アルゴリズム 59−4
(1997. 11. 28)

逐次診断可能次数の上界

大塚 高　　上野 修一

東京工業大学　電子物理工学科
〒152　東京都目黒区大岡山 2-12-1
E-mail: {*takashi, ueno*}*@ss.titech.ac.jp*

概要

N点から成る n 次元格子とハイパーキューブの逐次診断可能次数の下界はそれぞれ$\Omega(N^{n/(n+1)})$と$\Omega(N \log\log N/\log N)$ であることが知られている．本論文では，N点から成る n 次元格子とハイパーキューブの逐次診断可能次数の上界はそれぞれ $O(N^{n/(n+1)})$ と $O(N \log\log N/\sqrt{\log N})$ であることを示す．

Upper Bounds for the Degree of Sequential Diagnosability

Takashi OHTSUKA and Shuichi UENO

Department of Physical Electronics
Tokyo Institute of Technology
2–12–1 O-okayama, Meguro-ku, Tokyo 152, Japan
E-mail: {*takashi, ueno*}*@ss.titech.ac.jp*

Abstract

It is known that the degree of sequential diagnosability for an n-dimensional grid with N vertices is at least $\Omega(N^{n/(n+1)})$, and the degree of sequential diagnosability for an N-vertex hypercube is at least $\Omega(N \log\log N/\log N)$. This paper shows that the degree of sequential diagnosability for an n-dimensional grid with N vertices is at most $O(N^{n/(n+1)})$, and the degree of sequential diagnosability for an N-vertex hypercube is at most $O(N \log\log N/\sqrt{\log N})$.

1 Introduction

The system diagnosis has been extensively studied in the literature in connection with fault-tolerant multiprocessor computer systems. An original graph-theoretical model for system diagnosis was introduced in a classic paper by Preparata, Metze, and Chien [6]. In this model, the testing assignment is represented by a digraph (directed graph) associated with the interconnection graph of the system. The model assumes that the processors can test each other along available communication links. A testing processor evaluates a tested processor as fault-free or faulty. The evaluation is reliable if and only if the testing processor is fault-free. A syndrome is a collection of test results. The model also assumes that the number of faulty processors is bounded.

Two strategies for the diagnosis were introduced and discussed in [6]. A system is said to be one-step t-diagnosable if all faulty processors can be identified uniquely from any syndrome provided that the number of faulty processors does not exceed t. A system is said to be sequentially t-diagnosable if at least one faulty processor can be identified from any syndrome provided that the number of faulty processors does not exceed t. The degree of one-step [sequential] diagnosability of a system is the maximal t such that the system is one-step [sequentially] t-diagnosable. A characterization of one-step t-diagnosable systems by Hakimi and Amin [2] implies that the degree of one-step diagnosability of any system is bounded by the minimum degree of a vertex in its interconnection graph. On the other hand, it is known that the degree of sequential diagnosability of a system is significantly larger. In particular, Khanna and Fuchs proved that the degree of sequential diagnosability of any system with N vertices is at least $\Omega(\sqrt[3]{N})$ [5]. Unfortunately, computing the degree of sequential diagnosability of a system is co-NP hard as proved by Raghavan and Tripathi [7].

The grid and hypercube are popular interconnection graphs for multiprocessor computer systems. The sequential diagnosis for hypercubes was first considered by Kavianpour and Kim [3]. They proved that the degree of sequential diagnosability for an N-vertex hypercube is at least $\Omega(\sqrt{N \log N})$ [3]. Khanna and Fuchs also showed the same lower bound by giving a linear time algorithm for sequential diagnosis for hypercubes [4]. Furthermore, they showed that the degree of sequential diagnosability for an N-vertex hypercube is at least $\Omega(N \log\log N/\log N)$ [5]. In the same paper [5], they proved that the degree of sequential diagnosability for an n-dimensional grid with N vertices is at least $\Omega(N^{n/(n+1)})$.

The purpose of the paper is to give a tight upper bound on the degree of sequential diagnosability for grids and a nearly tight upper bound on the degree of sequential diagnosability for hypercubes. More precisely, we prove that the degree of sequential diagnosability for an n-dimensional grid with N vertices is at most $O(N^{n/(n+1)})$ and the degree of sequential diagnosability for an N-vertex hypercube is at most $O(N \log\log N/\sqrt{\log N})$. These are the first nontrivial upper bounds for the degree of sequential diagnosability of systems, to the best of our knowledge.

2 Sequential Diagnosis

The interconnection network of a multiprocessor computer system is modeled by a graph, called an interconnection graph, with the processors represented by the vertices of the graph and the communication links by the edges. The testing assignment in the system is modeled by a digraph, called a testing digraph, with the processors represented by the vertices of the digraph and the tests by the arcs (directed edges). If (x, y) is an arc of the testing digraph then the processor x tests processor y. A test is performed along an edge of the interconnection graph.

We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. We also denote the vertex set and arc set of a digraph D by $V(D)$ and $A(D)$, respectively. The associated digraph $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e.

Let D be a testing digraph of a system. A syndrome for D is a mapping $\sigma : A(D) \to \{0,1\}$ defined as follows:

$$\sigma(x,y) = \begin{cases} 0 & \text{if } x \text{ tests } y \text{ with outcome } \textit{pass} \\ 1 & \text{if } x \text{ tests } y \text{ with outcome } \textit{fail}, \end{cases}$$

where we denote $\sigma((x,y))$ simply by $\sigma(x,y)$. A set $F \subseteq V(D)$ is said to be a consistent fault set for a syndrome σ if neither i) nor ii) below holds:

i) $\sigma(x,y) = 0$ where $x \in V(D) - F$ and $y \in F$,
ii) $\sigma(x,y) = 1$ where $x, y \in V(D) - F$.

For any syndrome σ for D and positive integer t, define

$$\mathcal{F}(\sigma,t) = \{F \mid F \subseteq V(D) \text{ is a consistent fault set for } \sigma \text{ and } |F| \le t\}.$$

A syndrome σ is said to be in a t-fault situation if $\mathcal{F}(\sigma,t) \neq \emptyset$. D is said to be sequentially t-diagnosable if $\bigcap\{F \mid F \in \mathcal{F}(\sigma,t)\} \neq \emptyset$ for any syndrome σ for D in a t-fault situation. The degree of sequential diagnosability for D, denoted by $\delta(D)$, is the largest integer t for which D is sequentially t-diagnosable.

