線状ロボットの d_1－最適な移動問題

浅野哲夫[1]、D. G. Kirkpatrick[2]、C. K. Yap[3]

[1]北陸先端科学技術大学院大学（石川県辰口町）
[2]University of British Columbia（Vancouver, Canada）
[3]New York University（New York, U.S.A.）

本論文では、多角形の障害物を避けて線状のロボットを移動するのに、線上に任意ではあるが固定された点（参照点とよぶ）の軌跡の長さを最小化する問題の計算複雑度を扱う。本文では、このような線状ロボットに回転や平行移動などの任意の動作を許した場合について参照点の軌跡を最小にする d1-最適な動作を特徴づける定理を示す。

d_1－Optimal Motion for a Rod

T. Asano[1]、D. G. Kirkpatrick[2]、C. K. Yap[3]

[1]JAIST (Tatsunokuchi-cho, Ishikawa)
[2]University of British Columbia（Vancouver, Canada）
[3]New York University（New York, U.S.A.）

We study the motion of a rod (line segment) in the plane in the presence of polygonal obstacles, under an optimality criterion based on minimizing the orbit length of a fixed but arbitrary point (called the *focus*) on the rod. In this paper, we present a local characterization of a d_1−optimal motion of a rod that minimizes the orbit length of a reference point, allowing arbitrary kinds of motions including rotation and translation.

1 Introduction

Although the *feasibility* of motion planning is very well studied, little is known about *optimal* motion planning except for the case where the robot body is a disc. In this paper we address the problem of characterizing and computing optimal motions for a *rod* (a directed line segment) in the plane. Of course, a rod is the next simplest planar body to study.

A non-trivial issue arising in the study of "optimal" motion of a rod is the choice of a reasonable yet tractable motion of optimality. One choice is that of maximizing the clearance (i.e., distance to the nearest obstacle). Here, an efficient algorithm based a generalization of the Voronoi diagram is known [9, 10]. Another interesting approach is to minimize the area swept by the motion of the rod. This problem attracted some interest in the past under the name "Kakeya's problem" (see [1]), but this turns out not to be a good idea (for example, one can sweep an arbitrarily small area while rotating a rod by 180°).

Turning to notions of optimality based on some distance or length concept, we may describe such optimal motions as "shortest". If X is any fixed point on the rod, the curve traced by X in any continuous motion μ of the rod is called the *orbit* of X in μ. One natural choice here is to minimize the *average* lengths of the orbits of the two endpoints of the rod. In the absence of obstacles, this has been called *Ulam's problem* [11]. Again it has an interesting history (see [3] and the references therein). The recent paper of Icking *et al.* [3] revisits this problem, introducing a simple tool based on Cauchy's surface area formula. They call the metric in

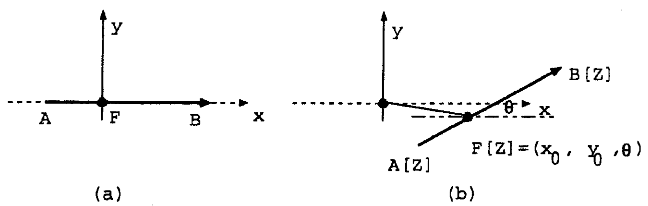

Figure 1: (a) Original position of AB. (b) Position at placement $Z = (x_0, y_0, \theta)$

Ulam's problem the d_2-metric. There is a natural generalization to d_n for any $n \geq 3$ or $n = \infty$ (corresponding to minimizing the average orbit lengths of n evenly distributed points on the rod).

As Icking *et al.* note, the "d_1-distance", based on minimizing the orbit length of the mid-point of the rod, is not a metric (rotation about the mid-point produces distinct rod placements with d_1-distance zero). It is nevertheless a rather natural measure of distance, capturing the idea of "charging" for translation but not for rotation about the mid-point. More importantly, it permits us to study optimal motions in the presence of obstacles, which previously we have not been able to do, even for d_2 optimal motions. (Notice that d_1-optimal motions, unlike the d_2-optimal motions, are trivial in the absence of obstacles.) In this paper, we generalize this "d_1-distance" to refer to the family of distance functions based on minimizing the orbit length of a fixed but arbitrary point F (the "focus") in the relative interior of the rod.

