comp. Sci., 1991, pp.812-821.

[7] H.N. Gabow, *Efficient splitting off algorithms for graphs*, Proceedings 26th ACM Symposium on Theory of Computing, 1994, pp. 696–705.

[8] T. Hsu and V. Ramachandran, *A linear time algorithm for triconnectivity augmentation*, Proc. 32nd IEEE Symp. Found. Comp. Sci., 1991, pp.548–559.

[9] T. Hsu, *On four-connecting a triconnected graph*, Proc. 33nd IEEE Symp. Found. Comp. Sci., 1992, pp.70–79.

[10] T. Hsu and V. Ramachandran, *Finding a smallest augmentation to biconnect a graph*, SIAM J. Computing, Vol.22, 1993, pp.889–912.

[11] T. Hsu, *Undirected vertex-connectivity structure and smallest four-vertex-connectivity augmentation*, Proc. 6th ISAAC, 1995.

[12] T. Jordán, *On the optimal vertex-connectivity augmentation*, J. Combinatorial Theory, Series B, Vol.63, 1995, pp.8–20.

[13] T. Jordán, *Some remarks on the unidirected connectivity*, working paper, 1996.

[14] W. Mader, *A reduction method for edge-connectivity in graphs*, Ann. Discrete Math., Vol.3, 1978, pp. 145-164.

[15] H. Nagamochi and T. Ibaraki, *A faster edge splitting algorithm in multigraphs and its application to the edge-connectivity augmentation problem*, Lectures Notes in Computer Science 920, Springer-Verlag, Egon Balas and Jens Clausen (Eds.), 4th Integer Programming and Combinatoric Optimization, IPCO'95 Copenhagen, May 1995, pp. 401-413.

[16] H. Nagamochi and T. Ibaraki, *Deterministic $\tilde{O}(nm)$ time edge-splitting in undirected graphs*, Proceedings 28th ACM Symposium on Theory of Computing, 1996, pp. 64–73 (also to appear in J. Combinatorial Optimization, 1, (1997), pp. 1–42).

[17] T. Watanabe and A. Nakamura, *Edge-connectivity augmentation problems*, J. Comp. System Sci., Vol.35, 1987, pp.96-144.

[18] T. Watanabe and A. Nakamura, *A smallest augmentation to 3-connect a graph*, Discrete Appl. Math., Vol.28, 1990, pp.183-186.