The degree of sequential diagnosability for various systems are considered in [5]. In particular, it is shown in [5] that $\delta(D) = \Omega(\sqrt[3]{N})$ if D is the associated digraph of an N-vertex graph and $\delta(D) = \Omega(\sqrt{N})$ if D

is the associated digraph of an N-vertex graph with bounded vertex-degree.

An n-dimensional m-sided grid $\mathcal{G}_n(m)$ is a graph with $N = m^n$ vertices defined as follows:

$$\begin{aligned} V(\mathcal{G}_n(m)) &= \{0,1,\cdots,m-1\}^n, \\ E(\mathcal{G}_n(m)) &= \{((x_1,\cdots,x_n),(y_1,\cdots,y_n))|(\exists j) \\ &\quad [|x_j - y_j| = 1], (\forall i \neq j)[x_i = y_i]\}. \end{aligned}$$

An n-dimensional m-sided torus $\mathcal{T}_n(m)$ is a graph with $N = m^n$ vertices defined as follows:

$$\begin{aligned} V(\mathcal{T}_n(m)) &= \{0,1,\cdots,m-1\}^n, \\ E(\mathcal{T}_n(m)) &= \{((x_1,\cdots,x_n),(y_1,\cdots,y_n))| \\ &\quad (\exists j)[|x_j - y_j| = 1 \text{ or } m-1], \\ &\quad (\forall i \neq j)[x_i = y_i]\}. \end{aligned}$$

An n-dimensional cube $\mathcal{H}_n$ is a graph with $N = 2^n$ vertices defined as follows:

$$\begin{aligned} V(\mathcal{H}_n) &= \{0,1\}^n, \\ E(\mathcal{H}_n) &= \{(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x},\boldsymbol{y} \in V(\mathcal{H}_n), d_H(\boldsymbol{x},\boldsymbol{y}) = 1\}, \end{aligned}$$

where $d_H(\boldsymbol{x},\boldsymbol{y})$ denotes the Hamming distance between $\boldsymbol{x}$ and $\boldsymbol{y}$. The following lower bounds can be found in the literature.

Theorem I [5] $\delta(D(\mathcal{G}_n(m))) = \Omega\left(N^{n/(n+1)}\right)$.

Corollary II [5] $\delta(D(\mathcal{T}_n(m))) = \Omega\left(N^{n/(n+1)}\right)$.

Theorem III [5] $\delta(D(\mathcal{H}_n)) = \Omega(\frac{N \log\log N}{\log N})$.

The purpose of this paper is to show the following upper bounds.

Theorem 1 $\delta(D(\mathcal{T}_n(m))) = O\left(N^{n/(n+1)}\right)$.

Theorem 2 $\delta(D(\mathcal{H}_n)) = O(\frac{N \log\log N}{\sqrt{\log N}})$ *if n is a power of* 2.

The following corollary is a direct consequence of Corollary II and Theorem 1.

Corollary 3 $\delta(D(\mathcal{T}_n(m))) = \Theta\left(N^{n/(n+1)}\right)$.

We also have the following corollary from Theorem I and Theorem 1.

Corollary 4 $\delta(D(\mathcal{G}_n(m))) = \Theta\left(N^{n/(n+1)}\right)$.

Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

3 Proof of Theorem 1

We prove the theorem by showing that there exist an integer $t = O(N^{n/(n+1)})$ and a syndrome σ_Ψ for $D(\mathcal{T}_n(m))$ such that $\bigcap\{F|F \in \mathcal{F}(\sigma_\Psi, t) \neq \emptyset\} = \emptyset$.

3.1 Partition of $V(D(\mathcal{T}_n(m)))$

Let Ψ be an integer such that $\Psi|m$, and $\rho = m/\Psi$. For each $\boldsymbol{t} \in \{0,\cdots,\Psi-1\}^n$, define $P(\boldsymbol{t})$ and $Q(\boldsymbol{t})$ as follows:

$$\begin{aligned} P(\boldsymbol{t}) &= \{\boldsymbol{x}|\boldsymbol{x} \in V(D(\mathcal{T}_n(m))), (\forall i)[\lfloor x_i/\rho \rfloor = t_i \\ &\quad \text{and } 1 \leq x_i \bmod \rho \leq \rho - 2]\}, \\ Q(\boldsymbol{t}) &= \{\boldsymbol{x}|\boldsymbol{x} \in V(D(\mathcal{T}_n(m))), (\forall i)[\lfloor x_i/\rho \rfloor = t_i] \\ &\quad \text{and } (\exists j)[x_j \bmod \rho = 0 \text{ or } \rho - 1]\}, \end{aligned}$$

where we denote the ith component of a vector $\boldsymbol{v}$ by v_i. It is easy to see that $(P((0,\cdots,0)),\cdots,P((\Psi-1,\cdots,\Psi-1)),(Q((0,\cdots,0)),\cdots,Q((\Psi-1,\cdots,\Psi-1))$ is a partition of $V(D(\mathcal{T}_n(m)))$. Let $\mathcal{P} = \bigcup_{\boldsymbol{t}} P(\boldsymbol{t})$ and $\mathcal{Q} = \bigcup_{\boldsymbol{t}} Q(\boldsymbol{t})$.

3.2 Syndrome and Fault Sets

The syndrome σ_Ψ for $D(\mathcal{T}_n(m))$ is defined as follows:

$$\sigma_\Psi(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 0 & \text{if } \begin{cases} 1.\ \boldsymbol{x},\boldsymbol{y} \in P(\boldsymbol{t}) \text{ for some } \boldsymbol{t}, \text{ or} \\ 2.\ \boldsymbol{x},\boldsymbol{y} \in Q(\boldsymbol{t}) \text{ for some } \boldsymbol{t}, \end{cases} \\ 1 & \text{otherwise.} \end{cases}$$

We define Ψ^n fault sets as follows:

$$F(\boldsymbol{t}) = P(\boldsymbol{t}) \cup (\mathcal{Q} - Q(\boldsymbol{t}))(\boldsymbol{t} \in \{0,\cdots,\Psi-1\}^n).$$

We prove Theorem 1 by showing the following claims.

Claim 1 *For any $\boldsymbol{t} \in \{0,\cdots,\Psi-1\}^n$, $F(\boldsymbol{t})$ is a consistent fault set for σ_Ψ.*

Claim 2 $\bigcap_{\boldsymbol{t} \in \{0,\cdots,\Psi-1\}^n} F(\boldsymbol{t}) = \emptyset$.