Other than [3], there are few previous papers on d_n-optimal motions. Papadimitriou and Silverberg [5] studied d_1-optimal motions with the focus F at one endpoint of the rod. However, they severely restricted the motions so that F travels only in straight lines between obstacle vertices. Their results were improved by Sharir [8]. O'Rourke [4] studied d_∞-optimal motions restricted to either pure translations or rotations by $\pm 90°$.

In contrast to these last cited papers, we are interested in *unrestricted* motions of the rod, except of course when the rod collides with obstacles. Motion planning problems come in two variations depending on whether the two endpoints of the rod are distinguished (in which case the rod is said to be *directed*) or not (in which case the rod is said to be *undirected*). Clearly, motion planning problem for undirected rods can be reduced to two instances of the motion planning problem for directed rods. (The converse relationship is not known.) Unless otherwise stated we assume that rods are directed.

In ths paper we give a local characterization of d_1-optimal motion. We use the language of "constraints" to classify optimal submotions. There are considerable geometric details here. This local characterization theorem leads to the NP-hardness result of this optimization problem, which will be given in another SIGAL meeting.

1.1 Notation and basic definitions

This section is mainly for establishing our terminology. Most of this is in the literature.

All points are in the Euclidean plane. If P, Q, R are points, then $\overline{PQ}$ denotes the line through P and Q, and $[P, Q]$ denotes the closed line segment between P and Q, and $|PQ|$ is the length of the segment. The ray from P through Q is denoted $\vec{PQ}$. The angle as one sweeps counter-clockwise from $\vec{QP}$ to $\vec{QR}$ is denoted $\angle(P, Q, R)$. Thus $\angle(P, Q, R) = -\angle(R, Q, P)$.

A (directed) rod is denoted AB and in figures, we direct the line segment from its B-end to its A-end. (Mnemonically, A indicates the *a*rrow head, and B the *b*ase of the rod.) We assume a fixed point F called the *focus* in the relative interior of the rod (see Fig. 1).

We are given a closed planar set $\Omega \subseteq \mathbf{R}^2$ in which the rod is free to move. Its complement $\mathbf{R}^2 - \Omega$ is called the *obstacle set.* The boundary of the obstacle set is assumed polygonal and is partitioned into a pairwise disjoint collection of *(obstacle) features*, where a feature is either an isolated point called a *corner* or an open line segment called a *wall.* We assume non-degeneracy conditions on these features as convenient.

We use the language of "placements" [12]: a *placement* is simply a pair $Z = (p, \theta) \in \mathbf{R}^2 \times \mathbf{S}^1$ where p is a point in the plane and θ an angle. For any Z, we also write $Z = (p(Z), \theta(Z))$. Horizontal placements, those with $\theta = 0$, play a special role in our NP-hardness constructions and have their own special notation: $H_p = (p, 0)$. For any set $S \subseteq \mathbf{R}^2$, we write $S[Z] \subseteq \mathbf{R}^2$ for position of the set S after rotating the plane containing S about the origin by θ, followed by translating the plane by p, viewed on a fixed reference plane. Hence we call $S[Z]$ the *position* of S in placement Z. In other words, $[Z]$ defines an Euclidean transformation of the plane. In particular, we have $AB[Z]$ for the position of the rod in placement Z; $A[Z]$ and $B[Z]$ denote the two endpoints of this position. *We shall choose (the original position of) AB in such a way that the focus F is at the origin; this means that the position $F[Z]$ is precisely $p(Z)$.*

If $S, T \subseteq \mathbf{R}^2$ are two closed sets, $d_H(S,T) \geq 0$ denotes the *Hausdorff distance* between S and T. The *clearance* of a placement Z is just $d_H(AB[Z], \mathbf{R}^2 - \Omega)$, and denoted $h(Z)$ (so AB and Ω are left implicit). We make the set of placements into a metric space with metric d_{AB} where $d_{AB}(Z, Z')$ is defined to be $d_H(AB[Z], AB[Z'])$.

A placement Z is *free* if $AB[Z] \subseteq \Omega$ (recall that Ω is a closed set). Let FP denote the set of free placements. Consider the continuous function $\mu : [s,t] \to FP$ where $[s,t]$ is a real interval. Usually, $s = 0$, $t = 1$ unless otherwise specified. For any point $X \in \mathbf{R}^2$, let $X_\mu : [s,t] \to \mathbf{R}^2$ denote the function $X_\mu(x) = X[\mu(x)]$. We call X_μ the *X-orbit* of μ. In case X is the focus of the rod, the X-orbit is called the *trace* of μ. We call μ a *motion* if both its A-orbit and B-orbit are rectifiable (i.e., has a definite arc length). This implies the trace of μ has a length, which we call the *d_1-distance* of μ. A motion μ is *optimal* (or, d_1-optimal) if its d_1-distance is minimum among all rectifiable motions between $\mu(0)$ and $\mu(1)$.