Claim 3 *For any $\boldsymbol{t} \in \{0,\cdots,\Psi-1\}^n$,*

$$|F(\boldsymbol{t})| = O\left(N^{n/(n+1)}\right).$$

3.3 Proof of Claim 1

We will prove the claim by showing that neither i) nor ii) below holds for any $\boldsymbol{t} \in \{0,\cdots,\Psi-1\}^n$:

i) $\sigma_\Psi(\boldsymbol{x},\boldsymbol{y}) = 0$ where $\boldsymbol{x} \in V(D(\mathcal{T}_n(m))) - F(\boldsymbol{t})$ and $\boldsymbol{y} \in F(\boldsymbol{t})$,

ii) $\sigma_\Psi(\boldsymbol{x},\boldsymbol{y}) = 1$ where $\boldsymbol{x},\boldsymbol{y} \in V(D(\mathcal{T}_n(m))) - F(\boldsymbol{t})$.

Let $F(\boldsymbol{t})$ be a fault set. Let $\boldsymbol{x} \in V(D(\mathcal{T}_n(m))) - F(\boldsymbol{t})$ and $(\boldsymbol{x},\boldsymbol{y}) \in A(D(\mathcal{T}_n(m)))$.

Case 1 $\boldsymbol{x} \in P(\boldsymbol{t}')$ for some $\boldsymbol{t}' \neq \boldsymbol{t}$ **:** The vertices adjacent to $\boldsymbol{x}$ are contained in $P(\boldsymbol{t}') \cup Q(\boldsymbol{t}')$.

Case 1.1 $\boldsymbol{y} \in F(\boldsymbol{t})$ **:** $\boldsymbol{y} \in Q(\boldsymbol{t}')$ and so $\sigma_\Psi(\boldsymbol{x},\boldsymbol{y}) = 1$.

Case 1.2 $\boldsymbol{y} \in V(D(\mathcal{T}_n(m))) - F(\boldsymbol{t})$ **:** $\boldsymbol{y} \in P(\boldsymbol{t}')$ and so $\sigma_\Psi(\boldsymbol{x},\boldsymbol{y}) = 0$.

Case 2 $\boldsymbol{x} \in Q(\boldsymbol{t})$ **:** The vertices adjacent to $\boldsymbol{x}$ are contained in $P(\boldsymbol{t}) \cup \mathcal{Q}$.

Case 2.1 $\boldsymbol{y} \in F(\boldsymbol{t})$ **:** $\boldsymbol{y} \notin Q(\boldsymbol{t})$ and so $\sigma_\Psi(\boldsymbol{x},\boldsymbol{y}) = 1$,

Case 2.2 $\boldsymbol{y} \in V(D(\mathcal{T}_n(m))) - F(\boldsymbol{t})$ **:** $\boldsymbol{y} \in Q(\boldsymbol{t})$ and so $\sigma_\Psi(\boldsymbol{x},\boldsymbol{y}) = 0$.

Thus, neither i) nor ii) holds for any arc $(\boldsymbol{x},\boldsymbol{y})$.

3.4 Proof of Claim 2

The claim follows from the fact that $Q(\boldsymbol{t}) \cap F(\boldsymbol{t}) = \emptyset$ for any $\boldsymbol{t}$ and $P(\boldsymbol{t}) \cap F(\boldsymbol{t}') = \emptyset$ for any distinct $\boldsymbol{t}$ and $\boldsymbol{t}'$.

3.5 Proof of Claim 3

$$\begin{aligned}
|P(\boldsymbol{t})| &< |\{\boldsymbol{x}|\boldsymbol{x} \in V(D(\mathcal{T}_n(m))), (\forall i)[\lfloor x_i/\rho \rfloor = t_i]\}| \\
&= \left(\frac{m}{\Psi}\right)^n. \\
|Q| &= |\{\boldsymbol{x}|\boldsymbol{x} \in V(D(\mathcal{T}_n(m))), \\
&\quad (\exists i)[x_i \bmod \rho = 0 \text{ or } \rho - 1]\}| \\
&< \sum_{i=1}^{n}\sum_{j=0}^{\Psi-1}(|\{\boldsymbol{x}|x_i = j\rho\}| + \\
&\quad |\{\boldsymbol{x}|x_i = (j+1)\rho - 1\}|) \\
&= 2n\Psi m^{n-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|F(\boldsymbol{t})| &= |P(\boldsymbol{t}) \cup (Q - Q(\boldsymbol{t}))| < |P(\boldsymbol{t}) \cup Q| \\
&< \left(\frac{m}{\Psi}\right)^n + 2n\Psi m^{n-1}.
\end{aligned}$$

If we choose $\Psi = (m/n)^{1/(n+1)}$, we have

$$\begin{aligned}
|F(\boldsymbol{t})| &< m^n\left(\frac{n}{m}\right)^{n/(n+1)} + 2n\left(\frac{m}{n}\right)^{1/(m+1)} m^{n-1} \\
&= O\left(N^{n/(n+1)}\right).
\end{aligned}$$

4 Proof of Theorem 2

We prove the theorem by showing that if n is a power of 2 then there exist a syndrome σ_Φ for $D(\mathcal{H}_n)$ and an integer t such that:

$$t = O(N \log\log N/\sqrt{\log N}), \text{ and}$$

$$\bigcap\{F|F \in \mathcal{F}(\sigma_\Phi, t) \neq \emptyset\} = \emptyset.$$

4.1 Partition of $V(D(\mathcal{H}_n))$

Let k be a non-negative integer. $\text{bin}(k,m)$ is the m-bit binary representation of k, and $\text{bin}(k,m,i)$ is the ith least significant bit of $\text{bin}(k,m)(0 \le k \le 2^m - 1, 1 \le i \le m)$. If $\boldsymbol{x} = \text{bin}(k,m)$ then we denote $k = \text{dec}(\boldsymbol{x})$. Let Ψ be an integer such that $1 \le \Psi \le n$, and $\Phi = 2^{\Psi+1}$. A sequence of n w's is denoted by w^n ($w \in \{0,1\}$). The concatination of binary strings $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $\boldsymbol{x}\cdot\boldsymbol{y}$. For an integer a such that $1 \le a \le \Psi + 1$, $\boldsymbol{r}(a)$ is a binary string of length n defined as follows:

$$\boldsymbol{r}(a) = \begin{cases} \left(0^{n/2^{\Psi-a+1}} \cdot 1^{n/2^{\Psi-a+1}}\right)^{2^{\Psi-a}} & \text{if } 1 \le a \le \Psi \\ 0^n & \text{if } a = \Psi + 1. \end{cases}$$

We consider $\boldsymbol{r}(a)$ as a vertex of $D(\mathcal{H}_n)$ in a natural way. Define $p_a(\boldsymbol{x})$ and $q_a(\boldsymbol{x})$ as follows:

$$p_a(\boldsymbol{x}) = \begin{cases} 0 & \text{if } 0 \le d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \le n/2 - 2 \\ 1 & \text{if } n/2 + 2 \le d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \le n \\ -1 & \text{if } n/2 - 1 \le d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \le n/2 + 1, \end{cases}$$

$$q_a(\boldsymbol{x}) = \begin{cases} 0 & \text{if } 0 \le d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \le n/2 - 1 \\ 1 & \text{if } n/2 + 1 \le d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \le n \\ -1 & \text{if } d_H(\boldsymbol{x}, \boldsymbol{r}(a)) = n/2. \end{cases}$$

It should be noted that if $p_a(\boldsymbol{x}) \in \{0,1\}$ then $q_a(\boldsymbol{x}) = p_a(\boldsymbol{x})$ by definition.

For an integer b such that $0 \le b \le \Phi - 1$, define subsets $P(b), Q(b)$, and $R(b)$ of $V(D(\mathcal{H}_n))$ as follows:

$$\begin{aligned}
P(b) &= \{\boldsymbol{x}|(\forall a)[p_a(\boldsymbol{x}) \in \{0,1\}], \\
&\quad \text{dec}(p_{\Psi+1}(\boldsymbol{x})\cdots p_1(\boldsymbol{x})) = b\}, \\
Q(b) &= \{\boldsymbol{x}|(\exists a')[p_{a'}(\boldsymbol{x}) = -1], (\forall a)[q_a(\boldsymbol{x}) \in \{0,1\}], \\
&\quad \text{dec}(q_{\Psi+1}(\boldsymbol{x})\cdots q_1(\boldsymbol{x})) = b\}, \\
R(b) &= \{\boldsymbol{x}|(\exists a)[q_a(\boldsymbol{x}) = -1], T(\boldsymbol{x}) = b\},
\end{aligned}$$

where $T(\boldsymbol{x})$ is the decimal representation of the most significant $\Psi + 1$ bits of $\boldsymbol{x}$. Define $\mathcal{P} = \bigcup_b P(b), \mathcal{Q} = \bigcup_b Q(b)$, and $\mathcal{R} = \bigcup_b R(b)$.

Lemma 1 *$\Pi = (P(0), \cdots, P(\Phi-1), Q(0), \cdots, Q(\Phi-1), R(0), \cdots, R(\Phi-1))$ is a partition of $V(D(\mathcal{H}_n))$.*

Proof : We will prove the lemma by showing the following:

i) for any distinct blocks U and U' of Π, $U \cap U' = \emptyset$;

ii) $\mathcal{P} \cup \mathcal{Q} \cup \mathcal{R} = V(D(\mathcal{H}_n))$.

Proof of i) : First of all, observe that $\mathcal{P} \cap \mathcal{Q} = \mathcal{Q} \cap \mathcal{R} = \mathcal{R} \cap \mathcal{P} = \emptyset$ by definition. We will show that $P(b) \cap P(b') = \emptyset$ for any distinct b and b' $(0 \le b, b' \le \Phi - 1)$. Assume contrary that $P(b) \cap P(b') \neq \emptyset$ for some distinct b and b'. There exists a such that $\text{bin}(b, \Psi+1, a) \neq \text{bin}(b', \Psi+1, a)$. Suppose without loss of generality that $\text{bin}(b, \Psi+1, a) = 0$ and $\text{bin}(b', \Psi+1, a) = 1$. Let $\boldsymbol{x} \in P(b) \cap P(b')$. Since $\boldsymbol{x} \in P(b)$, we have $p_a(\boldsymbol{x}) = 0$ and $d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \le n/2 - 2$. However, since $\boldsymbol{x} \in P(b')$, we also have $p_a(\boldsymbol{x}) = 1$ and $d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \ge n/2 + 2$, a contradiction. Thus, $P(b) \cap P(b') = \emptyset$ for any distinct b and b'. Similarly, it can be shown that $Q(b) \cap Q(b') = \emptyset$ for any distinct b and b'. It is easy to see that $R(b) \cap R(b') = \emptyset$ for any distinct b and b'.

Proof of ii) : Suppose $\boldsymbol{x} \in V(D(\mathcal{H}_n))$. For any a such that $1 \le a \le \Psi + 1$, we have $0 \le d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \le n$. If $d_H(\boldsymbol{x}, \boldsymbol{r}(a)) = n/2$ for some a then $q_a(\boldsymbol{x}) = -1$, and so $\boldsymbol{x} \in R(b)$ for b with $T(\boldsymbol{x}) = b$. If $d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \neq n/2$ for any a and $d_H(\boldsymbol{x}, \boldsymbol{r}(a')) = n/2 \pm 1$ for some a' then $q_a(\boldsymbol{x}) \in \{0,1\}$ and $p_{a'}(\boldsymbol{x}) = -1$, and so $\boldsymbol{x} \in Q(b)$ for b with $\text{dec}(q_{\Psi+1}(\boldsymbol{x})\cdots q_1(\boldsymbol{x})) = b$. If $d_H(\boldsymbol{x}, \boldsymbol{r}(a)) \notin \{n/2, n/2 \pm 1\}$ for any a then $p_a(\boldsymbol{x}) \in \{0,1\}$, and so $\boldsymbol{x} \in P(b)$ for b with $\text{dec}(p_{\Psi+1}(\boldsymbol{x})\cdots p_1(\boldsymbol{x})) = b$. Thus, we conclude that if $\boldsymbol{x} \in V(D(\mathcal{H}_n))$ then $\boldsymbol{x} \in \mathcal{P} \cup \mathcal{Q} \cup \mathcal{R}$ and we have $V(D(\mathcal{H}_n)) = \mathcal{P} \cup \mathcal{Q} \cup \mathcal{R}$. ∎

4.2 Syndrome and Fault Sets

The syndrome σ_Φ for $D(\mathcal{H}_n)$ is defined as follows:

$$\sigma_\Phi(x,y) = \begin{cases} 0 & \text{if } \begin{cases} 1.\ x,y \in P(b) \text{ for some } b, \\ 2.\ x \in Q(b) \text{ and } y \in R(b) \text{ for some } b, \text{ or} \\ 3.\ x \in R(b) \text{ and } y \in Q(b) \text{ for some } b, \end{cases} \\ 1 & \text{otherwise.} \end{cases}$$

We define Φ fault sets as follows:

$$F(b) = P(b) \cup (\mathcal{Q} - Q(b)) \cup (\mathcal{R} - R(b)) (0 \leq b \leq \Phi - 1).$$

We prove Theorem 2 by showing the following claims.