2 The Structure of Optimal Motions

2.1 Constraints

We begin with an illustration. Let W be a wall and Z a placement. If the B-end of the rod lies on W (see Fig. 2(a)), we say that Z satisfies the constraint $B@W$. Similarly, if B (resp. the interior of the segment BF, the focus F) intersects the corner C (see Fig. 2(b) and (c)), we say that Z satisfies the constraint $B@C$ (resp. $BF@C$, $F@C$). For technical reasons, we insist that when $B@W$ (resp. $B@C$) $AB[Z]$ is not perpendicular to W (resp. $AB[Z]$ forms an non-acute angle with both of the walls incident on C) at the same time. We denote the latter constraint by $B \perp W$ (resp. $B \perp C$), see Fig. 2(d) (resp. Fig. 2(e)). Exchanging the roles of A and B, we also have the constraint "$A@W$" etc..

More formally, *constraints* are synthetic categories (expressions) of the form

$X@S$ (read "X at S") or,
$X \perp S$ (read "X is perpendicular at S")

where X is either one of the points A, B or F, or one of the open segments AF or FB, and S is either a wall or a corner. For any constraint ξ and placement Z, the relation "*Z satisfies ξ*" either holds or does not hold. We define this relation at the same time as classifying the constraints.

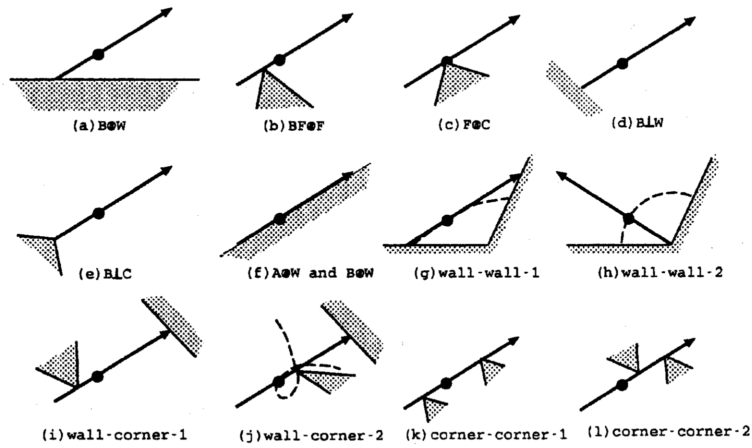

Figure 2: Constraints.

A fundamental operation that we perform is to rotate a rod about its focus F. (It follows from our characterization theorem that rotations centred at points other than F do not figure in optimal motions, hence if we speak of a rotation it is understood that its centre is F.) A *left rotation* refers to a counter-clockwise rotation and a *right rotation* is a clockwise rotation. If Z is a placement that satisfies an @-constraint, then note that the rod cannot rotate in one of the two directions (this was the reason for excluding placements that satisfy $\perp$-constraints). If Z cannot rotate left because it touches a feature S, we call S a *left-stop* of Z. Similarly for *right-stop*.

If Z cannot rotate in either direction, we say that Z is *constricted*. Note that a necessary but not sufficient condition for Z to be constricted is that Z satisfies two @-constraints. We call Z *semi-constricted* in case it can rotate in one direction but it also satisfies two @-constraints.

2.2 Classification of Constraints

Each constraint is either *simple* or *composite*. Intuitively, simple constraints reduce one degree of freedom while composite constraints reduce two degrees of freedom.

A. Simple Constraints

A.1 Simple Wall Constraint

These have the form $A@W$ or $B@W$, where W is a wall (see Fig. 2(a))

We say "Z satisfies $A@W$" if either of the following condition holds:

(a) $A[Z]$ is incident on W but $AB[Z]$ is not perpendicular to W.

(b) $A[Z]$ is incident on an endpoint of W and some portion of $AB[Z]$ projects perpendicularly onto W.

A.2 Simple Corner Constraint

These have the form $AF@C$ or $BF@C$, where C is a corner (see Fig. 2(b)).

We say "Z satisfies $AF@C$" provided C is incident on the relative interior of $AF[Z]$.