Claim 4 *For any $b(0 \leq b \leq \Phi - 1)$, $F(b)$ is a consistent fault set for σ_Φ.*

Claim 5 $\bigcap_{0 \leq b \leq \Phi - 1} F(b) = \emptyset.$

Claim 6 $|F(b)| = O(\frac{N \log\log N}{\sqrt{\log N}})$ *for any b $(0 \leq b \leq \Phi - 1)$.*

4.3 Proof of Claim 4

Before proving the claim, we need a couple of lemmas.

Lemma 2 *For any adjacent vertices $x, y \in V(D(\mathcal{H}_n))$,*

1. *if $x \in \mathcal{Q}$ then $y \notin \mathcal{Q}$.*
2. *if $x \in \mathcal{R}$ then $y \notin \mathcal{R}$.*

Proof : We will show *1*. Assume contrary that $x, y \in \mathcal{Q}$. Then, there exist a and a' such that $p_a(x) \neq -1, q_a(x) = -1, p_{a'}(y) \neq -1$, and $q_{a'}(y) = -1$. We also have that $d_H(x, r(a)) = n/2 \pm 1$ and $d_H(y, r(a')) = n/2 \pm 1$. Since $d_H(r(a), r(\Psi+1)), d_H(r(\Psi+1), r(a')) = n/2$ or 0, we conclude that $d_H(x, r(a)) + d_H(r(a), r(\Psi+1)) + d_H(r(\Psi+1), r(a')) + d_H(r(a'), y)$ is even. However, since x and y are adjacent, $d_H(x, y) = 1$, which is odd, a contradiction.

We can show *2* by a similar argument. ∎

Lemma 3 *For any $b(0 \leq b \leq \Phi - 1)$,*

1. *The vertices adjacent to $x \in P(b)$ are contained in $P(b) \cup Q(b)$.*
2. *The vertices adjacent to $x \in Q(b)$ are contained in $P(b) \cup \mathcal{R}$.*
3. *The vertices adjacent to $x \in R(b)$ are contained in $\mathcal{Q}$.*

Proof : We will show *1*. Let $x \in P(b)$ and y be a vertex adjacent to x. Then $|d_H(y, r(a)) - d_H(x, r(a))| = 1$ for any a. If $p_a(x) = 0$ then $0 \leq d_H(x, r(a)) \leq n/2 - 2$. Thus we have $0 \leq d_H(y, r(a)) \leq n/2 - 1$, and so $q_a(y) = 0$. If $p_a(x) = 1$ then $n/2 + 2 \leq d_H(x, r(a)) \leq n$. Thus we have $n/2+1 \leq d_H(y, r(a)) \leq n$, and so $q_a(y) = 1$. Thus we conclude that $q_a(y) = p_a(x)$ for any a and so $\mathrm{dec}(q_{\Psi+1}(y) \cdots q_1(y)) = b$. If there exists a' such that $d_H(y, r(a')) = n/2 \pm 1$ then $y \in Q(b)$. Otherwise, $p_a(y) = q_a(y)$ for any a, and so $y \in P(b)$.

2 and *3* follow from *1* and Lemma 2. ∎

We will prove Claim 4 by showing that neither i) nor ii) below holds for any b:

i) $\sigma_\Phi(x,y) = 0$ where $x \in V(D(\mathcal{H}_n)) - F(b)$ and $y \in F(b)$,

ii) $\sigma_\Phi(x,y) = 1$ where $x, y \in V(D(\mathcal{H}_n)) - F(b)$.

Let $F(b)$ be a fault set. Let $x \in V(D(\mathcal{H}_n)) - F(b)$ and $(x,y) \in A(D(\mathcal{H}_n))$.

Case 1 $x \in P(b')$ for some $b' \neq b$: From Lemma 3, the vertices adjacent to x are contained in $P(b') \cup Q(b')$.

Case 1.1 $y \in F(b)$: $y \in Q(b')$ and so $\sigma_\Phi(x,y) = 1$.

Case 1.2 $y \in V(D(\mathcal{H}_n)) - F(b)$: $y \in P(b')$ and so $\sigma_\Phi(x,y) = 0$.

Case 2 $x \in Q(b)$: From Lemma 3, the vertices adjacent to x are contained in $P(b) \cup \mathcal{R}$.

Case 2.1 $y \in F(b)$: $y \notin R(b)$ and so $\sigma_\Phi(x,y) = 1$,

Case 2.2 $y \in V(D(\mathcal{H}_n)) - F(b)$: $y \in R(b)$ and so $\sigma_\Phi(x,y) = 0$.

Case 3 $x \in R(b)$: From Lemma 3, the vertices adjacent to x are contained in $\mathcal{Q}$.

Case 3.1 $y \in F(b)$: $y \notin Q(b)$ and so $\sigma_\Phi(x,y) = 1$,

Case 3.2 $y \in V(D(\mathcal{H}_n)) - F(b)$: $y \in Q(b)$ and so $\sigma_\Phi(x,y) = 0$.

Thus, neither i) nor ii) holds for any arc (x,y).

4.4 Proof of Claim 5

The claim follows from the fact that $Q(b) \cap F(b) = R(b) \cap F(b) = \emptyset$ for any b and $P(b) \cap F(b') = \emptyset$ for any distinct b and b'.

4.5 Proof of Claim 6

We will prove the claim by a series of lemmas.

Lemma 4 $|\mathcal{Q}| < 2(\Psi + 1) \binom{n}{n/2 - 1}.$

Proof :

$$\begin{aligned} |\mathcal{Q}| &= |\{x | (\exists a)[d_H(x, r(a)) = n/2 \pm 1]\}| \\ &< |\{x | (\exists a)[d_H(x, r(a)) = n/2 - 1]\}| \\ &\quad + |\{x | (\exists a)[d_H(x, r(a)) = n/2 + 1]\}| \\ &< \sum_{i=1}^{\Psi+1} (|\{x | d_H(x, r(i)) = n/2 - 1\}| \end{aligned}$$

$$+|\{x|d_H(x,r(i)) = n/2+1\}|)$$
$$= 2\sum_{i=1}^{\Psi+1}\binom{n}{n/2-1}$$
$$= 2(\Psi+1)\binom{n}{n/2-1}.$$

∎

Lemma 5 $|\mathcal{R}| < (\Psi+1)\binom{n}{n/2}$.

Proof :

$$|\mathcal{R}| = |\{x|(\exists a)[d_H(x,r(a)) = n/2]\}|$$
$$< \sum_{i=1}^{\Psi+1}|\{x|d_H(x,r(i)) = n/2\}|$$
$$= \sum_{i=1}^{\Psi+1}\binom{n}{n/2}$$
$$= (\Psi+1)\binom{n}{n/2}.$$

∎

Lemma 6 *For any b and b' ($0 \le b, b' \le \Phi - 1$), $|P(b)| = |P(b')|$.*

Proof : For any integers k and a ($0 \le k \le 2^{\Psi} - 1, 1 \le a \le \Psi$), let $\mathrm{ex}(k,a)$ denote the integer such that $\mathrm{bin}(\mathrm{ex}(k,a),\Psi)$ and $\mathrm{bin}(k,\Psi)$ differ just in the ath least significant bit. It should be noted that $b = \mathrm{ex}(\mathrm{ex}(b,a),a)$.

We prove the lemma by showing the following.

Claim 7 $|P(b)| = |P(ex(b,a))|$ *for any b and a ($0 \le b \le \Phi - 1, 1 \le a \le \Psi + 1$).*

Proof of Claim 7 : Before proving the claim, we need some preliminaries. For any x, let $x_u = (x_{n/2^{\Psi}\times(u+1)-1}\cdots x_{n/2^{\Psi}\times u})$ be an $n/2^{\Psi}$-bit substring of x. For any distinct a and $a'(1 \le a, a' \le \Psi)$ and $w, w' \in \{0,1\}$, let

$$W_{awa'w'}(x) = \sum\{w_H(x_u)|\mathrm{bin}(u,\Psi,a) = w \text{ and } \mathrm{bin}(u,\Psi,a') = w'\},$$
$$W_{aw}(x) = \sum\{w_H(x_u)|\mathrm{bin}(u,\Psi,a) = w\}$$
$$= W_{awa'1}(x) + W_{awa'0}(x),$$

where $w_H(x_u)$ denotes the Hamming weight of x_u. For any x and a ($1 \le a \le \Psi$), let

$$e_a(x) = x_{\mathrm{ex}(2^{\Psi}-1,a)}\cdot x_{\mathrm{ex}(2^{\Psi}-2,a)}\cdots x_{\mathrm{ex}(0,a)}.$$

It should be noted that e_a is a one-to-one mapping and that $W_{a1}(e_a(x)) = W_{a0}(x)$ and $W_{a0}(e_a(x)) = W_{a1}(x)$ for any a ($1 \le a \le \Psi$).

Claim 8 *For any a and u ($1 \le a \le \Psi$, $0 \le u \le 2^{\Psi} - 1$), $r(a)_u = 1^{n/2^{\Psi}}$ if and only if $bin(u,\Psi,a) = 0$.*

Proof of Claim 8 : By the definition of $r(a)$, it is easy to see that if $\mathrm{bin}(u,\Psi,a) = 1$ then $r(a)_u = 0^{n/2^{\Psi}}$ and if $\mathrm{bin}(u,\Psi,a) = 0$ then $r(a)_u = 1^{n/2^{\Psi}}$.

∎

Claim 9 *For any x and $a(1 \le a \le \Psi+1)$,*

1. *if $1 \le a \le \Psi$ then $d_H(x,r(a)) = W_{a1}(x)+(n/2-W_{a0}(x))$.*
2. *if $a = \Psi+1$ then $d_H(x,r(a)) = W_{a'1}(x) + W_{a'0}(x)$ for any $a'(1 \le a' \le \Psi)$.*

Proof of Claim 9 : (Proof of *1*) Suppose $1 \le a \le \Psi$. By the definition of $W_{aw}(x)$ and Claim 8, we have

$$d_H(x,r(a))$$
$$= \sum\{d_H(x_u,r(a)_u)|\mathrm{bin}(u,\Psi,a) = 1\}$$
$$+\sum\{d_H(x_u,r(a)_u)|\mathrm{bin}(u,\Psi,a) = 0\}$$
$$= \sum\{d_H(x_u,0^{n/2^{\Psi}})|\mathrm{bin}(u,\Psi,a) = 1\}$$
$$+\sum\{d_H(x_u,1^{n/2^{\Psi}})|\mathrm{bin}(u,\Psi,a) = 0\}$$
$$= \sum\{w_H(x_u)|\mathrm{bin}(u,\Psi,a) = 1\}$$
$$+(n/2-\sum\{w_H(x_u)|\mathrm{bin}(u,\Psi,a) = 0\})$$
$$= W_{a1}(x) + (n/2 - W_{a0}(x)).$$

(Proof of *2*) If $a = \Psi+1$ then $r(a) = 0^n$ by definition and so $d_H(x,r(a)) = w_H(x) = W_{a'1}(x) + W_{a'0}(x)$ for any $a'(1 \le a' \le \Psi)$.

Claim 10 *For any distinct a and $a'(1 \le a \le \Psi$, $1 \le a' \le \Psi+1)$,*

1. *$d_H(e_a(x),r(a)) = n - d_H(x,r(a))$.*
2. *$d_H(e_a(x),r(a')) = d_H(x,r(a'))$.*

Proof of Claim 10 : (Proof of *1*) Since $W_{a1}(e_a(x)) = W_{a0}(x)$ and $W_{a0}(e_a(x)) = W_{a1}(x)$ as mentioned earlier, we have from Claim 9 that

$$d_H(e_a(x),r(a))$$
$$= W_{a1}(e_a(x)) + (n/2 - W_{a0}(e_a(x)))$$
$$= W_{a0}(x) + (n/2 - W_{a1}(x))$$
$$= n - \{W_{a1}(x) + (n/2 - W_{a0}(x))\}$$
$$= n - d_H(x,r(a)).$$