B. Composite Constraints

B.1 **Pivot Constraint**

These have the form $F@C$ where C is a corner. We say "Z satisfies $F@C$" provided C coincides with $F[Z]$. We call such a Z *pivotal* (see Fig. 2(c)).

B.2 **Composite Wall Constraint**

These have the form $A \perp W$ or $B \perp W$ where W is a wall (see Fig. 2(d)).

We say "Z satisfies $A \perp W$" provided $A[Z]$touches W and is perpendicular to W. We call such a Z *reflecting* at W. The reason for this unusual terminology will be clear later — but essentially, a shortest motion can make a reflection at this point.

B.3 **Composite Corner Constraint**

These have the form $A \perp C$ or $B \perp C$ where C is a corner (see Fig. 2(e)).

We say "Z satisfies $A \perp C$" provided C coincides with $A[Z]$ and the two rays normal to the walls incident on C defines a closed cone that contains $AB[Z]$. Again, we call such a Z *reflecting* at C.

Constricting Pairs of Constraints

Wall-Wall cases

- **WW** A placement Z that satisfies two constraints of the form
  $A@W$ and $B@W$
  is necessarily constricted. The wall W is both a left- and a right-stop. (see Fig. 2(f))

- $\mathbf{W_1W_2}$ A placement Z that satisfies the constraints
  $A@W_1$ and $B@W_2$
  may be constricted, depending in part on the angle formed by the two walls. If this angle is acute then placements of the form illustrated in Fig. 2(h) are all constricting but those illustrated in Fig. 2(g) are constricting for only a portion of the associated ellipse. On the other hand, if the angle is obtuse, all placements of the form illustrated in Fig. 2(g) are constricting, while those illustrated in Fig. 2(h) are constricting for only a portion of the associated circle.

Wall-Corner Cases

- **WFC** A placement Z that satisfies the constraints
  $A@W$ and $BF@C$
  (or with the roles of A, B interchanged) may be constricted (see Fig. 2(i)).

- **WCF** A placement Z that satisfies the constraints
  $A@W$ and $AF@C$
  (or with the roles of A, B interchanged) may be constricted (see Fig. 2(j)).

Corner-Corner Cases

- $\mathbf{C_1FC_2}$ A placement Z that satisfies the constraints
  $AF@C_1$ and $BF@C_2$
  (or with the roles of A, B interchanged) is constricted iff C_1 and C_2 both lie on the same side of the rod AB. Here C_1, C_2 are both not at F (see Fig. 2(k)).

- $\mathbf{C_1C_2F}$ A placement Z that satisfies the constraints
  $AF@C_1$ and $AF@C_2$
  (or with the roles of A, B interchanged) is constricted iff C_1 and C_2 lie on the different sides of the rod AB. Here C_1, C_2 are distinct and C_2 is not at F (see Fig. 2(l)).

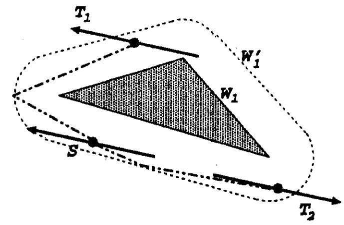

Figure 3: B-displaced features and reflecting motions

2.3 Special curves

Essential to our characterization is the observation that there are a finite number of "stopovers" which the focus must pass through in optimal motions. As in the well-studied case of a point robot, the identification of stopovers serves to reduce the search for optimal motions to finite number of possibilities because we can decompose any optimal motion into a finite number N of optimal submotions that connect pairs of stopovers. (N is no more than quadratic in the number of stopovers.) In the case of a point robot, stopovers can be identified with corners of obstacles. In our case of a (non-degenerate) rod, however, they include other curve segments as well.

When Z is constricted by a pair of constraints, it still has one degree of freedom to move while maintaining these two constraints. The trace of the motion that results when Z exercises this degree of freedom is a curve that has a very special nature. These curves constitute our new stopovers:

Stopover Curves Defined by Composite Constraints.

1. **Wall-Wall Cases** (Fig. 2(f)) In the WW subcase, the trace is clearly a straight line segment along W. In the W_1W_2 subcases, the trace is part of an ellipse (Fig. 2(g)) or a circle (Fig. 2(h)).

2. **Wall-Corner Cases** The trace is part of a conchoid. In the WFC subcase (Fig. 2(i)), the trace is in the "lower portion" of the conchoid. In the WCF subcase (Fig. 2(j)), the trace is part of the loop of a conchoid (see [7]).