(Proof of *2*) If $1 \le a' \le \Psi$ then we have from Claim 9 that

$$d_H(e_a(x),r(a'))$$
$$= W_{a'1}(e_a(x)) + (n/2 - W_{a'0}(e_a(x)))$$

$$
\begin{aligned}
&= (W_{a'1a1}(e_a(\boldsymbol{x})) + W_{a'1a0}(e_a(\boldsymbol{x}))) \\
&\quad +\{n/2 - (W_{a'0a1}(e_a(\boldsymbol{x})) + W_{a'0a0}(e_a(\boldsymbol{x})))\} \\
&= (W_{a'1a0}(\boldsymbol{x}) + W_{a'1a1}(\boldsymbol{x})) \\
&\quad +\{n/2 - (W_{a'0a0}(\boldsymbol{x}) + W_{a'0a1}(\boldsymbol{x}))\} \\
&= W_{a'1}(\boldsymbol{x}) + (n/2 - W_{a'0}(\boldsymbol{x})) \\
&= d_H(\boldsymbol{x}, \boldsymbol{r}(a')).
\end{aligned}
$$

If $a' = \Psi + 1$ then we have from Claim 9 that

$$
\begin{aligned}
d_H(e_a(\boldsymbol{x}), \boldsymbol{r}(\Psi+1)) &= W_{a1}(e_a(\boldsymbol{x})) + W_{a0}(e_a(\boldsymbol{x})) \\
&= W_{a0}(\boldsymbol{x}) + W_{a1}(\boldsymbol{x}) \\
&= d_H(\boldsymbol{x}, \boldsymbol{r}(\Psi+1)).
\end{aligned}
$$

This completes the proof of Claim 10

Now we are ready to prove Claim 7. We distinguish two cases.

Case 1 $1 \le a \le \Psi$ **:** We first show the following.

Claim 11 *For any b and $a(0 \le b \le \Phi - 1,\ 1 \le a \le \Psi)$,*

$$\boldsymbol{x} \in P(b) \Rightarrow e_a(\boldsymbol{x}) \in P(ex(b, a)).$$

Proof of Claim 11 : From Claim 10, if $\boldsymbol{x} \in P(b)$ then $p_a(e_a(\boldsymbol{x})) \notin \{n/2, n/2 \pm 1\}$ and so $e_a(\boldsymbol{x}) \in \mathcal{P}$. We can also see from Claim 10 that if $\boldsymbol{x} \in P(b)$ then $p_a(e_a(\boldsymbol{x})) \neq p_a(\boldsymbol{x})$ and $p_{a'}(e_a(\boldsymbol{x})) = p_{a'}(\boldsymbol{x})$ for any distinct a and a' $(1 \le a \le \Psi, 1 \le a' \le \Psi + 1)$. Thus,

$$
\begin{aligned}
&\mathrm{dec}(p_{\Psi+1}(e_a(\boldsymbol{x})) \cdots p_1(e_a(\boldsymbol{x}))) \\
&\quad = \mathrm{dec}(p_{\Psi+1}(\boldsymbol{x}) \cdots \overline{p_a(\boldsymbol{x})} \cdots p_1(\boldsymbol{x})) \\
&\quad = \mathrm{ex}(b, a),
\end{aligned}
$$

where $\bar{\boldsymbol{v}}$ is the compliment of $\boldsymbol{v}$. Thus, we conclude that if $\boldsymbol{x} \in P(b)$ then $e_a(\boldsymbol{x}) \in P(\mathrm{ex}(b, a))$ for any b and $a(0 \le b \le \Phi - 1, 1 \le a \le \Psi)$. This completes the proof of Claim 11.

Since e_a is a one-to-one mapping and $\mathrm{ex}(\mathrm{ex}(b, a), a) = b$ as mentioned above, we have from Claim 11 that $|P(b)| = |P(\mathrm{ex}(b, a))|$ for any b and $a(0 \le b \le \Phi - 1, 1 \le a \le \Psi)$.

Case 2 $a = \Psi + 1$ **:** We need more preliminaries. Let $\mathcal{E} : V(D(\mathcal{H}_n)) \to V(D(\mathcal{H}_n))$ be a mapping such that $\mathcal{E} : \boldsymbol{x} \mapsto e_\Psi \circ e_{\Psi-1} \circ \cdots \circ e_1(\bar{\boldsymbol{x}})$, where $\circ$ denotes the composition of mappings. It should be noted that $\mathcal{E}$ is a one-to-one mapping.

Claim 12 *For any $a(1 \le a \le \Psi + 1)$, $d_H(\bar{\boldsymbol{x}}, \boldsymbol{r}(a)) = n - d_H(\boldsymbol{x}, \boldsymbol{r}(a))$.*

Proof of Claim 12 : Since $W_{aw}(\bar{\boldsymbol{x}}) = n/2 - W_{aw}(\boldsymbol{x})$ for any $a(1 \le a \le \Psi)$ and $w \in \{0, 1\}$, if $1 \le a \le \Psi$ then

$$
\begin{aligned}
d_H(\bar{\boldsymbol{x}}, \boldsymbol{r}(a)) &= W_{a1}(\bar{\boldsymbol{x}}) + (n/2 - W_{a0}(\bar{\boldsymbol{x}})) \\
&= (n/2 - W_{a1}(\boldsymbol{x})) \\
&\quad +\{n/2 - (n/2 - W_{a0}(\boldsymbol{x}))\} \\
&= n - \{W_{a1}(\boldsymbol{x}) + (n/2 - W_{a0}(\boldsymbol{x}))\} \\
&= n - d_H(\boldsymbol{x}, \boldsymbol{r}(a)).
\end{aligned}
$$

If $a = \Psi + 1$ then $\boldsymbol{r}(a) = 0^n$ and $d_H(\bar{\boldsymbol{x}}, \boldsymbol{r}(a)) = w_H(\bar{\boldsymbol{x}}) = n - w_H(\boldsymbol{x}) = n - d_H(\boldsymbol{x}, \boldsymbol{r}(a))$.

Claim 13

1. *$d_H(\mathcal{E}(\boldsymbol{x}), \boldsymbol{r}(a)) = d_H(\boldsymbol{x}, \boldsymbol{r}(a))$ for any $a(1 \le a \le \Psi)$.*
2. *$d_H(\mathcal{E}(\boldsymbol{x}), \boldsymbol{r}(\Psi+1)) = n - d_H(\boldsymbol{x}, \boldsymbol{r}(\Psi+1))$.*

Proof of Claim 13 : (Proof of *1*) From Claims 10 and 12,

$$
\begin{aligned}
d_H(\mathcal{E}(\boldsymbol{x}), \boldsymbol{r}(a)) &= d_H(e_a(\bar{\boldsymbol{x}}), \boldsymbol{r}(a)) \\
&= n - d_H(\bar{\boldsymbol{x}}, \boldsymbol{r}(a)) \\
&= n - (n - d_H(\boldsymbol{x}, \boldsymbol{r}(a))) \\
&= d_H(\boldsymbol{x}, \boldsymbol{r}(a)).
\end{aligned}
$$

(Proof of *2*) From Claims 10 and 12,

$$
\begin{aligned}
d_H(\mathcal{E}(\boldsymbol{x}), \boldsymbol{r}(\Psi+1)) &= d_H(\bar{\boldsymbol{x}}, \boldsymbol{r}(\Psi+1)) \\
&= n - d_H(\boldsymbol{x}, \boldsymbol{r}(\Psi+1)).
\end{aligned}
$$

Now we are ready to prove Claim 7 for Case 2. We first show the following.

Claim 14 *For any $b(0 \le b \le \Phi - 1)$,*

$$\boldsymbol{x} \in P(b) \Rightarrow \mathcal{E}(\boldsymbol{x}) \in P(ex(b, \Psi + 1)).$$

Proof of Claim 14 : From Claim 13, if $\boldsymbol{x} \in P(b)$ then $p_a(\mathcal{E}(\boldsymbol{x})) \notin \{n/2, n/2 \pm 1\}$ and so $\mathcal{E}(\boldsymbol{x}) \in \mathcal{P}$. It is also seen from Claim 13 that if $\boldsymbol{x} \in P(b)$ then $p_a(\mathcal{E}(\boldsymbol{x})) = p_a(\boldsymbol{x})$ for any $a(1 \le a' \le \Psi)$ and $p_{\Psi+1}(\mathcal{E}(\boldsymbol{x})) \neq p_{\Psi+1}(\boldsymbol{x})$. Thus,

$$
\begin{aligned}
&\mathrm{dec}(p_{\Psi+1}(\mathcal{E}(\boldsymbol{x})) \cdots p_1(\mathcal{E}(\boldsymbol{x}))) \\
&\quad = \mathrm{dec}(\overline{p_{\Psi+1}(\boldsymbol{x})} p_\Psi(\boldsymbol{x}) \cdots p_1(\boldsymbol{x})) \\
&\quad = \mathrm{ex}(b, \Psi + 1).
\end{aligned}
$$

Thus, we conclude that if $\boldsymbol{x} \in P(b)$ then $\mathcal{E}(\boldsymbol{x}) \in P(\mathrm{ex}(b, \Psi + 1))$ for any $b(0 \le b \le \Phi - 1)$. This completes the proof of Claim 14.

Since $\mathcal{E}$ is a one-to-one mapping and $\mathrm{ex}(\mathrm{ex}(b, \Psi + 1), \Psi + 1) = b$ as mentioned above, we have from Claim 14 that $|P(b)| = |P(\mathrm{ex}(b, \Psi + 1))|$ for any $b(0 \le b \le \Phi - 1)$.

∎

Lemma 7 $|P(b)| < 2^n/\Phi$ *for any $b(0 \le b \le \Phi - 1)$.*

Proof : From lemma 6, we have

$$|P(b)| = |\mathcal{P}|/\Phi, \tag{1}$$

for any $b(0 \le b \le \Phi - 1)$. We also have

$$|\mathcal{P}| < |V(D(\mathcal{H}_n))| = 2^n, \tag{2}$$

from Lemma 1. From (1) and (2), we have the lemma. ∎

Lemma 8 $|F(b)| = O\left(\frac{N \log\log N}{\sqrt{\log N}}\right)$.

Proof : From lemmas 4, 5 and 7,

$$\begin{aligned} |F(b)| &= |P(b)| + |(\mathcal{Q} - Q(b))| + |(\mathcal{R} - R(b))| \\ &< |P(b)| + |\mathcal{Q}| + |\mathcal{R}| \\ &< \frac{2^n}{\Phi} + 2(\Psi + 1)\binom{n}{n/2-1} \\ &\quad + (\Psi + 1)\binom{n}{n/2}. \end{aligned}$$

It is well-known [1] that

$$\binom{n}{n/2}, \binom{n}{n/2-1} = O\left(\frac{2^n}{\sqrt{n}}\right).$$

Thus, we have

$$|F(b)| = O\left(\frac{2^n}{\Phi} + (\Psi + 1)\frac{2^n}{\sqrt{n}}\right).$$

If we choose $\Psi = (\log n)/2 - \log\log n - 1$ then $\Phi = \sqrt{n}/\log n$ and we have

$$|F(b)| = O\left(\frac{2^n \log n}{\sqrt{n}}\right) = O\left(\frac{N \log\log N}{\sqrt{\log N}}\right).$$

∎

Acknowledgement

The authors are grateful to Prof. Y. Kajitani for his encouragements. The research is a part of CAD21 Project at TIT.

References

[1] T. H. Cormen, C. E. Leiserson, and R. L. Rivest. *Introduction to Algorithms.* The MIT Press, 1990.

[2] S. L. Hakami and A. T. Amin. Characterization of connection assignment of diagnosable systems. *IEEE Transactions on Computers*, Vol. 23, pp. 86–88, 1974.

[3] A. Kavianpour and K. H. Kane Kim. A comparative evaluation of four basic system-level diagnosis strategies for hypercubes. *IEEE Transactions on Reliability*, Vol. 41, pp. 26–37, 1992.

[4] S. Khanna and W. K. Fuchs. A linear time algorithm for sequential diagnosis in hypercubes. *Journal of Parallel and Distributed Computing*, Vol. 26, No. 1, pp. 48–53, February 1995.

[5] S. Khanna and W. K. Fuchs. A graph partitioning approach to sequential diagnosis. *IEEE Transactions on Computers*, Vol. 46, pp. 39–47, 1997.

[6] F. P. Preparata, G. Metze, and R. T. Chien. On the connection assignment problem of diagnosable systems. *IEEE Transactions on Electronic Computers*, Vol. 16, pp. 848–854, 1967.

[7] V. Raghavan and A. Tripathi. Sequential diagnosis is co-NP complete. *IEEE Transactions on Computers*, Vol. 40, pp. 584–595, 1991.