3. **Corner-Corner Cases** (Fig. 2(k) and (l)) This trace is a clearly a straightline segment.

Mirrors. In addition to having more general stopover configurations, the optimal motion of a rod between successive stopovers is more involved than in the case of a point robot. This additional complexity arises from the need to include, in optimal motions, placements that satisfy $\perp$-constraints.

The set of points that are at distance $|AF|$ from a corner C forms a circular arc which we call the *A-displaced corner.* Similarly the circular arc centered at C at distance $|FB|$ from C is called the *B-displaced corner.* Again, the set of points that are at distance $|XF|$ ($X = A, B$) from a wall W is a line segment called the *X-displaced wall.* (Fig. 3 illustrates B-displaced walls and corners.)

Because of the role played by X-displaced corners and walls (where X is either A or B) in optimal motions, we refer to them as *mirrors* and we refer to placements satisfying $\perp$-constraints as *reflecting.* If the mirror is a displaced corner, we call it a *circular mirror*; otherwise it is a

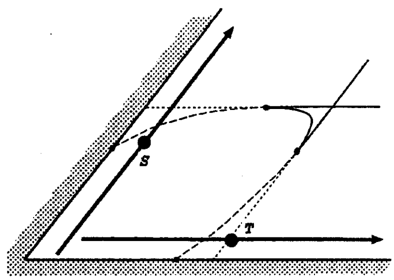

Figure 4: What is an d_1−optimal motion from S to T?

straight mirror coming from a displaced wall. The reason for this terminology will be clearer in the next subsection.

Since stopover curves are associated with pairs of features, there are in total $O(n^2)$ stopover curves and mirrors generated by a given obstacle set of complexity n.

An example of an optimal motion

In the following we give a nontrivial example of an optimal motion of a rod to convince the readers of the difficulty of the problem. Situation we consider is simple enough but the optimal motion is complicated enough.

Situation is as follows: Given a rod placement in an acute angle as shown in Fig. 4 we want to move it to the target placement shown in the figure. For simplicity we assume that the focus is fixed at the middle of the rod. An optimal motion in this case is as follows: First the rod is translated until it hits the straight mirror for the lower wall. Then, reflecting at the mirror according the Snell's law, it is translated again along a tangent line from the reflecting point to the ellipse defined by the two walls. Once it reaches the ellipse, sliding along the two walls (by tracing the ellipse), it leaves the ellipse along another tangent line from the mirrored point of the target point. After it reaches the straight mirror, it is again translated to the target point.

2.4 Local Characterization of Optimal Motion

Intuitively, the trace of an optimal motion must travel in a straight line unless it must bend around a convex corner or it is constricted (in which case there is no choice but to trace the conchoid or elliptic curves). But there is one other possibility, namely, the trace can reflect off a displaced feature in accordance to Snell's law. (Fig. 3 illustrates such reflecting motions from position S to positions T_1 and T_2 respectively.) Our characterization result, whose proof will appear in an expanded version of this paper, is that these possibilities are exhaustive.

The following theorem shows exactly what happens at a point that is not locally straight.

Theorem 1 *Let $\mu : [0,1] \to FP$ be an optimal motion and $0 \le t_0 \le 1$. If $F\mu$ is not locally straight at t_0 then one of the following three must hold:*
(i) $\mu(t_0)$ is pivotal at a corner C and $F\mu$ is locally "bending" around C.
(ii) $\mu(t_0)$ is constricted and the trace $F\mu$ is locally tangent to the stopover curve at $F\mu(t_0)$.
(iii) For some $t_1 \in [0,1]$, $F\mu(t) = F\mu(t_0)$ for all t between t_1 and t_0, and $\mu(t_1)$ is reflecting at some reflecting curve γ, and $F\mu$ is reflecting off γ according to Snell's law.

Using the above characterization, it is straightforward to prove the following:

Corollary 1 *Any optimal motion μ can be transformed into a motion μ' such that $F\mu = F\mu'$ and μ' consists of a finite sequence of $O(n^4)$ submotions in which each submotion has one of*

the following forms:
1. pure rotation around a pivot
2. pure translation along a straight line segment
3. dragging an endpoint along a wall in a straight trace
4. dragging the rod along a convex corner in a straight trace
5. sliding along two walls in an elliptical trace
6. sliding along a wall and a corner in a conchoidal trace.

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